Review (Jahresbericht der DMV): Modern Classical Homotopy Theory, Jeffrey Strom

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Homotopy theory is a very broad subject. The basic idea is easy to describe: the objects of study are (nicely behaved) topological spaces but the crucial point is that continuous maps between spaces are considered up to homotopy, that is, up to continuous deformation. The concept of homotopy allows for classifications that are, in general, much coarser than for instance the classification of spaces up to homeomorphism or classifications in differential or complex geometry.

The subject has a long history starting around 1950 (with some important earlier contributions) and homotopy theory is thriving today. Concepts and constructions from homotopy theory influence many areas of mathematics, for instance motivic homotopy theory plays an important role in algebraic geometry, Hopkins' spectrum of topological modular forms has connections with the classical theory of modular forms, with elliptic curves and with conformal field theories, and algebraic K-theory connects homotopy theory and algebraic number theory. In the foundations of mathematics, homotopy theory is used in homotopy type theory and in geometric topology methods and results from homotopy theory are used to gain genuine geometric information. Simplicial methods and Quillen model category structures by now belong to the standard toolkit of many mathematicians. An important recent result is the solution of the Kervaire invariant problem by Hill, Hopkins and Ravenel. The original problem is a question about smooth framed manifolds but the solution is given in terms of stable homotopy theory.

Homotopy theory is a very diverse subject. The Mathematics Subject Classification mentions topics such as cofibrations and fibrations, homotopy equivalences, classification of homotopy types, Eilenberg-Mac Lane spaces, Spanier-Whitehead and Eckman-Hilton duality, (infinite) loop spaces and suspensions, stable homotopy theory and spectra, spectra with extra structure, operads, localization and completion, string topology, rational homotopy theory, shape theory and proper homotopy theory and equivariant homotopy theory. This list is not complete at all and it does not include technical tools, such as spectral sequences or cohomology operations, or calculational aspects. One of the features of homotopy theory is a mix of methods, ranging from genuinely homotopy theoretic ones to geometric and algebraic methods.

The long list of topics above indicates that even a book on homotopy theory like Jeffrey Strom's with more than 800 pages cannot give a comprehensive introduction into the subject. Any author of such a volume has to choose which topics to discuss and of course this choice depends on personal taste: Strom focusses on the concept of homotopy limits and homotopy colimits and the model category of topological spaces with a thorough discussion of cofibrations and fibrations. Most of the standard topics of unstable homotopy theory are covered by the book.

The book starts with a short introduction to category theoretical concepts, in particular it contains a chapter on limits and colimits. Part two contains the basic concepts of homotopy theory, starting with a discussion what properties a nice category of topological spaces should have, introducing the concept of homotopy, the notions of (co)fibrations, introducing homotopy (co)limits, discussing (co-)H-spaces and Lusternik-Schnirelmann category and finally treating Quillen model category structures. Connectivity, n-equivalences, the Seifert-van-Kampen theorem and cellular approximation are dealt with in the part three. In this part Strom also explains what one can say about pullbacks of cofibrations. As cofibrations behave well with respect to colimits, but in general not with respect to limits, this is a non-standard topic. The result, namely that cofibrations are preserved under pullbacks along fibrations, is useful to know. Notions like coverings and bundles, Serre fibrations and quasifibrations are subsumed under the chapter "Related Topics"; this does not quite reflect their importance.

Part four features some of the main results in classical homotopy theory: This part starts with skeleta of spaces, Postnikov towers and classifying spaces, and then deals with loop spaces and suspensions. The Freudenthal suspension theorem and the Blakers-Massey theorem are proven and some consequences for homotopy groups and Eilenberg-MacLane and Moore spaces are discussed. Part four closes with a chapter on "Further Topics" which collects themes ranging from Lusternik-Schnirelmann category to infinite symmetric products.

Cohomology and homology show up in Part five. Strom introduces cohomology in the represented form, then gives the general definition of a cohomology theory before discussing concrete examples. He states Brown representability and then describes basic properties of homology theories. Cohomology operations, the structure of the Steenrod algebra and cohomology and homology via the cellular and singular (co)chain complexes are other topics of this part.

Spectral sequences are the main topic of Part six. An extensive discussion of filtrations is the starting point and the spectral sequence associated to a filtration and the Leray-Serre spectral sequence are described in some detail with applications, ranging from some classical cohomology calculations to Bott periodicity.

The book closes with Part seven, which is called 'Vistas'. Four main topics are presented: Localizations and completions of spaces, exponents for homotopy groups, classes of spaces and a theme and variations on Miller's theorem of the triviality of the space of pointed maps from the classifying space of a cyclic group of prime order to a finite-dimensional CW complex.

Strom clearly states his preferences in the introduction: "I have generally used topological or homotopy-theoretical arguments rather than algebraic ones." If readers think that this preference results in a plenitude of geometrical arguments, they will be disappointed. There are a lot of diagrams in the book, but the only figures are of a schematic nature, explaining homotopy extensions, for instance. Homotopy theory often transfers topological questions into algebraic ones, thus a certain amount of algebraic arguments is intrinsic to the subject and cannot be avoided. Cohomological methods and spectral sequences appear rather late in the book (in Chapters 21 and 30 out of a total of 37 chapters). Many theorems that are typically proved using these methods are treated in the chapters before Chapter 21, and are proved using homotopy-theoretic arguments on space level.

The book is not a classical textbook whose content is structured as a sequence of results followed by proofs with some remarks and examples. Rather it encourages learning-by-doing. Strom says that "theorems are followed by multi-part problems that guide the readers to find the proofs for themselves". In these problems, proofs are broken down into smaller portions. Only a reader who works on the problems and exercises will gain something from this book.

Having no outright proofs at all in a book might have its drawbacks: Some things are difficult to learn if you never see them done. For instance, finding cartoon proofs in homotopy theory that you might translate to a full proof later (or you don't because you are happy with them as they are), is something you learn from role models; otherwise this aspect of homotopy theory might just be lost on you. As Strom does not provide detailed references to the literature, a reader who does not manage to solve the problems might find it hard to fill in gaps.

According to Strom, the intended readership of the book consists of people who had an introductory course in topology, but have not necessarily seen the fundamental group. While no knowledge in topology is required that goes beyond that, some experience with arguments and proofs in topology is necessary, in order to solve the problems and do the exercises.

Strom's book is certainly different from the existing literature on homotopy theory. His book is *not* suited for someone who just wants to apply homotopy theory, get a quick impression what the subject is about and how the proofs work. But for someone with a serious interest in the deeper aspects of those topics in homotopy theory that are presented in the book, the book can help to learn them in an active way.