Topological Hochschild homology

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1. Model structure on symmetric spectra

We first have to finish the discussion of model structures on symmetric spectra from last time. For some of the technical details the sources refer to symmetric spectra in based simplicial sets.

Definition A symmetric spectrum X is *injective* if for every monomorphism which is also a level equivalence $i\colon A \to B$ and every morphism $f\colon A \to X$ there is a morphism $g\colon B \to X$ with $f = g \circ i$.

Every symmetric spectrum is level equivalent to an injective spectrum [HSS, Corollary 5.1.3]. We can now describe the stable model structure on symmetric spectra.

Definition A morphism $f\colon E \to F$ of symmetric spectra is a *stable equivalence* if $W^0(f)\colon W^0(F) \to W^0(E)$ is an isomorphism for all injective Ω spectra W in Sp^Σ.

(Here, W is an Ω-spectrum if it is a symmetric spectrum consisting of Kan complexes W_n such that the adjoints of the structure maps are weak equivalences of simplicial sets.)

One can describe a corresponding stable model structure on symmetric spectra via a Bousfield localization of the level structure, so that stable cofibrations are level cofibrations, weak equivalences are stable equivalences and fibrations are what they have to be.

The good news is then that there is a Quillen equivalence between the homotopy category of the stable model structure on symmetric spectra and the homotopy category of the stable model structure on Bousfield-Friedlander spectra [HSS, Theorem 4.2.5]. So we do get a model for the stable homotopy category together with a decent smash product.

1.1. Model structures on ring spectra and modules.

As Sp^Σ is a symmetric monoidal category with respect to the smash product, we can define ring spectra and modules over ring spectra without any difficulties.

Definition We call a symmetric spectrum R a *ring spectrum*, if it is a monoid in $(Sp^\Sigma, \wedge)$. Explicitly, this means that there are maps

$$\mu\colon R \wedge R \to R,\ \eta\colon S \to R$$

such that η is a unit and μ is associative.

It is a *commutative ring spectrum* if it is a commutative monoid.

Similarly, a symmetric spectrum M is an *R-module spectrum* for a ring spectrum R, if there is a map in Sp^Σ

$$\nu\colon R \wedge M \to M$$

such that the following diagrams commute:

$$\begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{\mu \wedge M} & R \wedge M \\ {\scriptstyle R\wedge\nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ R \wedge M & \xrightarrow{\nu} & M, \end{array} \qquad \begin{array}{ccc} M & = & M \\ {\scriptstyle \cong}\downarrow & & \uparrow{\scriptstyle \nu} \\ S \wedge M & \xrightarrow{\eta \wedge M} & R \wedge M. \end{array}$$

Right R-modules and R-bimodule spectra can be defined analogously.

We can also work relative to a commutative ring spectrum R and consider $M \wedge_R N$ for R-modules M and N defined as the coequalizer

$$M \wedge R \wedge N \rightrightarrows M \wedge N \longrightarrow M \wedge_R N.$$

We can then define (commutative) R-algebra spectra as (commutative) monoids with respect to this relative smash product.

For R-modules there is a model category structure in which the weak equivalence and fibrations are defined via the forgetful functor to Sp^Σ. If R is a commutative ring spectrum, then the same is true for the category of associative R-algebras [MMSS, Theorem 12.1]. For commutative R-algebra spectra one has to consider a *positive* variant [MMSS, §15].

Shipley established a 'convenient' model structure on commutative R-algebra spectra in Sp^Σ that uses a different model structure on the category of modules. Her structure has the remarkable property that cofibrant objects in the model structure for commutative R-algebras still have underlying cofibrant objects.

2. Definitions of topological Hochschild homology

For a ring spectrum R and an R-bimodule spectrum M we can form the following simplicial spectrum:

$$M \leftleftarrows M \wedge R \leftleftarrows M \wedge R \wedge R \leftleftarrows \ldots,$$

where the degeneracy map $s_i\colon M \wedge R^{\wedge n} \to M \wedge R^{\wedge n+1}$ inserts the unit map $\eta\colon S \to R$ after the ith factor of R and the face maps $d_i\colon M \wedge R^{\wedge n} \to M \wedge R^{\wedge n-1}$ for $0 < i < n$ are given by the multiplication in R of the ith and $(i+1)$st smash factor. The zeroth face map d_0 uses the right R-module structure on M and the last face map d_n cyclically permutes the smash factors to bring the last one to the front and then uses the left module structure on M.

One should assume that R and M are cofibrant. Then the geometric realization of the above simplicial spectrum is $\mathsf{THH}(R; M)$. Working with the relative smash product one can define $\mathsf{THH}^R(A; M)$ in a similar manner.

Some facts (see [EKMM, Chapter IX]):

- Assuming enough cofibrancy, one can show that this definition of $\mathsf{THH}^R(A; M)$ is equivalent to $M \wedge_{A^e} A$ where $A^e = A \wedge_R A^{op}$. So that's an analogue of the Tor-interpretation of Hochschild homology in the flat case.
- If R is a (discrete) commutative ring and A is an R-algebra such that A is flat as an R-module, then for every A-bimodule M one has:

  $$\mathsf{HH}^R_*(A; M) \cong \mathsf{THH}^{HR}_*(HA; HM).$$

  If A is commutative, then this is an isomorphism of graded A-algebras.

  If A is not flat as an R-module, then $\mathsf{THH}^{HR}_*(HA; HM)$ is isomorphic to a derived version of Hochschild homology, which is called *Shukla homology*.

  Note that there is always a change-of-rings map from $\mathsf{THH}_*(HR)$ to $\mathsf{THH}^{H\mathbb{Z}}_*(HR)$ induced by the unique map of ring spectra $S \to H\mathbb{Z}$. One can prolong this to a map to Hochschild homology by using maps of the type $HR \wedge_{H\mathbb{Z}} HR \to H(R \otimes_{\mathbb{Z}} R)$.

  The map $\mathsf{THH}_*(HR) \to \mathsf{THH}^{H\mathbb{Z}}_*(HR)$ is an iso in degrees $* = 0, 1, 2$; the map to $\mathsf{HH}_*(R)$ is 2-connected. This can be seen by using a spectral sequence developed by Pirashvili-Waldhausen [PiWa]

  $$\mathrm{Shukla}_p(R; \mathsf{THH}_q(\mathbb{Z}; R)) \Rightarrow \mathsf{THH}_{p+q}(R)$$

and the calculations of $\mathsf{THH}_*(\mathbb{Z})$ by Bökstedt.

So we now have four different models for the 'same' thing: MacLane homology, $\mathsf{HML}_*(R)$, stable K-theory, $K_*^{st}(R)$, topological Hochschild homology, $\mathsf{THH}_*(HR)$ and the theory defined in terms of functor homology, $\mathsf{Tor}_*^{\mathcal{F}(R)}(R,R) \cong H_*(F(R), \mathrm{Hom}(R,R))$ and also the variants where one might take coefficients in an R-bimodule. Thus we have several different tools at hand, ranging from homological algebra to unstable and stable homotopy theory.

3. Examples

There are different levels of THH-calculations. Maybe one can identify the homotopy groups $\pi_*\mathsf{THH}(R) =: \mathsf{THH}_*(R)$ or $\pi_*\mathsf{THH}^R(A) =: \mathsf{THH}_*^R(A)$ but sometimes one might even identify topological Hochschild homology as a spectrum or in the commutative case even as a commutative ring spectrum.

(1) A classical example of a THH-calculation is the one of $H\mathbb{Z}$ and $H\mathbb{F}_p$ by Marcel Bökstedt (see [L, Chapter 13]):

$$\mathsf{THH}_*(H\mathbb{F}_p) \cong \mathbb{F}_p[\mu], \quad |\mu| = 2.$$

$$\mathsf{THH}_i(H\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/j\mathbb{Z}, & i = 2j-1, \\ 0, & \text{otherwise.} \end{cases}$$

The change-of-rings map $\mathsf{THH}_*(H\mathbb{F}_p) \to \mathsf{THH}_*^{H\mathbb{Z}}(H\mathbb{F}_p)$ maps $\mathbb{F}_p[\mu]$ to the free divided power algebra on a generator in degree 2, so the algebraic structure is way simpler on $\mathsf{THH}_*(H\mathbb{F}_p)$.

(2) If we apply THH to Eilenberg-Mac Lane spectra of number rings, Lindenstrauss and Madsen show that THH detects arithmetic properties:

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Then

$$\mathsf{THH}_n(H\mathcal{O}_K) = \begin{cases} \mathcal{O}_K, & n = 0, \\ \mathcal{D}_{\mathcal{O}_K}^{-1}/\ell\mathcal{O}_K, & n = 2\ell-1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\mathcal{D}_{\mathcal{O}_K}^{-1}$ is the inverse different. This is the set of all $x \in K$ such that the trace $tr(xy)$ is an integer for all $y \in \mathcal{O}_K$. The inverse different detects ramified primes.

(3) For a suspension spectrum on a based (Moore) loop space, $\Sigma_+^\infty \Omega_M X$, the cyclic bar construction reduces to the suspension spectrum of the cyclic bar construction on $\Omega_M X$ and Goodwillie identifies the latter with the free loop space on X, LX. Hence one obtains

$$\mathsf{THH}(\Sigma_+^\infty \Omega_M X) \simeq \Sigma_+^\infty LX.$$

(4) If $A \to B$ is a G-Galois extension of commutative ring spectra in the sense of Rognes, then the canonical map $B \to \mathsf{THH}^A(B)$ is a weak equivalence of commutative B-algebras.

(5) At an odd prime $KU_{(p)}$ splits as

$$KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} L.$$

Here, L is the Adams summand of $KU_{(p)}$ with $\pi_*(L) \cong \mathbb{Z}_{(p)}[v_1^{\pm 1}]$ and $|v_1| = 2p-2$. For consistency we set $L = KU_{(2)}$ at the prime 2.

For any spectrum E one can form its connective cover e. This comes with a map $\varrho\colon e \to E$ and

$$\pi_i(e) = 0 \text{ for } i < 0; \; \pi_i(e) \cong \pi_i(E) \text{ for } i \geq 0,$$

where the isomorphism is induced by ϱ. We denote by ku, ℓ and ko the connective covers of KU, L and KO.

McClure and Staffeldt determine the mod p-homotopy of $\mathsf{THH}(\ell)$ at odd primes and they show that $\mathsf{THH}(L)_p \simeq L_p \vee (\Sigma L_p)_{\mathbb{Q}}$.

Angeltveit, Hill and Lawson show that for all primes,

$$\mathsf{THH}_*(\ell) \cong \ell_* \oplus \Sigma^{2p-1} F \oplus T$$

as ℓ_*-modules, where F is a torsionfree summand and T is an infinite direct sum of torsion modules concentrated in even degrees. They describe F explicitly using a rational calculation. Determining the torsion is way more involved.

Again, things are way easier for the periodic versions:

$$\mathsf{THH}(KO) \simeq KO \vee \Sigma KO_{\mathbb{Q}}, \quad \mathsf{THH}(KU) \simeq KU \vee \Sigma KU_{\mathbb{Q}}.$$

(6) Blumberg-Cohen-Schlichtkrull determine THH of Thom spectra. This yields for instance an explicit description of $\mathsf{THH}(MU)$.

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