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Robinson–Whitehouse complex and stable homotopy

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Abstract

There is a version of André–Quillen homology for commutative algebras called Γ-homology H^*_Γ which was introduced by A. Robinson and S. Whitehouse. We will prove that a generalized variant of H^*_Γ calculates the homotopy of every abelian Γ-group. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Recently, Alan Robinson and Sarah Whitehouse introduced a brave new algebra version of André–Quillen homology theory, called Γ-homology. One version of Γ-homology for Eilenberg–MacLane spectra of commutative rings, has a purely algebraic description (see [9] and Section 2). The goal of this paper is to give a construction, which is a little bit more general and which allows us to prove the following result. Let Γ be the small category of finite pointed sets. For any $n \geqslant 0$, let $[n]$ be the set $\{0, 1, \dots, n\}$ with basepoint 0. We assume that the objects of Γ are the sets $[n]$. Let A be a commutative k-algebra over a commutative ring k and let M be an A-module. According to Loday [4] there exists a functor $\mathscr{L}(A, M): \Gamma \to k\text{-}mod$, which assigns $M \otimes A^{\otimes n}$ to $[n]$. Here all tensor products are taken over k. On the other hand any functor $T: \Gamma \to \{pointed\ spaces\}$ gives rise to a spectrum (see [8,1] and Section 3), thus $\mathscr{L}(A, M)$ gives a spectrum as well. Our result

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claims that the homotopy groups of this spectrum are isomorphic to the Γ-homology of A with coefficients in M as defined in [9]. Actually we prove a more general result: The Robinson-Whitehouse complex, as it is defined in Section 2, calculates the homotopy of any abelian Γ-group.

2. Robinson–Whitehouse complex

A left Γ-module is a covariant functor from Γ to k-*mod*. For any left Γ-module $T:\Gamma\to k-mod$ we define the chain complex $C^{\Gamma}_{*}(T)$, which coincides with the Robinson–Whitehouse complex $C^{\Gamma}_{*}(A, M)$ when $T=\mathscr{L}(A, M)$. Let Ω be the category of all finite nonempty sets and surjections. We will assume that the objects of Ω are the sets

$$\underline{n}:=\{1,\dots,n\},\quad n\geqslant 1.$$

Let $N\Omega_q(\underline{n},\underline{1})$ be the set of composable morphisms $[f_q|f_{q-1}|\cdots|f_1]$ in Ω of length q, starting at $\underline{n}$ and ending at $\underline{1}$. So, we assume that the domain of f_1 is $\underline{n}$ and the codomain of f_q is $\underline{1}$. Let $kN\Omega_q(\underline{n},\underline{1})$ be the module generated by the set $N\Omega_q(\underline{n},\underline{1})$. For any arrow $g:\underline{n}\to\underline{m}$ and $i\in\underline{m}$ one denotes by $g^i:\underline{n^i}\to\underline{1}$ the component of g at i. Here n^i is the number of elements in $g^{-1}(i)$. Similarly, given a string of k morphisms $[f_k|f_{k-1}|\cdots|f_1]$ of Ω ending at $\underline{m}$, one decomposes this into m strings of k morphisms each ending at $\underline{1}$. One denotes by $[f^{(i)}_k|f^{(i)}_{k-1}|\cdots|f^{(i)}_1]$ the ith component of $[f_k|f_{k-1}|\cdots|f_1]$. Let $T:\Gamma\to Vect$ be a Γ-module. Any map $g:\underline{n}\to\underline{m}$ has a unique extension as a pointed map $[n]\to[m]$. By abuse of notation we still denote this map by g. Following Sarah Whitehouse [9] we define the Robinson–Whitehouse chain complex $C^{\Gamma}_{*}(T)$ by

$$C^{\Gamma}_0(T)=T([1])$$

$$C^{\Gamma}_q(T)=\bigoplus_{n\geqslant 1} kN\Omega_q(\underline{n},\underline{1})\otimes T([n])\quad \text{for } q\geqslant 1.$$

The boundary map $d:C^{\Gamma}_q(T)\to C^{\Gamma}_{q-1}(T)$ is the alternating sum of face maps $\partial_i:C^{\Gamma}_q(T)\to C^{\Gamma}_{q-1}(T)$, $0\leqslant i\leqslant q$. For $[f_q|f_{q-1}|\cdots|f_1]\in N\Omega_q(\underline{n},\underline{1})$ and $x\in T([n])$ one defines

$$\partial_0([f_q|\cdots|f_1]\otimes x)=[f_q|\cdots|f_2]\otimes f_{1*}(x),$$

$$\partial_i([f_q|\cdots|f_1]\otimes x)=[f_q|\cdots|f_{i+1}f_i|\cdots|f_1]\otimes x\quad \text{for } 0<i<q.$$

In order to describe the last face map we need to fix additional notation. Let $\underline{r}$ be the domain of f_q. Thus $f_{q-1}\cdots f_1:\underline{n}\to\underline{r}$. Moreover, for any $1\leqslant j\leqslant r$ let r_j be the number of elements in the preimage of j under $f_{q-1}\cdots f_1$. Let l_j: $[n]\to[r_j]$ be the map, which is nonzero only on the preimage of j under $f_{q-1}\cdots f_1$, where it is an ordering preserving bijection. Now one defines

$$\partial_q([f_q|\cdots|f_1]\otimes x)=\sum_{j=1}^{r}[f^{(j)}_{q-1}|\cdots|f^{(j)}_1]\otimes l_{j*}x\quad \text{if } q>1$$

and $\partial_1([\underline{n}\to\underline{1}]\otimes x)=\sum_{j=1}^{n} g_{j*}x$ for $q=1$. Here g_j: $[n]\to[1]$ is the map, for which $g_j(j)=1$ and $g_j(i)=0$ for $i=j$. Straightforward calculation shows that $\partial_i\partial_j=\partial_{j-1}\partial_i$ if $i<j$. Hence one obtains a chain complex, whose homology is denoted by $H^{\Gamma}_{*}(T)$.

For $T = \mathscr{L}(A, M)$ this complex was defined by S. Whitehouse. In this case one writes $H^{\Gamma}_*(A, M)$ instead of $H^{\Gamma}_*(T)$ and $H^{\Gamma}_*(A, M)$ is called the Γ-homology of A with coefficients in M.

3. Homotopy of Γ-spaces

Let $Sets_*$ be the category of all pointed sets and F be a left Γ-module. One can prolong F by direct limits to a functor $Sets_* \to k\text{-}mod$. Then by degreewise action one obtains a functor from the category of simplicial sets with basepoint $s.Sets_*$ to the category of simplicial modules. By abuse of notation we will still denote this functor by F. By [1] one knows that the homotopy of the spectrum corresponding to the Γ-space F can be described as

$$\pi^{st}_*(F) := \operatorname{colim} \pi_{*+n} F(S^n).$$

Here S^n denotes a simplicial model of the n-dimensional sphere. By [1] this definition does not depend on the model one chooses for the sphere. Mimicking Korollar 6.12 in [2], one can prove that this limit always stabilizes and one has the isomorphism

$$\pi_i(F) \cong \pi_{i+n} F(S^n) \quad \text{if } n > i. \tag{3.1}$$

Theorem 1. *Let F be a left Γ-module. Then there are natural isomorphisms*

$$\pi^{st}_*(F) \cong H^{\Gamma}_*(F).$$

Remark. It is already proved by the first author (see [5] or E.13.2.2 of [4]) that $\pi^{st}_*(F)$ is isomorphic to $Tor^{\Gamma}_*(t, F)$. Here $t : \Gamma^{op} \to Ab$ maps a finite pointed set S_+ to the free abelian group generated by the elements of S.

Proof. Let Γ-*mod* be the category of all Γ-modules. Clearly Γ-*mod* is an abelian category with enough projective objects. Moreover π^{st}_* and H^{Γ}_* define exact connected sequences of functors from Γ-modules to k-modules. Therefore it is enough to show that both sequences vanish on projectives in positive dimensions and are isomorphic to each other in dimension zero. Since π^{st}_* and H^{Γ}_* commute with direct sums it is enough to consider projective generators. According to Section 4 and Lemma 2 it suffices to consider the left modules $L^{\otimes n}$, $n \geqslant 0$. That π^{st}_* vanishes on projective left Γ-modules is clear from the remark we made above. Lemma 3 below gives an independent proof for this fact. The vanishing result for H^{Γ}_* is proved in Lemma 4. The isomorphism in dimension zero can be directly seen; it is also consequence of Lemmas 3 and 4. □

Lemma 2. *For left Γ-modules F, T one has an isomorphism*

$$\pi^{st}_*(F \otimes T) \cong \pi^{st}_*(F) \otimes T([0]) \oplus F([0]) \otimes \pi^{st}_*(T).$$

Proof. One of the models of S^n has only two nondegenerate simplexes, one in dimension 0 and a second one in dimension n. Therefore for $n > 0$ the group $\pi_j F(S^n)$ is $F(0)$ for $j = 0$ and is zero for $0 < j < n$. Having this fact in mind the Lemma is a consequence of the isomorphism (3.1) and the Eilenberg–Zilber theorem. □

4. Projective generators in *Γ-mod*

For any $n \geqslant 0$ one defines

$$\Gamma^n := k[Hom_\Gamma([n], -)].$$

Here $k[S]$ denotes the free k-module generated by a set S. It is a consequence of the Yoneda lemma that the functors Γ^n are projective generators in Γ-*mod* for $n \geqslant 0$. Clearly Γ^0 is the constant functor with the value k and $\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m}$. Here for any two left Γ-modules F and T we define

$$(F \otimes T)([n]) := F([n]) \otimes T([n]).$$

Moreover $\Gamma^1 \cong \Gamma^0 \oplus L$, where L takes $[n]$ to the free k-module generated by the set $[n]$ modulo the subspace generated by $0 \in [n]$. Hence the $L^{\otimes n}, n \geqslant 0$, are also projective generators. The Γ-homology and the homotopy of these projective generators are described in the following two lemmas.

Lemma 3. *The left Γ-modules $F = L^{\otimes n}$ have the following homotopy*:

$$\pi_*^{st}(F) = 0 \quad \text{if } n \neq 1,$$

and for $n = 1$ one has

$$\pi_i(L) = 0, \text{ for } i \geqslant 1 \quad \text{and} \quad \pi_0(L) \cong k.$$

Proof. If $n = 0$, then F is a constant functor. Therefore $F(S^n)$ is a constant simplicial module and the result follows. Now assume $n \geqslant 1$. Thanks to Lemma 6, it is enough to consider the case $F = L$, because $L([0]) = 0$. In this case one can use the isomorphism (3.1) and the fact that the chain complex associated to the simplicial module $L(S^n)$ is nothing but the reduced chains of S^n with coefficients in k □.

Lemma 4. *The Γ-homology of the left Γ-modules $F = L^{\otimes m}$ is as follows*:

$$H_*^\Gamma(F) = 0 \quad \text{if } m \neq 1$$

and for $m = 1$ one has

$$H_i^\Gamma(L) = 0, \text{ for } i \geqslant 1 \quad \text{and} \quad H_0^\Gamma(L) \cong k.$$

Proof. Using the fact that the generalized Robinson–Whitehouse complex is a semisimplicial module, we are going to construct homotopies to prove the claim.

In the case $m = 0$ we have the constant functor with the value k. We denote a generator of $C_q^\Gamma(F)$ by

$$[f_q| \dots |f_1] \otimes (1).$$

Here (1) is the unit in k. The homotopy in degree zero from the identity map to zero is easy to guess: We take

$$h_0(1) := [\underline{2} \to \underline{1}] \otimes (1) - [\underline{1} \to \underline{1}] \otimes (1).$$

For the homotopies in higher degrees we need to describe some additional maps. If we have two maps $f: \underline{n} \to \underline{m}$ and $g: \underline{k} \to \underline{l}$ in Ω, we can build their sum $f \sqcup g$ in the obvious way, such that $f \sqcup g: \underline{n+k} \to \underline{m+l}$. Moreover we can define a folding map $\delta: \underline{2n} \to \underline{n}$ just by

$$\delta(i) = i \quad \text{for } 1 \leqslant i \leqslant n \quad \text{and} \quad \delta(n+i) = i \text{ again for } 1 \leqslant i \leqslant n$$

Now the presimplicial homotopy $h = \sum_{i=0}^{q} (-1)^i h_i([f_q| \dots |f_1] \otimes (1))$ can be defined as follows:

$$h_0([f_q| \dots |f_1] \otimes (1))$$
$$:= [f_q| \dots |f_1|\delta] \otimes (1) - [f_q| \dots |f_1|id] \otimes (1)$$

and

$$h_i([f_q| \dots |f_1] \otimes (1))$$
$$:= [f_q| \dots |f_{i+1}|\delta|f_i \sqcup f_i| \dots |f_1 \sqcup f_1] \otimes (1)$$
$$- [f_q| \dots |f_{i+1}|id|f_i| \dots |f_1] \otimes (1).$$

A straightforward calculation shows that this yields a homotopy between the identity map on $C^{\Gamma}_*(T)$ and the zero map.

In the cases $m \geqslant 1$ we can define the homotopy as follows: The chain complex consists of strings of composable morphisms tensorized with m-tuples $(a_1, \dots, a_m)$ of $a_i \in \underline{n_1}$ when the first map in this string starts in $\underline{n_1}$.

Let $\varepsilon(i)$ with $1 \leqslant i \leqslant n$ denote the map $\varepsilon(i): \underline{n+1} \to \underline{n}$ which takes $n+1$ to i and is the identity on all other values. Then we can define the maps h_j as

$$h_0([f_q| \dots |f_1] \otimes (a_1, \dots, a_m) := [f_q| \dots |f_1|\varepsilon(a_m)] \otimes (a_1, \dots, a_{m-1}, n_1 + 1)$$

and

$$h_j([f_q| \dots |f_1] \otimes (a_1, \dots, a_m) := [f_q| \dots |f_{i+1}|\varepsilon(f_j \cdots f_1(a_m))|f_i \sqcup id| \dots |f_1 \sqcup id]$$
$$\otimes (a_1, \dots, a_{m-1}, n_1 + 1).$$

Here f_1 is supposed to start in $\underline{n_1}$.

In the cases $m > 1$ we obtain a homotopy between the identity map and the zero map. For $m = 1$ the homotopy connects the identity and the constant chain map η

$$\eta([f_q| \dots |f_1] \otimes (a)) := [id| \dots |id] \otimes (1).$$

These facts can be seen by direct but tedious calculation. □

5. Relation with Harrison theory

In 3.3 of [3] Loday defined the Harrison homology of a left Γ-module F, which is denoted by $Harr_*(F)$. For $F = \mathscr{L}(A, M)$ one recovers the usual Harrison homology of commutative algebras (see [4]). It is a well-known fact that, in the characteristic zero case, Harrison homology is isomorphic to André–Quillen homology (see [6]) up to a shift in dimension. It follows from the very definition that $Harr_0(F) \cong F([0])$ and $Harr_1(F) \cong \pi_0(F)$.

It is not hard to show that in the characteristic zero case one has an isomorphism $Harr_{*-1}(F) \cong \pi_*^{st}(F)$ (see [5] for this and more general results). Thus $Harr_{*-1}(F) \cong H_*^{\Gamma}(F)$. This was also proved in [9] based on the combinatorical and homotopical analysis of the space of fully grown trees [7]. In positive characteristic the sequence of functors $Harr_* : \Gamma\text{-}mod \to k\text{-}mod$ does not form an exact connected sequence of functors, but still Harrison homology vanishes on projective Γ-modules. The proof of this fact is a bit technical and will be given elsewhere.

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