An Atiyah-Hirzebruch spectral sequence for topological André-Quillen homology

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Abstract

We show how topological André-Quillen homology can be related to the usual algebraic André-Quillen homology. To this end we construct an Atiyah-Hirzebruch spectral sequence starting with the algebraic version and converging to the topological theory. This determines topological André-Quillen homology in classical cases of étale and smooth algebras.

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1 Definitions and background

In the last years several definitions of topological André-Quillen homology were given. For an E_∞ ring spectrum k, an E_∞ algebra spectrum A over k and an A module spectrum M one wishes to define a homology theory $\mathsf{TAQ}_*(A|k;M)$ which is analogous to André-Quillen homology. But note that "analogous" does not mean that the theory is isomorphic to usual André-Quillen homology of a commutative algebra A over a ring k with coefficients in an A-module M if we consider TAQ for the corresponding Eilenberg-MacLane spectra $\mathsf{TAQ}_*(HA|Hk;HM)$ and if the arising modules are flat. But the way how this homology theory is defined is similar to usual André-Quillen homology and these homology theories have analogous properties as the algebraic version such as cofibrant-base-change properties and an analog of the Jacobi-Zariski sequence.

One version of topological André-Quillen homology is defined by Maria Basterra [2]; another approach can be found in the work of Alan Robinson and Sarah Whitehouse [12]. They called their homology groups "Gamma-homology".

For Eilenberg-MacLane spectra Hk, HA and HM there is an identification of Gamma-homology with stable homotopy: Let Γ be the skeleton of the category of finite pointed sets with objects $\{0,1,\dots,n\}$ and basepoint 0 and let k be a field. In [9] Teimuraz Pirashvili and the present author extended the definition of Gamma-homology to arbitrary Γ-modules, i.e., to functors from Γ to a category of k-vector spaces. With $H_*^\Gamma(F)$ we denote the Γ-homology of the functor F.

The main result of [9] was the identification of Gamma-homology as stable homotopy: Gamma-homology $H_i^\Gamma(F)$ of a Γ-module F is isomorphic to its stable homotopy $\pi_i^{st}(F)$ and stable homotopy itself is isomorphic to the derived functor

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of the tensor product of Γ-modules $\mathrm{Tor}_i^\Gamma(t, F)$ (see [8, Proposition 2.2]). Here $t : \Gamma^{op} \to k$-vector spaces is the contravariant functor which takes an object $[n] \in \Gamma$ to the k-vector space that has the elements of $\{1, \ldots, n\}$ as a basis.

In particular Gamma-homology of a commutative algebra A with coefficients in an A-module M is Gamma-homology of the Γ-module $\mathcal{L}(A, M)$ which takes an object $[n]$ to $M \otimes A^{\otimes n}$. A morphism $f : [n] \to [m]$ maps a tensor monomial $a_0 \otimes \cdots \otimes a_n$ to $b_0 \otimes \cdots \otimes b_m$ where $b_i = \prod_{f(j)=i} a_j$.

In [3] Maria Basterra and Randy McCarthy proved that the two approaches coincide in the algebraic case, i.e., Gamma-homology and topological André-Quillen homology coincide in the case of Eilenberg-MacLane spectra. In order to unify notation we will only use the expression "topological André-Quillen homology" from now on and we will abbreviate this theory by TAQ.

The aim of this note is to exploit the above isomorphisms to get a better understanding of topological André-Quillen homology. We recall arguments of [11] to identify its value on polynomial algebras and on truncated polynomial algebras in prime characteristic p with p^n-truncation. The general case of a truncated polynomial algebra $k[x]/x^{n+1}$ needs different arguments. Here we see the Steenrod splitting on the level of Γ-modules. The TAQ homology groups of truncated polynomial algebras can be calculated with the help of an Atiyah-Hirzebruch spectral sequence which we develop in the next section.

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2 The spectral sequence

We construct an Atiyah-Hirzebruch spectral sequence starting with algebraic André-Quillen homology of an augmented commutative algebra and having topological André-Quillen homology as abutment. The polynomial algebra in one generator plays the rôle of the basepoint. Let A an arbitrary commutative augmented algebra over a field k and let $P_* \to A$ be a free simplicial resolution of A. The P_ℓ are of the form $P_\ell = k[x_e | e \in E_\ell]$, for some set E_ℓ.

Theorem 2.1 *For a commutative augmented algebra A there is a spectral sequence*

$$E^2_{p,q} = \mathsf{TAQ}_q(k[x]|k; k) \otimes \mathsf{AQ}_p(A|k; k) \Longrightarrow \mathsf{TAQ}_{p+q}(A|k; k).$$

Here $\mathsf{AQ}_*(-)$ abbreviates André-Quillen homology.

Proof Two Γ-modules F and G can be tensorized to give a new Γ-module $F \otimes G$ which is defined as $(F \otimes G)[n] := F[n] \otimes G[n]$. Concerning this pointwise tensor product topological André-Quillen homology is additive (compare [9], p.3):

$$\mathsf{TAQ}_*(F \otimes G) \cong \mathsf{TAQ}_*(F) \otimes G[0] \oplus F[0] \otimes \mathsf{TAQ}_*(G).$$

The functor $\mathcal{L}(k[x,y],k)$ is isomorphic to the pointwise tensor product $\mathcal{L}(k[x],k) \otimes \mathcal{L}(k[x],k)$. Thus we can calculate topological André-Quillen homology of a polynomial ring in several variables via TAQ_* of the polynomial ring in a single variable:

$$\mathsf{TAQ}_*(k[x_1,\ldots,x_{i_\ell}]|k;k) \cong \mathsf{TAQ}_*(k[x_1]|k;k) \oplus \cdots \oplus \mathsf{TAQ}_*(k[x_{i_\ell}]|k;k).$$

As everything in sight commutes with colimits we get an analogous formula for the terms $P_\ell = k[x_e|e \in E_\ell]$ in the free simplicial resolution. Therefore topological André-Quillen homology of this resolution gives

$$\mathsf{TAQ}_*(P_*|k;k) \cong \mathsf{TAQ}_*(k[x]|k;k) \otimes (\Omega^1_{P_*|k} \otimes_{P_*} k)$$

because the module of Kähler differentials tensorized with k over the algebra P_ℓ counts the number of generators of P_ℓ. As an E^1-term of the spectral sequence we obtain

$$E^1_{p,q} = \mathsf{TAQ}_q(k[x]|k;k) \otimes (\Omega^1_{P_p|k} \otimes_{P_p} k) \Longrightarrow \mathsf{TAQ}_{p+q}(A|k;k).$$

The differentials in this E^1-tableau are induced by the differentials of the resolution P_*, thus they compute the homotopy of the Kähler differentials and we get the desired form of the E^2-term. □

Corollary 2.2 *With the help of this spectral sequence one can transfer calculations of classical André-Quillen homology directly to the topological context. For instance:*

- *For A étale over k the topological André-Quillen homology of A over k vanishes because André-Quillen homology is trivial in all degrees. This result is already stated in [12, Theorem 3.5].*
- *For A smooth over k topological André-Quillen homology is not at all trivial but computable: The spectral sequence is concentrated in one line and degenerates at the E^2-term. The terms that arise are tensor products of the form $\mathsf{TAQ}_q(k[x]|k;k) \otimes (\Omega^1_{A|k} \otimes_A k)$. We will determine $\mathsf{TAQ}_*(k[x]|k;k)$ in the next section.*

Remark 2.3 *For sake of simplicity we stated result 2.1 for Γ-vector spaces. But one obtains a similar spectral sequence for arbitrary commutative rings:*

$$E^2_{p,q} \cong \mathsf{AQ}_p(A|k; \mathsf{TAQ}_*(k[x]|k;k)) \Longrightarrow \mathsf{TAQ}_{p+q}(A|k;k).$$

Remark 2.4 *In [13, 5.5], Stefan Schwede constructs an Atiyah-Hirzebruch spectral sequence of the form*

$$E^2_{p,q} = H_p(A, \pi_q M) \Longrightarrow M_{p+q}(A)$$

for a T-algebra A and a right-T^s-module M. Here T is an algebraic theory and T^s-modules are Quillen equivalent to connective spectra of T-algebras. For the theory of augmented commutative algebras over a commutative ring k and $M = T^s$ (called Dk in [13, 7.9]) the E^2-term of this spectral sequence coincides with 2.3 and it converges to stable homotopy of the algebra A defined via an algebraic suspension spectrum [13, 5.1]. The abutment is isomorphic to $\mathsf{TAQ}_*(A|k;k)$.

3 Stable derived functors

In this section we identify topological André-Quillen homology of the basepoint $k[x]$ and we will determine TAQ of p^n-truncated polynomial algebras over a field of characteristic p.

In the following considerations we will make use of the relationship between the stable homotopy of Γ-modules and the stable derived functors in the sense of Dold-Puppe [5]. We briefly recall the definition of stable homotopy for Γ-modules: Every Γ-module F can be prolonged to a functor from pointed sets to k-vector spaces via colimits. Given a simplicial set X_* we can evaluate F degreewise to make it a functor from pointed simplicial sets to simplicial vector spaces. The important point is that for two pointed simplicial sets X_* and Y_* there are always maps from $X_* \wedge F(Y_*)$ to $F(X_* \wedge Y_*)$. For every $x \in X_*$ we obtain a map $Y_* \to X_* \wedge Y_*$ and with F applied to this map we obtain the desired transformation. In particular there are maps from $\mathbb{S}^1 \wedge F(\mathbb{S}^n)$ to $F(\mathbb{S}^{n+1})$; hence every Γ-module gives rise to a spectrum. With $\pi_i^{st}(F)$ we denote the homotopy of this spectrum, i.e., $\pi_i^{st}(F) = \mathrm{colim}_n \pi_{i+n} F(\mathbb{S}^n)$.

The i-th stable derived functor of an endofunctor of k-vector spaces F on k is by definition (see [5, 8.3]) isomorphic to the stable homotopy of the same functor F precomposed with the functor L, which takes an object $[n] \in \Gamma$ to the k-vector space with basis $\{1, \dots, n\}$:

$$L_i^{st}(F)(k) \cong \pi_i^{st}(F \circ L) \cong \mathrm{Tor}_i^\Gamma(t, F \circ L).$$

The results whose proof we give here are not new. In [11] we gained them for arbitrary coefficients. But as we restrict to the case of augmentation coefficients, the proofs become easier.

Let the polynomial algebra $k[x]$ act on k via the standard augmentation which keeps just the constant summand of a polynomial. In [10, 4.3.3] we identified the functor $\mathcal{L}(k[x];k)$ with the composed functor $\mathsf{Sym} \circ L$ which takes on an object $[n] \in \Gamma$ first the vector space on the elements $1, \dots, n$ and builds the symmetric algebra on this vector space. That both functors coincide on objects, is trivial; for the agreement on maps of finite pointed sets the action of $k[x]$ on k via the augmentation is crucial: Whenever a map of finite pointed sets maps an element $i \neq 0$ to zero, the corresponding polynomial in the varible x_i is sent to its constant term. Hence we obtain

Proposition 3.1 *Topological André-Quillen homology of the basepoint $k[x]$ is*

$$\mathsf{TAQ}_i(k[x]|k;k) \cong \pi_i^{st}(\mathsf{Sym} \circ L) \cong L_i^{st}(\mathsf{Sym})(k) \cong Hk_i(H\mathbb{Z}).$$

The last observation uses the work of Dold and Thom [6], identifying the stable derived functors of the symmetric product with the usual homology groups. For a nice description of the used isomorphims see Stanislaw Betley's paper [4].

Corollary 3.2 *With this identification we can describe the topological André-Quillen homology of a smooth algebra A more thoroughly as*

$$\mathsf{TAQ}_*(A|k;k) \cong Hk_*H\mathbb{Z} \otimes (\Omega^1_{A|k} \otimes_A k).$$

For instance TAQ *of a polynomial algebra on n generators is given by n copies of $Hk_*H\mathbb{Z}$.*

A similar identification describes topological André-Quillen homology of truncated polynomial algebras over a field of characteristic p:

Proposition 3.3 *Let k denote a field of characteristic p. Topological André-Quillen homology of the p^n-truncated polynomial algebra $k[x]/x^{p^n}$ over k with coefficients in k is isomorphic to the k-homology of the Eilenberg-MacLane spectrum of $\mathbb{Z}/p^n\mathbb{Z}$.*

$$\mathsf{TAQ}_i(k[x]/x^{p^n}|k;k) \cong Hk_i(H\mathbb{Z}/p^n\mathbb{Z}).$$

Proof We gain an isomorphism of the functor $\mathcal{L}(k[x]/x^{p^n};k)$ and the functor of the p^n reduced symmetric product $\mathsf{Sym}_{p^n} \circ L$, which sends a finite pointed set $[m]$ to the truncated polynomial algebra $k[x_1,\ldots,x_m]/(x_1^{p^n},\ldots,x_m^{p^n})$. As we work in characteristic p this functor is isomorphic to the one which takes the corresponding truncation by polynomials instead of variables. But this is the same as the chains on the p^n-truncated symmetric product (compare [4]) and the result follows. □

4 Truncated polynomial algebras

We cannot calculate topological André-Quillen homology of arbitrary truncated algebras $k[x]/x^{n+1}$ via stable derived functors in a similar easy manner as in the p^n-truncated case over characteristic p, because the isomorphism in Theorem 3.3 relies on the identification of p^n-th powers of polynomials and p^n-th powers of variables. Instead we will describe the Steenrod splitting on the level of Γ-modules and give the calculation via the Atiyah-Hirzebruch spectral sequence.

4.1 Steenrod splitting for $\mathcal{L}(k[x]/x^{n+1}; k)$

The Steenrod splitting for the infinite symmetric product (see [14, §22]) gives a decomposition for the Γ-module $\mathsf{Sym} \circ L$ and hence for $\mathcal{L}(k[x], k)$: The i-th decomposition part consists of all monomials with total degree i. Let us denote this part with Υ_i. As we work with coefficients in k with the action given by the augmentation map the maps of finite pointed sets induce either multiplications or the zero map. Thus every Υ_i is actually a subfunctor and we obtain

$$\mathcal{L}(k[x], k) \cong \bigoplus_{i \geq 0} \Upsilon_i.$$

Here the zeroth part is the constant functor having value k. For the case of truncated polynomials this descends to a splitting where the single varibles do not appear with a power greater than n.

$$\mathcal{L}(k[x]/x^{n+1}; k) \cong \bigoplus_{i \geq 0} \Upsilon_i^{(n)}.$$

Example The Γ-vector space $\Upsilon_2^{(2)}$ evaluated on the object [2] has the basis x_1^2, x_2^2, x_1x_2. The non-injective map $f : [2] \to [1]$ which maps 1 and 2 to 1 and zero to zero sends the element x_1x_2 to x_1^2. In $\Upsilon_4^{(2)}$ the generator $x_1^2x_2^2$ is sent to zero by the same map, because the exponent would be $4 > 2$.

Remark 4.1 *The decomposition parts $\Upsilon_i^{(n)}$ stabilize with n growing larger:*

$$\Upsilon_i^{(n)} \cong \Upsilon_i^{(n+1)} \qquad \forall n \geq i$$

Hence the first part $\Upsilon_1^{(n)}$ of the decomposition is independant of n and this Γ-vector space is projective, because it is nothing but the functor L and this is a splitting cokernel $L = coker(\Gamma^0 \to \Gamma^1)$. Here Γ^i is the representable functor $\Gamma^i([n]) = k\{\Gamma([i], [n])\}$.

Remark 4.2

1. *This decomposition corresponds to the x-weight composition of Hochschild homology of truncated algebras (compare [7, 5.4.14]), because Hochschild homology of commutative algebras is isomorphic to the homotopy of the functor $\mathcal{L}$ evaluated on the standard model of $\mathbb{S}^1$ (see [8]).*

2. *Obviously the i-th part of the decomposition is a functor of degree i, because it has its origin in the i-th symmetric power.*

4.2 The Atiyah-Hirzebruch spectral sequence applied to $k[x]/x^{n+1}$

For the calculation of topological André-Quillen homology of truncated polynomial algebras we will use the Atiyah-Hirzebruch spectral sequence from section 2 which in this case is

$$E^2_{p,q} = \mathsf{TAQ}_q(k[x]|k;k) \otimes \mathsf{AQ}_p(k[x]/x^{n+1}|k;k) \Longrightarrow \mathsf{TAQ}_{p+q}(k[x]/x^{n+1}|k;k).$$

To this end we exploit the properties of usual André-Quillen homology:

Lemma 4.3 *André-Quillen homology of truncated polynomial algebras with coefficients in k is trivial in dimensions different from $0, 1$.*

Proof Let us abbreviate the truncated polynomial algebra $k[x]/x^{n+1}$ by T. The sequence $k \longrightarrow k[x] \longrightarrow T$ gives rise to a Jacobi-Zariski sequence. André-Quillen homology of T over $k[x]$ with field coefficients vanishes in all degrees different from one (see [1, VI, Lemma 22]), because x^{n+1} is a regular element in the polynomial algebra. The smoothness of $k[x]$ over k then proves the claim. □

Corollary 4.4 *The Atiyah-Hirzebruch spectral sequence collapses for truncated polynomial algebras in the E^2-term; hence $\mathsf{TAQ}_m(k[x]/x^{n+1}|k;k)$ is given as a sum*

$$Hk_*(H\mathbb{Z}) \otimes \mathsf{AQ}_0(k[x]/x^{n+1}|k;k) \oplus Hk_{*-1}(H\mathbb{Z}) \otimes \mathsf{AQ}_1(k[x]/x^{n+1}|k;k)$$

.

Here the zeroth homology groups $\mathsf{AQ}_0(k[x]/x^{n+1}|k;k)$ is given by the module of Kähler differentials $\Omega^1_{T|k} \otimes_T k \cong k\{dx\}$ and AQ_1 is determined by the Jacobi-Zariski sequence in low degrees:

$$0 \longrightarrow \mathsf{AQ}_1(T|k;k) \longrightarrow J/J^2 \otimes_T k \longrightarrow k\{dx\} \longrightarrow \Omega^1_{T|k} \otimes_T k \longrightarrow 0$$

with J denoting the ideal generated by x^{n+1}; the isomorphism $\mathsf{TAQ}_1(T|k[x];k) \cong J/J^2 \otimes_T k$ can be found in [1, VI, Proposition 1]. The tensor product $J/J^2 \otimes_T k$ is one-dimensional, because every generator except x^{n+1} on the left hand side is equivalent to zero. As the map $k\{dx\} \longrightarrow \Omega^1_{T|k} \otimes_T k \cong k\{dx\}$ is an isomorphism we obtain that the first homology group $AQ_1(T|k;k)$ is isomorphic to k, hence André-Quillen homology of a truncated polynomial algebra with augmentation coefficients does not depend on the degree of the truncation; thus the same holds for TAQ.

To summarize the result, we obtain

$$\mathsf{TAQ}_*(k[x]/x^{n+1}|k;k) \cong Hk_*(H\mathbb{Z}) \otimes k \oplus Hk_{*-1}(H\mathbb{Z}) \otimes k.$$

In particular the result for the p^n-truncated polynomial algebra over $\mathbb{F}_p$ coming out of this spectral sequence then reads

$$\begin{aligned}(H\mathbb{F}_p)_*(H\mathbb{Z}/p^n\mathbb{Z}) &\cong \mathsf{TAQ}(\mathbb{F}_p[x]/x^{p^n}|\mathbb{F}_p;\mathbb{F}_p)\\ &\cong (H\mathbb{F}_p)_*(H\mathbb{Z})\otimes\mathbb{F}_p\oplus(H\mathbb{F}_p)_{*-1}(H\mathbb{Z})\otimes\mathbb{F}_p,\end{aligned}$$

and this is the usual splitting of $(H\mathbb{F}_p)_*(H\mathbb{Z}/p^n\mathbb{Z})$ into two copies of $(H\mathbb{F}_p)_*(H\mathbb{Z})$ which one usually obtains by the universal coefficient theorem or the Bockstein sequence.

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