Stability of Loday constructions

Birgit Richter

Joint work with Ayelet Lindenstrauss

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connecting the algebraic K-theory of R to the topological Hochschild homology of R.

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Suslin: $K(\mathbb{C})_p \simeq ku_p$, p-completed connective complex topological K-theory.
Ausoni, Rognes: $K(ku)$ is a form of elliptic cohomology.

If we end up in ring spectra anyway, we can also start with them, so from now on R will be a commutative ring spectrum (such as an Eilenberg-MacLane spectrum of a commutative ring, topological K-theory, cobordism theories such as MU, MO or topological modular forms or...).

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The simplicial model of the circle S^1 has $n+1$ points in S^1_n:

$$\begin{tikzcd}
\{0\} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \{0,1\} \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow] & \{0,1,2\} & \cdots
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Placing R at each point and smashing these copies together gives $\mathrm{THH}(R)$:

$$\begin{tikzcd} R \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & R \wedge R \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow] & R \wedge R \wedge R & \cdots \end{tikzcd}$$

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Direct inspection gives

$$\mathcal{L}_X^R(\mathcal{L}_Y^R(A)) \simeq \mathcal{L}_{X \times Y}^R(A).$$

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Theorem [Dundas-Lindenstrauss-R 2018; Mandell]
For all $n \geq 2$:

$$\pi_* \mathcal{L}_{S^n}(\mathbb{F}_p) \cong \mathrm{Tor}_{*,*}^{\pi_* \mathcal{L}_{S^{n-1}}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

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What if it just depended on the homotopy type of $\Sigma(X)$?

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Calculating $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^m}(\mathbb{Q}[t]/t^m;\mathbb{Q})$ for all $m \geq 2$ shows that $\pi_*\mathcal{L}^{\mathbb{Q}}_{X}(\mathbb{Q}[t]/t^m;\mathbb{Q})$ doesn't just depend on the homotopy type of ΣX.

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Similar results hold for $\mathbb{Z}[t]/t^m$ for all $m \geq 2$ and $\mathbb{F}_p[t]/t^m$ for $2 \leq m < p$.

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Can we give more examples?

Notions of stability

- Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We call $R \to A$ *stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ implies that $\mathcal{L}^R_X(A) \simeq \mathcal{L}^R_Y(A)$.

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- Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We call $R \to A$ *multiplicatively stable* (m-stable) if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in $sSets_*$ implies that $\mathcal{L}^R_X(A) \simeq \mathcal{L}^R_Y(A)$ as commutative augmented A-algebra spectra.

- Let $S \longrightarrow R \xrightarrow{\alpha} A \xrightarrow{\beta} B$ be a sequence of cofibrations of commutative S-algebras. Then we call $R \to A \to B$ *multiplicatively stable* (m-stable) if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in $sSets_*$ implies that $\mathcal{L}^R_X(A;B) \simeq \mathcal{L}^R_Y(A;B)$ and $\mathcal{L}^R_X(B) \simeq \mathcal{L}^R_Y(B)$ as commutative augmented B-algebras such that the diagram

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An example:

Proposition

If B is an augmented commutative A-algebra, then $B \to A$ and $A \to \mathcal{L}^A_{\Sigma X}(B; A) \to A$ are m-stable.

Sketch of proof:
Bobkova-Höning-Lindenstrauss-Poirier-R-Zakharevich 2019:

$$\mathcal{L}^A_{\Sigma X}(B;A) \simeq \mathcal{L}^A_{\Sigma X}(A;A) \wedge^L_{\mathcal{L}^A_X(A)} \mathcal{L}^B_X(A) \simeq \mathcal{L}^B_X(A).$$

Therefore $\Sigma X \simeq \Sigma Y$ implies $\mathcal{L}^B_X(A) \simeq \mathcal{L}^B_Y(A)$.

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Therefore $\Sigma X \simeq \Sigma Y$ implies $\mathcal{L}^B_X(A) \simeq \mathcal{L}^B_Y(A)$.
For the second claim observe that

$$\mathcal{L}^A_Y(\mathcal{L}^A_{\Sigma X}(B; A); A) \simeq \mathcal{L}^A_{Y \wedge \Sigma X}(B; A) = \mathcal{L}^A_{\Sigma Y \wedge X}(B; A).$$

Some examples

- Schlichtkrull: Determines $\mathcal{L}_X(M(f))$ if $M(f)$ is the Thom spectrum of an Ω^∞-map that starts on a grouplike space. This implies that such $M(f)$ are m-stable.

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- $HR \to HR/(a_1, \ldots, a_n)$ is m-stable if R is a commutative ring and $(a_1, \ldots, a_n)$ is a regular sequence.

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- Generalization by Rasekh, Stonek, Valenzuela to generalized Thom spectra arising from maps of E_∞-groups $f\colon G \to \mathrm{Pic}(R)$ where R is a commutative ring spectrum.
- Hedenlund, Klanderman, Lindenstrauss, R, Zou:
  If $\mathcal{H}$ is a commutative Hopf algebra spectrum and if $\Sigma(X_+) \simeq \Sigma(Y_+)$ is an equivalence in $\mathcal{S}_*$, then there is an equivalence $\mathcal{L}_X(\mathcal{H}) \simeq \mathcal{L}_Y(\mathcal{H})$ in CAlg. This generalizes a result by Berest, Ramadoss, Yeung.
- $HR \to HR/(a_1, \ldots, a_n)$ is m-stable if R is a commutative ring and $(a_1, \ldots, a_n)$ is a regular sequence.
- Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. Then $A \to \mathcal{L}^R_{\Sigma X}(A)$ is m-stable for all $X \in sSets_*$.

Inheritance properties

- If $f\colon A \to B$ is m-stable, then so is $C \wedge_R f\colon C \wedge_R A \to C \wedge_R B$.

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- If $R \to A$ is m-stable, then so is $R \to \mathcal{L}^R_Z(A)$ for any Z.
- If $S \to A$ and $S \to B$ are cofibrations of commutative S-algebras and if A and B are m-stable, then if X and Y are connected and $\Sigma X \simeq \Sigma Y$, then

$$\mathcal{L}^S_X(A \times B) \simeq \mathcal{L}^S_Y(A \times B)$$

as commutative S-algebras.

But beware, stability is not transitive:

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Let R be a commutative ring spectrum and let M be an R-module spectrum. Define

$$\mathbb{P}_R(M) = \bigvee_{n \geq 0} M^{\wedge_R n}/\Sigma_n$$

with the usual convention that $M^{\wedge_A 0}/\Sigma_0 = R$.
Then $\mathbb{P}_R(M)$ is the free commutative ring spectrum generated by M.

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in particular, if $\Sigma X \simeq \Sigma Y$, then $\mathcal{L}_X^R(\mathbb{P}_R(M)) \simeq \mathcal{L}_Y^R(\mathbb{P}_R(M))$ as commutative R-algebras.

- Let $R \to A \to B$ be a sequence of cofibrations of commutative S-algebras with R cofibrant. Then this sequence *satisfies étale descent* if for all connected X the canonical map

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  is an equivalence.
- We call a map of cofibrant S-algebras $\varphi\colon R \to A$ *really smooth* if it can be factored as $R \xrightarrow{i_R} \mathbb{P}_R(M) \xrightarrow{f} A$ where i_R is the canonical inclusion, M is an R-module, and $R \xrightarrow{i_R} \mathbb{P}_R(M) \xrightarrow{f} A$ satisfies étale descent.

Stability of really smooth algebras

Theorem

If $R \to A$ is really smooth then $\Sigma X \simeq \Sigma Y$ for connected X and Y implies

$$\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$$

as commutative R-algebras.

Étale extensions

Let $R \to A \to B$ be a sequence of cofibrations of commutative S-algebras with R cofibrant. If $R \to A$ is multiplicatively stable and if $R \to A \to B$ satisfies étale descent, then if $\Sigma X \simeq \Sigma Y$ in $sSets_*$ for connected X and Y, then there is a weak equivalence of augmented commutative B-algebras

$$\mathcal{L}_X^R(B) \simeq \mathcal{L}_Y^R(B).$$

KU is stable

Snaith: $KU \simeq (\Sigma^\infty_+ \mathbb{C}P^\infty)[\beta^{-1}]$.

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Theorem

If X and Y are connected and $\Sigma X \simeq \Sigma Y$ in $sSets_*$, then

$$\mathcal{L}^S_X(KU) \simeq \mathcal{L}^S_Y(KU)$$

as commutative augmented KU-algebra spectra.

KO is stable

Rognes: $KO \to KU$ is a C_2-Galois extension (i.e., $KO \simeq KU^{hC_2}$ and $KU \wedge_{KO} KU \simeq \prod_{C_2} KU$).

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Theorem

If X and Y are connected simplicial sets with $\Sigma X \simeq \Sigma Y$ then $\mathcal{L}^S_X(KO) \simeq \mathcal{L}^S_Y(KO)$ as commutative KO-algebras.

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The proof uses Galois descent. Mathew: $KO \to KU$ satisfies étale descent.

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Fix a prime p and consider the infinite loop space

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where $\nu(h) = \frac{p^{h+1}-1}{p-1}$.

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On the suspension spectrum of W_h they invert a generator x of the bottom non-trivial homotopy group $\pi_{2\nu(h)}(W_h) \cong \mathbb{Z}_{(p)}$ and obtain an E_∞-ring spectrum

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R_h has torsion-free homotopy groups that vanish in odd degrees, it is Landweber exact, and its Morava-$K(n)$ localization $L_{K(n)} R_h$ vanishes if and only if $n > h+1$, so R_h is of chromatic height $h+1$.

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Theorem

If X and Y are connected and $\Sigma X \simeq \Sigma Y$ in $sSets_*$, then $\mathcal{L}^S_X(R_h) \simeq \mathcal{L}^S_Y(R_h)$ as commutative augmented R_h-algebra spectra.

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Thank you!