A model for the stable homotopy category I

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1. Generalized cohomology theories

Singular cohomology satisfies the *Eilenberg-Steenrod axioms of a cohomology theory*:

(1) The assignment $(X, A) \mapsto H^n(X, A)$ is a contravariant functor from the category of CW-pairs to the category of abelian groups.
(2) For any subspace $A \subset X$ there is a natural homomorphism $\partial\colon H^n(A) \to H^{n+1}(X, A)$
(3) If $f, g\colon (X, A) \to (Y, B)$ are two homotopic maps of pairs of topological spaces, then $H^n(f) = H^n(g)\colon H^n(Y, B) \to H^n(X, A)$.
(4) For any subspace $A \subset X$ we get a long exact sequence

$$\ldots \xrightarrow{\partial} H^n(X, A) \longrightarrow H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial} \ldots$$

(5) Excision holds: If X is the union of two subcomplexes A, B, then the inclusion map $(A, A \cap B) \subset (X, B)$ induces an isomorphism $H^n(X, B) \cong H^n(A, A \cap B)$ for all n.
(6) Let pt be the one-point space, then

$$H^n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

This is called the axiom about the coefficients or the *dimension axiom*.
(7) Additivity:

$$H^n(\bigsqcup_{i \in I} X_i) \cong \prod_{i \in I} H^n(X_i).$$

For singular cohomology with coefficients in G we have an analoguous set of axioms. There are important so-called *generalized cohomology theories* like topological K-theory or cobordism theories that satisfy all axioms but the dimension axiom.

To every such generalized cohomology theory E^* we can associate a reduced variant by considering $\tilde{E}^*(X) := E^*(X, *)$ with $* \in X^0$.

Why should one study these? Generalized cohomology theories might be harder to calculate, but they can carry more information. For sake of simplicity we study the absolute versions $E^*(X, \emptyset)$.

Typical examples are:

- Topological K-theory. The most common variants are real or complex topological K-theory, $KO^*(X)$ and $KU^*(X)$. Here you study (real or complex) vector bundles of finite rank on X. More precisely, you take isomorphism classes of such bundles. They form an abelian monoid under direct sum of vector bundles and you take their group completion. That's $KU^0(X)$ ($KO^0(X)$) and you get the other groups by using suspensions. For KU that's easy because of Bott periodicity: First set $\widetilde{KU}^0(X) = \ker(KU^0(X) \to \mathbb{Z})$ and $\widetilde{KU}^{2i}(X) := \widetilde{KU}^0(X)$, $\widetilde{KU}^{2i+1}(X) := \widetilde{KU}^0(\Sigma X)$ for $i \in \mathbb{Z}$.

The Möbius bundle $\mu\colon E = [0,1] \times \mathbb{R}/\sim \to S^1$ with $(0,t) \sim (1,-t)$ and $\mu[x,t] = \exp(2\pi i x)$ gives rise to a generator $[\mu] - 1 \in \widetilde{KO}(S^1)$. (Here, 1 is the 1-dimensional trivial bundle on S^1, aka, the infinite cylinder.)

For more background on topological K-theory see [Sw, Chapter 11].

- There are various flavors of cobordism theories. For instance for unoriented bordism classes of manifolds you consider smooth closed manifolds of dimension n up to bordism, so you two such manifolds M_1 and M_2 if there is an $(n+1)$-dimensional manifold W whose boundary is $M_1 \sqcup M_2$. The corresponding *homology* theory considers maps $f\colon M_1 \to X$ and $g\colon M_2 \to X$ up to bordism, so if there is a map $H\colon W \to X$ whose restriction to the boundary components gives f and g then these maps are bordant. This gives $MO_n(X)$, the nth bordism group of X. (So instead of throwing standard simplices into your space, you probe it with manifolds.) There is a dual cohomology theory.

  Important other types of bordisms are oriented bordism, MSO, and stably complex bordism, MU. For the latter you consider smooth closed n-dimensional manifolds, M. Instead of asking for a complex structure on M, you require that for some k the sum $TM \oplus \mathbb{R}^k$ is a complex vector bundle. Here, $\mathbb{R}^k$ stands for the trivial bundle of dimension k over M. Then imposing a suitable equivalence relation on such objects gives manifolds with stably complex structure and a corresponding homology theory, $MU_*(X)$. See [Sw, Chapter 12] and [Ma, Chapter 25] for more details.
- There are several variants of elliptic cohomology theories, for instance the cohomology theory related to topolocial modular forms, tmf. This cohomology theory was used to decide immersion questions for real projective spaces into euclidean spaces. It plays an important role for realizing the Witten genus and has several other connections to mathematical physics.

2. Spectra

Working with cohomology theories is important, but just having a bunch of abelian groups, $E^n(X)$, doesn't give you many tools. What one wants to do, it do work with cohomology theories as topological objects. First we define spectra and their associated homology and cohomology theories and then later, we'll go backwards, representing cohomology theories by spectra.

The *suspension* of a space X, ΣX, is defined as the quotient of $X \times [0,1]$ by collapsing its top $X \times \{1\}$ and its bottom $X \times \{0\}$. If X has a basepoint $x \in X$, then one often considers the *reduced suspension*, where one builds $\Sigma X/\{x\} \times [0,1]$. We agree to use the reduced suspension in the pointed setting, but we'll still denote it by ΣX.

Definition A *spectrum* E is a sequence of pointed topological spaces $E_n, n \in \mathbb{N}_0$ together with structure maps

$$\sigma_n\colon \Sigma E_n \to E_{n+1}.$$

There are some technical issues here. Working with arbitrary spaces is not a good idea. At least they should be well-pointed, but if you want to be on the safe side, then you might want to assume that all the E_ns are CW complexes and that the basepoint is a zero simplex.

Definition A spectrum E is an Ω-*spectrum*, if the adjoints of the structure maps

$$\varrho_n\colon E_n \to \Omega E_{n+1}$$

are weak equivalences.

(1) You know an example of an Ω-spectrum: Let A be an abelian group. Then we defined the Eilenberg-Mac Lane space of type (A, n) as a topological space of the homotopy type of a CW complex $K(A, n)$ with

$$\pi_i(K(A,n)) = \begin{cases} A, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

By adjunction the homotopy groups of $\Omega K(A, n+1)$ are

$$\pi_i \Omega K(A, n+1) = \pi_{i+1}(K(A, n+1)) = \begin{cases} A, & i = n \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\Omega K(A, n+1)$ is a $K(A, n)$. One can jazz that argument up to obtain an Ω-spectrum HA whose nth space is a $HA_n = K(A, n)$. This spectrum is called the *Eilenberg-Mac Lane spectrum of A*.

(2) A spectrum that is not an Ω-spectrum is the *sphere spectrum* $S = (S^n, n \in \mathbb{N}_0)$. As $\Sigma S^n \cong S^{n+1}$, the structure maps σ_n are actually homeomorphisms in this case.

(3) If X is a pointed CW-complex, then we can define its *suspension spectrum* $\Sigma^\infty X$ as

$$(\Sigma^\infty X)_n := \Sigma^n X$$

where $\Sigma^n X$ denotes the n-fold reduced suspension of X. So $S = \Sigma^\infty S^0$.

Definition Let $E = (E_n)_{n \in \mathbb{N}_0}$ be a spectrum.
The *ith reduced homology group of a pointed CW space X with respect to E* is

$$\tilde{E}_i(X) := \mathrm{colim}_n \pi_{n+i}(E_n \wedge X).$$

If E is an Ω-spectrum, then the *ith reduced cohomology group of X with respect to E* is

$$\tilde{E}^i(X) = [X, E_i].$$

There is also a definition of E-cohomology if E is not an Ω-spectrum, but that needs more background.

Why do these definitions make sense? For homology, the maps that define the colimit are as follows. If $\alpha \colon S^{n+i} \to E_n \wedge X$ represents a class in $\pi_{n+i}(E_n \wedge X)$, then the suspension of α is a map $\Sigma(\alpha) \colon \Sigma S^{n+i} \to \Sigma E_n \wedge X$. We identify ΣS^{n+i} with S^{n+i+1} and use the structure map $\sigma_n \colon \Sigma E_n \to E_{n+1}$ to obtain a class in $\pi_{n+1+i}(E_{n+1} \wedge X)$

These theories have suspension isomorphisms: For cohomology, this is visible as

$$\tilde{E}^{i+1}(\Sigma X) = [\Sigma X, E_{i+1}] \cong [X, \Omega E_{i+1}] \cong [X, E_i] \cong \tilde{E}^i(X).$$

We'll see next, why this also holds for homology.

3. Stable homotopy groups of spheres

Definition A pointed space (X, x) is *k-connected*, if $\pi_i(X, x) = 0$ for $i \leq k$.

So a 0-connected space is path-connected and a 1-connected space is simply connected.

An n-sphere for $n \geq 1$ is $(n-1)$-connected; its bottom non-trivial homotopy group is $\pi_n(S^n) = \mathbb{Z}$. One can choose the generator as a suspension of the standard generator for

$\pi_1(S^1)$ whose representative is the identity on S^1, or if you prefer, the induced map on the quotient given on $[0,1]$ by $t \mapsto \exp(2\pi i t)$.

Freudenthal suspension theorem Let X be an $(n-1)$-connected CW complex for $n \geq 1$. Then the suspension Σ induces an isomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ for $i < 2n-1$ and an epimorphism for $i = 2n-1$.

Definition The *kth stable homotopy group of the sphere* is

$$\pi_k^s = \mathrm{colim}_n \pi_{n+k}(\mathbb{S}^n).$$

Thanks to the Freudenthal Theorem this limit stabilizes at a finite stage:

$$\pi_k^s \cong \pi_{n+k}(\mathbb{S}^n) \text{ for } n > k+1.$$

The first stable homotopy groups are

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
π_i^s	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$

Here, π_0^s comes from the suspensions of $\pi_1(S^1) \cong \mathbb{Z}$. Note that the generator of $\pi_3(S^2)$ comes from the Hopf fibration $\eta\colon S^3 \to S^2$ and generates a copy of $\mathbb{Z}$, coming from the long exact sequence of homotopy groups associated to the fibration $S^1 \to S^3 \to S^2$: $\pi_3(S^3) \cong \pi_3(S^2)$. So this class isn't stable yet but its suspensions generate π_1^s. The suspensions of the higher Hopf fibration $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ give the generators of π_3^s and π_7^s.

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