Seminar: From Hochschild homology to topological Hochschild homology
Winter term 2021/22

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The aim of this seminar is to generalize Hochschild homology to topological Hochschild homology. The latter has several equivalent descriptions. If you restrict to topological Hochschild homology of ordinary rings, then one can also study Mac Lane homology or stable K-theory. We start with ordinary Hochschild homology, then introduce Mac Lane homology and stable K-theory. The last third of the seminar is used to introduce topological Hochschild homology which is an analogue of Hochschild homology for ring spectra.

I assume that everybody who wants to give a talk in this seminar has some background in homological algebra and Hochschild homology. The first talk is just a brief recollection on Hochschild homology in order to recall some basics and set notation. For background on category theory and homology of small categories see [R1]. Some talks need methods and results from homotopy theory.

Beware! The level of the talks is pretty uneven. There are rather straightforward ones such as (1)-(4), (7) and then there are others. If you are not sure about a talk, just ask me. I will give talk (10) and all leftover talks.

Talks

(1) **Hochschild homology** Recall the basics of Hochschild homology: Its definition via the Hochschild chain complex, its derived functor definition in the flat case and some examples [L].

(2) **Mac Lane's cubical construction and HML_*** Define what the cubical construction is and also discuss its variant for sets. Define Mac Lane homology and its map to Hochschild homology. [L, §13.2 up to 13.2.9 (skip 13.2.2 for now)]

(3) **Generalization of Hochschild (co)homology to non-additive bimodules** Define and discuss $\mathsf{Tor}_*^{\mathcal{F}(R)}(I^*, T)$ and $\mathsf{Ext}^*_{\mathcal{F}(R)}(R, T)$. [L, 13.1 up to 13.1.3]

(4) **(Co)Homology of small categories and relationship to (3)** Define what the (co)homology of a small category with coefficients in a functor or bifunctor is [L, App. C]. Relate the Tor and Ext terms from talk (3) to such (co)homology groups [L, 13.1.4-13.1.7].

(5) **Mac Lane (co)homology and functor homology** Prove Jibladze's and Pirashvili's identification of Mac Lane (co)homology and suitable Tor and Ext terms [L, 13.2.10-13.2.17].

(6) **Mac Lane homology of $\mathbb{Z}/2$** Tell us what Eilenberg Mac Lane spaces are. State and prove [L, Theorem 13.2.2]. Use this to present the calculation of $\mathsf{HML}_*(\mathbb{Z}/2)$ [L, 13.4.1]. This talk needs background from homotopy theory.

(7) **Algebraic K-theory** Define $K_0(R)$, $K_1(R)$ and $K_2(R)$ and choose some important examples from [Ro]. Sketch the plus construction and define the higher algebraic K-groups. Tell us what the algebraic K-theory of finite fields is. If you want to do the plus construction in detail, then you need some background in topology.

(8) **Stable K-theory** Define the Dennis trace map [Ro, 6.2.14] and compare it to the definition in [L, 13.1.8,13.1.9]. Define stable K-theory and state the comparison theorem of Dundas and McCarthy, identifying stable K-theory with Mac Lane homology [L, 13.3 up to 13.3.3] and some background from [L] (without proof). This talk needs some topology.

(9) **A quick and dirty introduction to model categories** Tell us what a model category is and present the projective model structure on non-negatively graded chain complexes of R-modules for a ring R. Sketch the construction of the homotopy category associated to a model category [DS].

(10) **A model for the stable homotopy category I** This talk features the axioms of a generalized cohomology theory, a motivation and definition of the stable homotopy category, a definition of the stable homotopy groups of spheres and of the homology and cohomology with respect to a spectrum. Brown representability tells us how to connect generalized cohomology theories to spectra. (I will give this talk.)

(11) **A model for the stable homotopy category II** Define symmetric spectra and orthogonal spectra. As examples we need at least the sphere spectrum and Eilenberg-MacLane spectra [HSS],

[MMSS, Example 4.4, Proposition 8.7]. In this talk the emphasis is on introducing the models and explaining the model structures.

(12) **Topological Hochschild homology** Define $\mathsf{THH}(R; M)$ for a ring spectrum R and an R-bimodule spectrum M and also $\mathsf{THH}^R(A; M)$ where A is an R-algebra spectrum and M is an A-bimodule [EKMM, Definitions IX.1.1, IX.2.1] (state [EKMM, Proposition IX.2.5] without going into details). State the comparison theorem [EKMM, Theorem IX:1.7] between topological Hochschild homology over an Eilenberg Mac Lane spectrum and Hochschild homology in good cases. Remind us about the comparison diagram from [L, p. 400]. Give us some examples of THH of ring spectra [R2, 6.2].

Literatur

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