

# Divided power structures and chain complexes

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ABSTRACT. We interpret divided power structures on the homotopy groups of simplicial commutative rings as having a counterpart in divided power structures on chain complexes coming from a non-standard symmetric monoidal structure.

## 1. Introduction

Every commutative simplicial algebra has a divided power structure on its homotopy groups. The Dold-Kan correspondence compares simplicial modules to non-negatively graded chain complexes. It is an equivalence of categories, but its multiplicative properties do not interact well with commutativity: differential graded commutative algebras are sent to homotopy commutative simplicial algebras, but in general not to simplicial algebras that are commutative on the nose. The aim of this note is to gain a better understanding when suitable multiplicative structures on a chain complex actually do give rise to divided power structures. To this end, we use the equivalence of categories between simplicial modules and non-negatively graded chain complexes and transfer the tensor product of simplicial modules to a symmetric monoidal category structure on the category of chain complexes.

We start with a brief overview on divided power algebras in section 2. We prove a general transfer result for symmetric monoidal category structures in section 6. That such a transfer of monoidal structures is possible is a folklore result and constructions like ours are used in other contexts, see for instance [Sch01, p.263] and [Q∞]. We consider the case of chain complexes in section 7 where we use this monoidal structure to gain our main results: in Corollary 7.2 we give a criterion when a chain complex has a divided power structure on its homology groups and in Theorem 7.5 we describe when we can actually gain a divided power structure on the differential graded commutative algebra that interacts nicely with the differential.

As an example we give an alternative description of the well-known ([C54, BK94]) divided power structure on Hochschild homology: instead of working with

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the bar construction in the differential graded setting, we consider the simplicial bar construction of a commutative algebra. This is naturally a simplicial commutative algebra and hence the reduced differential graded bar construction inherits a divided power chain algebra structure from its simplicial relative.

## 2. Divided power algebras

Let  $R$  be a commutative ring with unit and let  $A_*$  be an  $\mathbb{N}_0$ -graded commutative algebra with  $A_0 = R$ . We denote the positive part of  $A_*$ ,  $\bigoplus_{i>0} A_i$ , by  $A_{*>0}$ .

DEFINITION 2.1. A *system of divided powers in  $A_*$*  consists of a collection of functions  $\gamma_n$  for  $n \geq 0$  that are defined on  $A_i$  for  $i > 0$  such that the following conditions are satisfied.

- (a)  $\gamma_0(a) = 1$  and  $\gamma_1(a) = a$  for all  $a \in A_{*>0}$ .
- (b) The degree of  $\gamma_i(a)$  is  $i$  times the degree of  $a$ .
- (c)  $\gamma_i(a) = 0$  if the degree of  $a$  is odd and  $i > 1$ .
- (d)  $\gamma_i(\lambda a) = \lambda^i \gamma_i(a)$  for all  $a \in A_{*>0}$  and  $\lambda \in R$ .
- (e) For all  $a \in A_{*>0}$

$$\gamma_i(a)\gamma_j(a) = \binom{i+j}{i} \gamma_{i+j}(a).$$

- (f) For all  $a, b \in A_{*>0}$

$$\gamma_i(a+b) = \sum_{k+\ell=i} \gamma_k(a)\gamma_\ell(b).$$

- (g) For all  $a, b \in A_{*>0}$

$$\gamma_i(ab) = i! \gamma_i(a)\gamma_i(b) = a^i \gamma_i(b) = \gamma_i(a)b^i.$$

- (h) For all  $a \in A_{*>0}$

$$\gamma_i(\gamma_j(a)) = \frac{(ij)!}{i!(j!)^i} \gamma_{ij}.$$

If we want to specify a fixed system of divided powers  $(\gamma_i)_{i \geq 0}$  on  $A_*$ , we use the notation  $(A_*, \gamma)$ . For basics about systems of divided powers see [C54, Exposé 7, 8], [GL69, section 7], [E95, Appendix 2], [Be74, Chapitre I] and [Ro68].

*Some properties.* Condition (e) implies that for all  $i > 1$  the  $i$ -fold power of an element  $a \in A_{*>0}$  is related to its  $i$ -th divided power via

$$(2.1) \quad a^i = i! \gamma_i(a).$$

Therefore, if the underlying  $R$ -modules  $A_i$  are torsion-free, then there is at most one system of divided powers on  $A_*$ , and if  $R$  is a field of characteristic zero, then the assignment  $\gamma_i(a) = a^i/i!$  defines a unique system of divided powers on every  $A_*$ .

If squares of odd degree elements are zero and if we are in a torsion-free context, then condition (c) is of course taken care of by condition (e). Some authors (e.g. [C54]) demand that the underlying graded commutative algebra is *strict*, i.e., that  $a^2 = 0$  whenever  $a$  has odd degree.

Not every  $\mathbb{N}_0$ -graded commutative algebra possesses a system of divided powers. Consider for instance the polynomial ring  $\mathbb{Z}[x]$  over  $\mathbb{Z}$ . The existence of  $\gamma_2(x)$  would imply that  $x^2$  were divisible by two.

Note, that the following useful product formula

$$(2.2) \quad \frac{(ij)!}{i!(j!)^i} = \prod_{r=2}^i \binom{rj-1}{j-1}$$

holds.

We saw that over the rationals, divided powers can be expressed in terms of the underlying multiplication of the  $\mathbb{N}_0$ -graded commutative algebra. If the ground ring  $R$  is a field of characteristic  $p$  for some prime number  $p \geq 2$ , then for any system of divided powers on  $A_*$  the relation

$$a^p = p! \gamma_p(a)$$

forces the  $p$ -th powers of elements in  $A_*$  to be trivial. There are more relations implied by divided power structures, for instance any iteration of the form  $\gamma_i(\gamma_p(a))$  is equal to  $\gamma_{ip} \frac{(ip)!}{i!(p!)^i}$ , but using relation (2.2) it is easy to see that the coefficient of  $\gamma_{ip}$  is congruent to one and hence

$$\gamma_i(\gamma_p(a)) = \gamma_{ip}(a).$$

For a more thorough treatment of divided powers in prime characteristic see [C54, Exposé 7, §§7,8], [A76], and [G90].

*Divided power structure with respect to an ideal.* The occurrence of divided power structures is not limited to the graded setting. In an  $\mathbb{N}_0$ -graded commutative  $R$ -algebra  $A_*$  the positive part is an ideal. In the context of ungraded commutative rings, divided power structures can be defined relative to an ideal. The following definition is taken from [Be74, Chapitre I, Définition 1.1.].

**DEFINITION 2.2.** Let  $A$  be a commutative ring and  $I$  an ideal in  $A$ . A *divided power structure on  $I$*  consists of a family of maps

$$\gamma_i: I \rightarrow A, i \geq 0$$

which satisfy the following conditions.

- (a) For all  $a \in I$ ,  $\gamma_0(a) = 1$  and  $\gamma_1(a) = a$ . The image of the  $\gamma_i$  for  $i \geq 2$  is contained in  $I$ .
- (b) For all elements  $a \in A$  and  $b \in I$ ,  $\gamma_i(ab) = a^i \gamma_i(b)$ .
- (c) Conditions (e), (f) and (h) of Definition 2.1 apply in an adapted sense.

An important example of divided power structures on ungraded rings is the case of discrete valuation rings of mixed characteristic. If  $p$  is the characteristic of the residue field,  $\pi$  is a uniformizer and  $p = u\pi^e$  with  $u$  a unit, then for the existence of a divided power structure on the discrete valuation ring it is necessary and sufficient that the ramification index  $e$  is less than or equal to  $p-1$  (see [Be74, Chapitre I, Proposition 1.2.2]).

*Morphisms and free objects.* Morphisms are straightforward to define:

**DEFINITION 2.3.** Let  $(A_*, \gamma)$  and  $(B_*, \gamma')$  be two  $\mathbb{N}_0$ -graded commutative algebras with systems of divided powers. A morphism of  $\mathbb{N}_0$ -graded commutative algebras  $f: A_* \rightarrow B_*$  is a *morphism of divided power structures*, if  $f(\gamma_i(a)) = \gamma'_i(f(a))$ , for all  $i \geq 2$  and  $a \in A_{>*}$ .

The analogous definition works in the ungraded case.

We will describe the free divided power algebra generated by an  $\mathbb{N}_0$ -graded module  $M_*$  whose components  $M_i$  are free  $R$ -modules.

DEFINITION 2.4. Consider the free  $R$ -module generated by an element  $x$  of degree  $m$ .

- If  $m$  is odd, then *the free divided power algebra on  $x$  over  $R$*  is the exterior algebra over  $R$  generated by  $x$ ,  $\Lambda_R(x)$ . In this case the  $\gamma_i$  are trivial for  $i \geq 2$ .
- If  $m$  is even, *the free divided power algebra on  $x$  over  $R$*  is

$$R[X_1, X_2, \dots]/I.$$

Here the  $X_n$  are polynomial generators in degree  $nm$  and  $I$  is the ideal generated by

$$X_i X_j - \binom{i+j}{i} X_{i+j}.$$

As the tensor product over  $R$  is the coproduct in the category of  $\mathbb{N}_0$ -graded commutative  $R$ -algebras, we get a notion of a free divided power algebra on a finitely generated module  $M_*$  whose  $M_i$  are free as  $R$ -modules by taking care of Condition (g) of Definition 2.1. If  $M_*$  is not finitely generated we take the colimit of the free divided power algebras on finitely many generators. Compare [GL69, Proposition 1.7.6].

If  $M_*$  is an  $\mathbb{N}_0$ -graded module that is freely generated by elements  $x_1, \dots, x_n$ , then it is common to denote the free divided power algebra over  $R$  on these generators by  $\Gamma_R(x_1, \dots, x_n)$ .

Occurrences of free divided power algebras are ample. For instance, the cohomology ring of the loop space on a sphere,  $\Omega(\mathbb{S}^n)$ , for  $n \geq 2$  is a free divided power algebra with

$$H^*(\Omega(\mathbb{S}^n); \mathbb{Z}) \cong \begin{cases} \Gamma_{\mathbb{Z}}(a) & |a| = n-1, n \text{ odd} \\ \Gamma_{\mathbb{Z}}(a, b) \cong \Lambda_{\mathbb{Z}}(a) \otimes \Gamma_{\mathbb{Z}}(b) & |a| = n-1, |b| = 2n-2, n \text{ even} \end{cases}$$

Often, free divided power algebras arise as duals of symmetric algebras [E95, A2.6]).

Let  $\Sigma_n$  denote the symmetric group on  $n$ -letters. If  $N_*$  is an  $\mathbb{N}_0$ -graded module with  $\Sigma_n$ -action, then we denote by  $N_*^{\Sigma_n}$  the invariants in  $N_*$  with respect to the  $\Sigma_n$ -action. If  $M$  is a free  $R$ -module, then one can describe the free divided power algebra on  $M$  as

$$(2.3) \quad \bigoplus_{n \geq 0} (M_*^{\otimes n})^{\Sigma_n}.$$

This is a classical result and is for instance proved in [C54, Exposé 8, Proposition 4]. See Roby [Ro68, Remarque p. 103] for an example where the two notions differ if one considers a module that is not free. Divided power structures can in fact be described via (2.3): a graded module with free components,  $M_*$ , with  $M_0 = 0$  has a divided power structure if there is a map

$$(2.4) \quad \bigoplus_{n \geq 1} (M_*^{\otimes n})^{\Sigma_n} \rightarrow M_*$$

that satisfies the axioms of a monad action (see [F00]). The monad structure that is applied in the description via (2.4) uses the invertibility of the norm map on reduced symmetric sequences of the form  $M_*^{\otimes n}$  [F00, 1.1.16 and 1.1.18]. The invertibility of the norm map in this case was discovered earlier by Stover [St93, 9.10].

2.0.1. *Divided power structures in the simplicial context.* On the homotopy groups of simplicial commutative rings there are divided power operations and it is this instance of divided power structures that we will investigate in this paper.

In the context of the action of the Steenrod algebra on cohomology groups of spaces, the top operation is the  $p$ -th power map. On the homotopy groups of simplicial commutative  $\mathbb{F}_2$ -algebras, there are analogous operations  $\delta_i$  of degree  $i \geq 2$  such that the highest operation is the divided square. These operations were investigated by Cartan [C54, Exposé no 8] and were intensely studied by many people ([Bo67, section 8], [D80], [G90, chapter 2], [T99]). In [Bo67, 8.8 onwards] a family of operations for odd primes is discussed as well.

**Notation.** With  $\Delta$  we denote the category whose objects are the sets  $[n] = \{0, \dots, n\}$  with their natural ordering and morphisms in  $\Delta$  are monotone maps. A simplicial object in a category  $\mathcal{C}$  is a functor from the opposite category of  $\Delta$ ,  $\Delta^{op}$ , to  $\mathcal{C}$ . We denote the category of simplicial objects in  $\mathcal{C}$  by  $s\mathcal{C}$ .

In the category of simplicial sets, the representable functors

$$\Delta(n): \Delta^{op} \rightarrow \text{Sets}$$

are the ones that send  $[m] \in \Delta$  to  $\Delta([m], [n])$ . If  $\delta_i: [n] \rightarrow [n+1]$  denotes the map that is the inclusion that misses  $i$  and is strictly monotone everywhere else and if  $\sigma_i: [n] \rightarrow [n-1]$  is the surjection that sends  $i$  and  $i+1$  to  $i$  and is strictly monotone elsewhere, then we denote their opposite maps by  $d_i = (\delta_i)^{op}$  and  $s_i = (\sigma_i)^{op}$ .

In the following we fix an arbitrary commutative ring with unit  $R$ . If  $S$  is a set, then we denote the free  $R$ -module generated by  $S$  by  $R[S]$ . The tensor product of two  $R$ -modules  $N$  and  $M$ ,  $N \otimes_R M$ , will be abbreviated by  $N \otimes M$ .

### 3. The Dold-Kan correspondence

The Dold-Kan correspondence [Do58, Theorem 1.9] compares the category of simplicial objects in an abelian category  $\mathcal{A}$  with the non-negatively graded chain complexes over  $\mathcal{A}$  via a specific equivalence of categories. We will focus on the correspondence between simplicial  $R$ -modules,  $\mathbf{smod}_R$ , and  $\mathbb{N}_0$ -graded chain complexes of  $R$ -modules,  $\mathbf{Ch}_{\geq 0}^R$ .

The equivalence is given by the *normalization functor*,  $N: \mathbf{smod}_R \rightarrow \mathbf{Ch}_{\geq 0}^R$ , and we denote its inverse by  $\Gamma$

$$N: \mathbf{smod}_R \xrightleftharpoons{\Gamma} \mathbf{Ch}_{\geq 0}^R : \Gamma.$$

In particular the functor  $N$  is a left adjoint to  $\Gamma$ . The value of  $N$  on a simplicial  $R$ -module  $X_\bullet$  in chain degree  $n$  is

$$N_n(X_\bullet) = \bigcap_{i=1}^n \ker(d_i : X_n \rightarrow X_{n-1})$$

where the  $d_i$  are the simplicial structure maps. The differential  $d : N_n(X_\bullet) \rightarrow N_{n-1}(X_\bullet)$  is given by the remaining face map  $d_0$ .

Recall that for a chain complex  $C_*$ ,

$$\Gamma_n(C_*) = \bigoplus_{p=0}^n \bigoplus_{\varrho: [n] \rightarrow [p]} C_p^{\varrho}$$

where  $\varrho$  is an order preserving surjection and  $C_p^\varrho = C_p$ . The normalized chain complex has an alternative description via a quotient construction where one reduces modulo degenerate elements [W94, lemma 8.3.7]. There is a canonical identification  $\varphi_{C_*}: N\Gamma C_* \cong C_*$ : if  $\varrho$  is not the identity map, then the simplicial structure of  $\Gamma(C_*)$  identifies elements in  $C_p^\varrho$  as being degenerate. For a simplicial  $R$ -module  $A_\bullet$  the isomorphism  $\psi_{A_\bullet}: \Gamma N(A_\bullet) \cong A_\bullet$  is induced by the map that sends  $N(A_\bullet)_p^\varrho \subset A_p$  via  $\varrho$  to  $A_n$ .

The tensor product of chain complexes  $(C_*, d)$  and  $(C'_*, d')$  is defined as usual via

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

with differential  $D(c \otimes c') = (dc) \otimes c' + (-1)^p c \otimes d'c'$  for  $c \in C_p$ ,  $c' \in C'_q$ . Let  $(R, 0)$  denote the chain complex that has  $R$  as degree zero part and that is trivial in all other degrees. There is a twist isomorphism  $\tau_{C_*, C'_*}: C_* \otimes C'_* \rightarrow C'_* \otimes C_*$  that is induced by  $\tau_{C_*, C'_*}(c \otimes c') = (-1)^{pq} c' \otimes c$  for  $c$  and  $c'$  as above. The structure  $(\text{Ch}_{\geq 0}^R, \otimes, (R, 0), \tau)$  turns  $\text{Ch}_{\geq 0}^R$  into a symmetric monoidal category.

For two arbitrary simplicial  $R$ -modules  $A_\bullet$  and  $B_\bullet$ , let  $A_\bullet \hat{\otimes} B_\bullet$  denote the degree-wise tensor product of  $A_\bullet$  and  $B_\bullet$ , *i.e.*,  $(A_\bullet \hat{\otimes} B_\bullet)_n = A_n \otimes B_n$ . Here, the simplicial structure maps are applied in each component; in particular, the differential on  $N_*(A_\bullet \hat{\otimes} B_\bullet)$  in degree  $n$  is  $d_0 \otimes d_0$ . The constant simplicial object  $\underline{R}$  which consists of  $R$  in every degree is the unit with respect to  $\hat{\otimes}$  and the twist

$$\hat{\tau}_{A_\bullet, B_\bullet}: A_\bullet \hat{\otimes} B_\bullet \longrightarrow B_\bullet \hat{\otimes} A_\bullet, \quad \hat{\tau}_{A_\bullet, B_\bullet}(a \otimes b) = b \otimes a$$

gives  $(\text{smod}_R, \hat{\otimes}, \underline{R}, \hat{\tau})$  the structure of a symmetric monoidal category. Note that  $N(\underline{R}) \cong (R, 0)$ .

There are natural maps, the *shuffle maps*,

$$\text{sh}: N(A_\bullet) \otimes N(B_\bullet) \longrightarrow N(A_\bullet \hat{\otimes} B_\bullet)$$

(see [ML95, VIII.8]) that turn the normalization into a lax symmetric monoidal functor, *i.e.*, the shuffle maps are associative in a suitable sense and the diagram

$$\begin{array}{ccc} N(A_\bullet) \otimes N(B_\bullet) & \xrightarrow{\text{sh}} & N(A_\bullet \hat{\otimes} B_\bullet) \\ \tau \downarrow & & \downarrow N(\hat{\tau}) \\ N(B_\bullet) \otimes N(A_\bullet) & \xrightarrow{\text{sh}} & N(B_\bullet \hat{\otimes} A_\bullet) \end{array}$$

commutes for all  $A_\bullet, B_\bullet \in \text{smod}_R$ . However, the inverse of  $N$ ,  $\Gamma$ , is *not* lax symmetric monoidal. In order to compare  $\Gamma(C_*) \hat{\otimes} \Gamma(C'_*)$  and  $\Gamma(C_* \otimes C'_*)$  one uses the Alexander-Whitney map

$$\text{aw}: N(A_\bullet \hat{\otimes} B_\bullet) \longrightarrow N(A_\bullet) \otimes N(B_\bullet)$$

and this natural map is not symmetric, *i.e.*, the diagram

$$\begin{array}{ccc} N(A_\bullet \hat{\otimes} B_\bullet) & \xrightarrow{\text{aw}} & N(A_\bullet) \otimes N(B_\bullet) \\ N(\hat{\tau}) \downarrow & & \downarrow \tau \\ N(B_\bullet \hat{\otimes} A_\bullet) & \xrightarrow{\text{aw}} & N(B_\bullet) \otimes N(A_\bullet) \end{array}$$

does *not* commute.

Schwede and Shipley proved that the Dold-Kan correspondence passes to a Quillen equivalence between the category of associative simplicial rings and the category of differential graded associative algebras that are concentrated in non-negative degrees. They consider the normalization functor and construct an adjoint on the level of monoids which then gives rise to a monoidal Quillen equivalence [SchSh03].

If one starts with a differential graded commutative algebra, then  $\Gamma$  sends this algebra to a simplicial  $E_\infty$ -algebra [R03, Theorem 4.1]. In general, the Dold-Kan correspondence gives rise to a Quillen adjunction between simplicial homotopy  $\mathcal{O}$ -algebras and differential graded homotopy  $\mathcal{O}$ -algebras for operads  $\mathcal{O}$  in  $R$ -modules [R06, Theorem 5.5.5].

#### 4. The divided power structure on homotopy groups of commutative simplicial algebras

In the following we view  $\Sigma_{ni}$  as the group of bijections of the set  $\{0, \dots, ni-1\}$ . For a permutation  $\sigma$  we use  $\varepsilon(\sigma)$  for its signum. We consider the set of shuffle permutations,

$$\text{Sh}_i(n) \subset \Sigma_{ni}.$$

This set consists of permutations  $\sigma \in \Sigma_{ni}$  such that

$$\sigma(0) < \dots < \sigma(n-1), \dots, \sigma((i-1)n) < \dots < \sigma(ni-1).$$

Let  $j$  denote the block of numbers  $(j-1)n < \dots < jn-1$  for  $1 \leq j \leq i$  and let  $[ni-1] \setminus \underline{j}$  denote the complement of  $j$  with its inherited ordering from the one of  $[ni-1]$ . We use the abbreviation  $s_{\sigma([ni] \setminus \underline{j})}$  for the composition of the degeneracy maps  $s_{\sigma(k)}$  where  $k \in [ni-1] \setminus \underline{j}$  and the order of the composition uses small indices first. For example, let  $\sigma \in \text{Sh}_3(2)$  be the permutation  $\sigma = (0, 2)(1, 4)(3, 5)$

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ & \searrow & \swarrow & \searrow & \swarrow & \searrow \\ & & & & & \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ 0 & 1 & 2 & 3 & 4 & 5. \end{array}$$

In this case,  $s_{\sigma([5] \setminus \underline{2})} = s_4 \circ s_3 \circ s_2 \circ s_1$ .

Let  $A_\bullet$  be a commutative simplicial  $R$ -algebra, *i.e.*, a commutative monoid in  $(\text{smod}_R, \hat{\otimes}, \underline{R}, \hat{\tau})$ . The homotopy groups of  $A_\bullet$ ,  $\pi_*(A_\bullet)$ , are the homology groups of the normalization of  $A_\bullet$ ,  $H_*(N(A_\bullet))$ . Starting with a cycle  $a \in N_n(A_\bullet)$  we can map  $a$  it to its  $i$ -fold tensor power

$$N_n(A_\bullet) \ni a \mapsto a^{\otimes i} \in N_n(A_\bullet)^{\otimes i}.$$

The  $i$ -fold iterated shuffle map sends  $a^{\otimes i}$  to

$$\sum_{\sigma \in \text{Sh}_i(n)} \varepsilon(\sigma) s_{\sigma([ni-1] \setminus \underline{1})}(a) \otimes \dots \otimes s_{\sigma([ni-1] \setminus \underline{n})}(a)$$

so that the outcome is an element in

$$A_{ni} \otimes \dots \otimes A_{ni} = (A_\bullet \hat{\otimes} \dots \hat{\otimes} A_\bullet)_{ni}.$$

As none of the degeneracy maps arises  $n$  times, we consider the image as an element of  $N_{ni}(A_\bullet \hat{\otimes} \dots \hat{\otimes} A_\bullet)$ . If we compose the  $i$ -fold diagonal map with the  $i$ -fold iterated shuffle map followed by the commutative multiplication in  $A_\bullet$ , we can view the composite as a map

$$P_i: N_n(A_\bullet) \longrightarrow N_{ni}(A_\bullet).$$

A tedious calculation shows that  $P_i$  is actually a chain map. On the level of homology, this composite sends a homology class to its  $i$ -fold power.

The group  $\Sigma_i$  acts on the set of shuffles  $\text{Sh}_i(n)$  by permuting the  $i$  blocks of size  $n$ . If  $\xi \in \Sigma_i$  we denote the corresponding block permutation by  $\xi^b$ . For a  $\sigma \in \text{Sh}_i(n)$  and  $\xi \in \Sigma_i$ ,  $\sigma \circ \xi^b$  is again an element of  $\text{Sh}_i(n)$ . As  $A_\bullet$  is commutative, we have that the multiplication applied to a summand  $s_{\sigma([ni-1]\underline{1})}(a) \otimes \dots \otimes s_{\sigma([ni-1]\underline{n})}(a)$  gives the same output as the multiplication applied to the summand corresponding to  $\sigma \circ \xi^b$ .

If the characteristic of the ground ring is not two, then  $i$ -fold powers for  $i \geq 2$  are trivial unless  $n$  is even. The signum of the permutation  $\sigma \circ \xi^b$  is the signum of  $\xi^b$  multiplied by  $\varepsilon(\sigma)$ . For each crossing in  $\xi$  the block permutation  $\xi^b$  has  $n^2$  crossings, so  $\xi^b$  is in the alternating group in this case. If the characteristic of  $R$  is 2, then signs do not matter.

**DEFINITION 4.1.** For a simplicial commutative  $R$ -algebra  $A_\bullet$ , the  $i$ -th divided power of  $[a] \in \pi_n(A_\bullet) = H_n(N(A_\bullet))$  is defined as the class of

$$\mu \circ \sum_{\sigma \in \text{Sh}_i(n)/\Sigma_i} \varepsilon(\sigma) s_{\sigma([ni]\underline{1})}(a) \otimes \dots \otimes s_{\sigma([ni]\underline{n})}(a)$$

where we choose a system of representing elements  $\sigma \in \text{Sh}_i(n)/\Sigma_i$ . We denote the  $i$ -th divided power of  $a \in \pi_n(A_\bullet)$  by  $\gamma_i(a)$ .

As we know that all elements  $\sigma'$  in the same coset as  $\sigma$  give rise to the same value under the map  $P_i$ , we obtain that

$$a^i = i! \gamma_i(a).$$

With the conventions  $\gamma_0(a) = 1$  and  $\gamma_1(a) = a$  we obtain the following (see for instance [F00, §2.2] for a proof).

**PROPOSITION 4.2.** *The system of divided powers in the homotopy groups of a commutative simplicial  $R$ -algebra  $A_\bullet$ ,  $(\pi_*(A_\bullet), \gamma)$  satisfies the properties from Definition 2.1.*

## 5. A large symmetric monoidal product on chain complexes

We will use the following product later in order to investigate divided power structures on chain complexes.

**DEFINITION 5.1.** We define the *large tensor product* of two chain complexes  $C_*$  and  $C'_*$  to be

$$C_* \tilde{\otimes} C'_* := N(\Gamma(C_*) \hat{\otimes} \Gamma(C'_*)).$$

Note that the large tensor product deserves its name: the degenerate elements in  $\Gamma(C_*) \hat{\otimes} \Gamma(C'_*)$  are only the ones that are images of the maps  $s_i \hat{\otimes} s_i$ , and in general  $N(\Gamma(C_*) \hat{\otimes} \Gamma(C'_*))$  is much larger than the ordinary tensor product  $C_* \otimes C'_* \cong N(\Gamma(C_*)) \otimes N(\Gamma(C'_*))$ .

As a concrete example, consider the normalized chain complex on the standard simplex  $\mathbb{Z}[\Delta(1)]$ . Let us denote a monotone map  $f: [n] \rightarrow [1]$  by an  $(n+1)$ -tuple corresponding to its image, so that for instance

$$f: [3] \rightarrow [1], f(0) = 0, f(1) = f(2) = f(3) = 1$$

is represented by  $(0, 1, 1, 1)$ .



As  $\mathbb{Z}[\Delta(1)]$  has non-degenerate simplices only in degrees zero and one corresponding to the monotone maps  $(0)$  and  $(1) \in \Delta([1], [0])$  and  $(0, 1) \in \Delta([1], [1])$ , its normalization  $C_* = N(\mathbb{Z}[\Delta(1)])$  is the chain complex

$$\mathbb{Z} \oplus \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \dots$$

and the boundary map sends the generator  $(0, 1)$  to  $(0) - (1)$ . Therefore,  $C_* \otimes C_*$  is a chain complex, that is concentrated in degrees zero, one and two with chain groups of rank four, four and one respectively. Note that

$$N(\Gamma(N(\mathbb{Z}[\Delta(1)])) \hat{\otimes} \Gamma(N(\mathbb{Z}[\Delta(1)]))) \cong N(\mathbb{Z}[\Delta(1)] \hat{\otimes} \mathbb{Z}[\Delta(1)]).$$

Thus for instance in degree one,  $C_* \hat{\otimes} C_*$  is of rank seven.

## 6. Equivalences of categories and transfer of monoidal structures

If  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{C}$  is a pair of functors that constitute an equivalence of categories and if  $(\mathcal{D}, \hat{\otimes}, 1, \hat{\tau})$  is symmetric monoidal, then we can transfer the symmetric monoidal structure on  $\mathcal{D}$  to one on  $\mathcal{C}$  in the following way.

- As for chain complexes, one defines a product  $\tilde{\otimes}$  via  $C_1 \tilde{\otimes} C_2 = G(FC_1 \hat{\otimes} FC_2)$  for objects  $C_1, C_2$  of  $\mathcal{C}$ .
- As an equivalence

$$\tilde{\tau}_{C_1, C_2}: C_1 \tilde{\otimes} C_2 = G(FC_1 \hat{\otimes} FC_2) \rightarrow G(FC_2 \hat{\otimes} FC_1) = C_2 \tilde{\otimes} C_1$$

we take  $G(\hat{\tau}_{FC_1, FC_2})$ .

- The unit for the symmetric monoidal structure is  $G(1)$ .

For later reference, we spell out some of the structural isomorphisms. Recall that any equivalence of categories gives rise to an adjoint equivalence [ML95, IV.4]; in particular the unit and counit of the adjunction are isomorphisms. We want to denote the natural isomorphism from  $GFC$  to  $C$  for  $C$  an object of  $\mathcal{C}$  by  $\varphi_C$  and the one from  $FGD$  to  $D$  by  $\psi_D$  for all objects  $D$  in  $\mathcal{D}$ . Then the identities

$$(6.1) \quad F(\varphi_C) = \psi_{FC} \text{ and } G(\psi_D) = \varphi_{GD}$$

hold for all  $C$  and  $D$ .

For the left unit we have to identify  $C$  with  $G(1) \tilde{\otimes} C$  and to this end we use the morphism

$$\tilde{\ell}: C \xrightarrow{\varphi^{-1}} G(F(C)) \xrightarrow{G(\hat{\ell})} G(1 \hat{\otimes} F(C)) \xrightarrow{G(\psi^{-1} \hat{\otimes} \text{id})} G(F(G(1)) \hat{\otimes} F(C)) = G(1) \tilde{\otimes} C$$

where  $\hat{\ell}$  is the left unit isomorphism for  $\hat{\otimes}$ . The right unit is defined similarly.

The associativity isomorphism  $\tilde{\alpha}$  is given in terms of the one for  $\hat{\otimes}$ ,  $\hat{\alpha}$  as

$$\tilde{\alpha} := G(\text{id} \hat{\otimes} \psi)^{-1} \circ G(\hat{\alpha}) \circ G(\psi \hat{\otimes} \text{id}):$$

$$\begin{array}{ccc} G(FG(F(C) \hat{\otimes} F(C)) \hat{\otimes} F(C)) & \xrightarrow{\tilde{\alpha}} & G(F(C) \hat{\otimes} FG(F(C) \hat{\otimes} F(C))) \\ \downarrow G(\psi \hat{\otimes} \text{id}) & & \downarrow G(\text{id} \hat{\otimes} \psi) \\ G((F(C) \hat{\otimes} F(C)) \hat{\otimes} F(C)) & \xrightarrow{G(\hat{\alpha})} & G(F(C) \hat{\otimes} (F(C) \hat{\otimes} F(C))) \end{array}$$

Then it is a tedious, but straightforward task to show the following result. A proof in the non-symmetric setting can be found in [ $\mathbf{Q}\infty$ , Theorem 3].

PROPOSITION 6.1. *The category  $(\mathcal{C}, \tilde{\otimes}, G(1), \tilde{\tau})$  is a symmetric monoidal category.*

If  $\mathcal{C}$  already has a symmetric monoidal structure, then we can compare the old one to the new one as follows.

PROPOSITION 6.2. *If  $(\mathcal{C}, \otimes, G(1), \tau)$  is a symmetric monoidal structure and if  $G: (\mathcal{D}, \hat{\otimes}, 1, \hat{\tau}) \rightarrow (\mathcal{C}, \otimes, G(1), \tau)$  is lax symmetric monoidal, then the identity functor*

$$\text{id}: (\mathcal{C}, \tilde{\otimes}, G(1), \tilde{\tau}) \longrightarrow (\mathcal{C}, \otimes, G(1), \tau)$$

*is lax symmetric monoidal.*

PROOF. We have to construct maps  $\lambda_{C_1, C_2}: C_1 \otimes C_2 \rightarrow C_1 \tilde{\otimes} C_2$  that are natural in  $C_1$  and  $C_2$  and that render the diagrams

$$\begin{array}{ccc} C_1 \otimes C_2 & \xrightarrow{\lambda_{C_1, C_2}} & C_1 \tilde{\otimes} C_2 \\ \tau \downarrow & & \downarrow \tilde{\tau}_{C_1, C_2} \\ C_2 \otimes C_1 & \xrightarrow{\lambda_{C_2, C_1}} & C_2 \tilde{\otimes} C_1 \end{array}$$

commutative for all  $C_1, C_2 \in \mathcal{C}$ . Let  $\Upsilon$  be the transformation that turns  $G$  into a lax symmetric monoidal functor. We define

$$\lambda_{C_1, C_2}: C_1 \otimes C_2 \xrightarrow{\varphi_{C_1}^{-1} \otimes \varphi_{C_2}^{-1}} GF(C_1) \otimes GF(C_2) \xrightarrow{\Upsilon_{F(C_1), F(C_2)}} G(F(C_1) \hat{\otimes} F(C_2))$$

$$\parallel$$

$$C_1 \tilde{\otimes} C_2.$$

□

COROLLARY 6.3. *Every commutative monoid in  $(\mathcal{C}, \tilde{\otimes}, G(1), \tilde{\tau})$  is a commutative monoid in  $(\mathcal{C}, \otimes, G(1), \tau)$ .*

The functor  $F$  compares commutative monoids in the categories  $\mathcal{C}$  and  $\mathcal{D}$  as follows.

THEOREM 6.4. *An object  $F(C)$  is a commutative monoid in  $(\mathcal{D}, \hat{\otimes}, 1, \hat{\tau})$  if and only if  $C$  is a commutative monoid in  $(\mathcal{C}, \tilde{\otimes}, G(1), \tilde{\tau})$ . Moreover, the assignment  $C \mapsto F(C)$  is a functor from the category of commutative monoids in  $(\mathcal{C}, \tilde{\otimes}, G(1), \tilde{\tau})$  to the category of commutative monoids in  $(\mathcal{D}, \hat{\otimes}, 1, \hat{\tau})$ .*

PROOF. If we assume that  $C$  is a commutative monoid in  $(\mathcal{C}, \tilde{\otimes}, G(1), \tilde{\tau})$ , then  $C$  has an associative multiplication

$$\tilde{\mu}: C \tilde{\otimes} C = G(F(C) \hat{\otimes} F(C)) \longrightarrow C$$

that satisfies  $\tilde{\mu} \circ \tilde{\tau} = \tilde{\mu}$  and there is a unit map  $j: G(1) \rightarrow C$ . We consider the composition  $\varphi^{-1} \circ \tilde{\mu}: G(F(C) \hat{\otimes} F(C)) \rightarrow GF(C)$ . As  $G$  is an equivalence of categories, it is a full functor, *i.e.*, the morphism  $\varphi^{-1} \circ \tilde{\mu}$  is of the form  $G(\hat{\mu})$  for some morphism  $\hat{\mu}: F(C) \hat{\otimes} F(C) \rightarrow F(C)$  in  $\mathcal{D}$ . We will show that  $\hat{\mu}$  turns  $F(C)$  into a commutative monoid. We define the unit map as  $i = F(j) \circ \psi^{-1}: 1 \rightarrow F(C)$ .

As  $\tilde{\tau} = G(\hat{\tau})$ , the commutativity of  $\hat{\mu}$  follows from the one of  $\tilde{\mu}$  and the fact that the functor  $G$  is faithful.

In order to check the unit property of  $i$  we have to show that the following diagram commutes:

$$(6.2) \quad \begin{array}{ccccc} F(C) & \xrightarrow{\hat{\ell}} & 1 \hat{\otimes} F(C) & \xrightarrow{\psi^{-1} \hat{\otimes} \text{id}} & F(G(1)) \hat{\otimes} F(C) \\ \parallel & & & & \downarrow F(j) \hat{\otimes} \text{id} \\ F(C) & \xleftarrow{\hat{\mu}} & & & F(C) \hat{\otimes} F(C). \end{array}$$

As  $j$  is a unit for the multiplication  $\tilde{\mu}$  we know that

$$\tilde{\mu} \circ (j \tilde{\otimes} \text{id}) \circ G(\psi^{-1} \hat{\otimes} \text{id}) \circ G(\hat{\ell}) \circ \varphi_C^{-1} = \text{id}_C.$$

Applying the faithful functor  $F$  to this identity and using the definition of  $\hat{\mu}$ , we get that

$$FG(\hat{\mu}) \circ F(j \tilde{\otimes} \text{id}) \circ FG(\psi^{-1} \hat{\otimes} \text{id}) \circ FG(\hat{\ell}) = \text{id}_{FGF(C)}.$$

By the very definition,  $F(j \tilde{\otimes} \text{id})$  is  $F(G(F(j) \hat{\otimes} \text{id}))$  and thus via the faithfulness of  $FG$  we can conclude that diagram (6.2) commutes. The analogous statement for the right unit can be shown similarly and hence  $i$  is a unit.

For the associativity of the multiplication  $\hat{\mu}$  we have to show that the inner pentagon in the following diagram commutes.

$$\begin{array}{ccc} G(FG(F(C) \hat{\otimes} F(C)) \hat{\otimes} F(C)) & \xrightarrow{\tilde{\alpha}} & G(F(C) \hat{\otimes} FG(F(C) \hat{\otimes} F(C))) \\ \downarrow G(\psi \hat{\otimes} \text{id}) & & \downarrow G(\text{id} \hat{\otimes} \psi) \\ G((F(C) \hat{\otimes} F(C)) \hat{\otimes} F(C)) & \xrightarrow{G(\hat{\alpha})} & G(F(C) \hat{\otimes} (F(C) \hat{\otimes} F(C))) \\ \downarrow G(\hat{\mu} \hat{\otimes} \text{id}) & & \downarrow G(\text{id} \hat{\otimes} \hat{\mu}) \\ G(F(C) \hat{\otimes} F(C)) & & G(F(C) \hat{\otimes} F(C)) \\ \downarrow G(\hat{\mu}) & & \downarrow G(\hat{\mu}) \\ & GF(C) & \\ \downarrow \varphi & & \\ & C & \end{array}$$

$\tilde{\mu} \hat{\otimes} \text{id}$  (left wing),  $\text{id} \hat{\otimes} \tilde{\mu}$  (right wing),  $\tilde{\mu}$  (bottom wings)

The outer diagram commutes because  $\tilde{\mu}$  is associative and the upper square commutes because  $\tilde{\alpha}$  is given in terms of  $\hat{\alpha}$  in this way. The only thing that remains to be proven is that the outer wings commute. We prove the claim for the left wing.

Using the definitions of the maps involved, we have to show that

$$\begin{array}{ccc}
 G(FG(F(C) \hat{\otimes} F(C)) \hat{\otimes} F(C)) & \xrightarrow{G(\psi_{F(C) \hat{\otimes} F(C)} \hat{\otimes} \text{id})} & G((F(C) \hat{\otimes} F(C)) \hat{\otimes} F(C)) \\
 \searrow^{\tilde{\mu} \hat{\otimes} \text{id} = G(F(\tilde{\mu}) \hat{\otimes} \text{id})} & & \swarrow_{G(\hat{\mu} \hat{\otimes} \text{id})} \\
 & & G(F(C) \hat{\otimes} F(C))
 \end{array}$$

commutes. We know that

$$FG(\hat{\mu}) = F(\varphi_C^{-1}) \circ F(\tilde{\mu})$$

and therefore

$$\psi_{F(C)} \circ FG(\hat{\mu}) = F(\tilde{\mu}).$$

The naturality of  $\psi$  implies that

$$\psi_{F(C)} \circ FG(\hat{\mu}) = \hat{\mu} \circ (\psi_{F(C) \hat{\otimes} F(C)})$$

and the claim follows.

If  $F(C)$  is a commutative monoid with respect to  $\hat{\otimes}$  with multiplication  $\hat{\mu}$  and unit  $i: 1 \rightarrow F(C)$ , then we claim that  $\tilde{\mu} := \varphi_C \circ G(\mu)$  and  $j := \varphi_C \circ G(i)$  give  $C$  the structure of a commutative monoid with respect to  $\tilde{\otimes}$ .

The fact that  $\tilde{\mu}$  is commutative follows directly because  $\tilde{\tau} = G(\hat{\tau})$ . The proofs of the unit axiom and of associativity use the same diagrams as above with the arguments reversed.

It remains to show that a morphism  $f: C_1 \rightarrow C_2$  of commutative monoids with respect to  $\tilde{\otimes}$  gives rise to a morphism  $F(f): F(C_1) \rightarrow F(C_2)$  of commutative monoids with respect to  $\hat{\otimes}$ . It suffices to prove that the outer diagram in

$$\begin{array}{ccc}
 G(F(C_1) \hat{\otimes} F(C_1)) & \xrightarrow{G(F(f) \hat{\otimes} F(f))} & G(F(C_2) \hat{\otimes} F(C_2)) \\
 \downarrow \tilde{\mu} & & \downarrow \tilde{\mu} \\
 C_1 & \xrightarrow{f} & C_2 \\
 \uparrow \varphi_{C_1} & & \uparrow \varphi_{C_2} \\
 GF(C_1) & \xrightarrow{GF(f)} & GF(C_2)
 \end{array}$$

commutes. The upper square commutes by assumption, the lower square commutes because  $\varphi$  is natural and the wings commute by the very definition of  $\tilde{\mu}$  in terms of  $\hat{\mu}$ . □

## 7. Divided power structures and commutative monoids

If we apply the above results to the Dold-Kan correspondence with  $\Gamma = F$ ,  $N = G$ ,  $\mathcal{C} = \text{Ch}_{\geq 0}^R$  and  $\mathcal{D} = \text{smod}_R$  and the large tensor product of chain complexes, then we obtain the following statements that we collect in one theorem.

**THEOREM 7.1.**

- (a) *The category of chain complexes with the large tensor product is a symmetric monoidal category with  $N(\underline{R})$  being the unit of the monoidal structure and*

$$\tilde{\tau} = N(\hat{\tau}_{\Gamma(C_*), \Gamma(C'_*)}): N(\Gamma(C_*) \hat{\otimes} \Gamma(C'_*)) \longrightarrow N(\Gamma(C'_*) \hat{\otimes} \Gamma(C_*))$$

as the twist.

(b) *The identity functor*

$$\text{id}: (\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau}) \longrightarrow (\text{Ch}_{\geq 0}^R, \otimes, (R, 0), \tau)$$

is lax symmetric monoidal.

(c) *Every commutative monoid in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$  is a differential graded commutative  $R$ -algebra.*

(d) *A simplicial  $R$ -module  $\Gamma(C_*)$  is a simplicial commutative  $R$ -algebra if and only if the chain complex  $C_*$  is a commutative monoid in the symmetric monoidal category  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$ . The assignment  $C_* \mapsto \Gamma(C_*)$  is a functor from the category of commutative monoids in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$  to the category of simplicial commutative  $R$ -algebras.*

**COROLLARY 7.2.** *Every commutative monoid  $C_*$  in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$  has a divided power structure on its homology.*

The converse of statement (c) of Theorem 7.1 is *not* true: not every differential graded commutative algebra possesses a divided power structure on its homology, so these algebras cannot be commutative monoids in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$ . For instance the polynomial ring  $\mathbb{Z}[x]$  over the integers with  $x$  of degree two and trivial differential provides an example.

**DEFINITION 7.3.** We denote by  $\text{Com}_{\tilde{\otimes}}^R$  the category whose objects are commutative monoids in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$  and whose morphisms are multiplicative chain maps, *i.e.*, if  $C_*, D_*$  are objects of  $\text{Com}_{\tilde{\otimes}}^R$  with unit maps  $j_{C_*}: N(\underline{R}) \rightarrow C_*$  and  $j_{D_*}: N(\underline{R}) \rightarrow D_*$ , then a morphism is a chain map  $f: C_* \rightarrow D_*$  with  $f \circ j_{C_*} = j_{D_*}$  and such that the diagram

$$\begin{array}{ccc} C_* \tilde{\otimes} C_* & \xrightarrow{f \tilde{\otimes} f} & D_* \tilde{\otimes} D_* \\ \tilde{\mu}_{C_*} \downarrow & & \downarrow \tilde{\mu}_{D_*} \\ C_* & \xrightarrow{f} & D_* \end{array}$$

commutes.

If we start with a  $C_* \in \text{Com}_{\tilde{\otimes}}^R$ , then in particular,  $C_*$  is a differential graded algebra, *i.e.*, the differential  $d$  on the underlying chain complex  $C_*$  is compatible with the product structure: it satisfies the Leibniz rule

$$d(ab) = d(a)b + (-1)^{|a|}ad(b), \quad \text{for all } a, b \in C_*.$$

If there are divided power structures on the underlying graded commutative algebra  $C_*$ , then we want these to be compatible with the differential.

**DEFINITION 7.4.** 1) A commutative differential algebra  $C_*$  with divided power operations is called a *divided power chain algebra*, if the differential  $d$  of  $C_*$  satisfies

- (a)  $d(\gamma_i(c)) = d(c) \cdot \gamma_{i-1}(c)$  for all  $c \in C_*$
- (b) If  $c$  is a boundary, then  $\gamma_i(c)$  is a boundary for all  $i \geq 1$ .

2) A morphism of commutative differential algebras  $f: C_* \rightarrow D_*$  is a *morphism of divided power chain algebras*, if  $f$  satisfies  $f(\gamma_i(c)) = \gamma_i(f(c))$  for all  $c \in C_{>0}$ ,  $i \geq 0$ .

The first condition in 1) ensures that divided powers respect cycles and together with the second condition this guarantees that the homology of  $C_*$  inherits a divided power structure from  $C_*$ .

A reformulation of the criterium for a divided power chain algebra is used in [AH86, definition 1.3]: they demand that every element of positive degree is in the image of a morphism of differential graded commutative algebras with divided power structure  $f: D_* \rightarrow C_*$  such that  $D_*$  satisfies condition (a) of Definition 7.4 and has trivial homology in positive degrees.

If  $A_\bullet$  is for instance a commutative simplicial  $\mathbb{F}_2$ -algebra, then the condition that  $\gamma_2$  sends boundaries to boundaries is automatically satisfied: Goerss, following Dwyer [D80], shows that the higher divided power operation  $\delta_{n-1}: H_n(NA_\bullet) = \pi_n(A_\bullet) \rightarrow \pi_{2n-1}(A_\bullet)$  is given on chain level by a map  $\Theta_{n-1}$  (see [G90, p. 37]) and that  $\Theta_{n-1}$  commutes with the boundary [G90, (3.3.1)]. If  $x$  has degree  $n$ , then  $\Theta_{n-1}(dx)$  is the highest divided power operation on  $dx$  and therefore equal to  $\gamma_2(dx)$  and we obtain  $d\Theta_{n-1}(x) = \Theta_{n-1}(dx) = \gamma_2(dx)$ .

For a commutative monoid with respect to  $\tilde{\otimes}$ ,  $C_*$ , the  $i$ -th power of an element  $c \in N\Gamma(C_*)$  is given via the following composition

$$\begin{array}{ccc}
c \in & N\Gamma(C_*)_n & \xrightarrow{c \mapsto c^{\otimes i}} N\Gamma(C_*)_n^{\otimes i} \subset (N\Gamma(C_*)^{\otimes i})_{ni} \\
\downarrow & & \downarrow \text{sh} \\
& & N(\Gamma(C_*) \hat{\otimes} \dots \hat{\otimes} \Gamma(C_*))_{ni} = C_* \tilde{\otimes} \dots \tilde{\otimes} C_* \\
& & \downarrow \tilde{\mu} \\
c^i \in & N\Gamma(C_*)_{ni} & \xrightarrow{\varphi} C_*
\end{array}$$

$N(\hat{\mu})$  (diagonal arrow from  $N\Gamma(C_*)_{ni}$  to  $N(\Gamma(C_*) \hat{\otimes} \dots \hat{\otimes} \Gamma(C_*))_{ni}$ )

We define a divided power structure on  $N\Gamma(C_*)$  by using a variant of the shuffle map as in Definition 4.1 sending  $c^{\otimes i}$  to

$$\sum_{\sigma \in \text{Sh}_i(n)/\Sigma_i} \varepsilon(\sigma) s_{\sigma([ni] \setminus \underline{1})}(c) \otimes \dots \otimes s_{\sigma([ni] \setminus \underline{n})}(c)$$

and applying  $N(\mu)$ .

**THEOREM 7.5.** *The composite  $N\Gamma$  is a functor from the category  $\text{Com}_{\tilde{\otimes}}$  to the category of divided power chain algebras.*

**PROOF.** Let  $C_*$  be a commutative monoid in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$ . Let us first prove that

$$d(\gamma_i(c)) = \gamma_{i-1}(c) \cdot d(c)$$

for all  $c \in C_*$  in positive degrees. If we apply the boundary  $d = d_0$  to

$$(7.1) \quad \gamma_i(c) = N(\mu) \left( \sum_{\sigma \in \text{Sh}_i(n)/\Sigma_i} \varepsilon(\sigma) s_{\sigma([ni-1] \setminus \underline{1})}(c) \otimes \dots \otimes s_{\sigma([ni-1] \setminus \underline{n})}(c) \right)$$

then we can use that  $d_0$  is a morphism in the simplicial category to obtain

$$d_0 \circ N(\mu) = N(\mu) \circ (d_0 \otimes \dots \otimes d_0).$$

Only one of the sets  $\sigma([ni-1] \setminus \underline{j})$  does not contain zero. Therefore the simplicial identities  $d_0 \circ s_i = s_{i-1} \circ d_0$  for  $i > 0$  and  $d_0 \circ s_0 = \text{id}$  ensure, that in the sum (7.1) there are  $i-1$  tensor factors containing just degeneracies applied to  $c$  and only one term containing degeneracies applied to  $d_0(c)$ .

A shuffle permutation in  $\text{Sh}_{i-1}(n)$  tensorized with the identity map followed by a shuffle in  $\text{Sh}(n(i-1), n)$  gives a shuffle in  $\text{Sh}_i(n)$  and every shuffle in  $\text{Sh}_i(n)$  is decomposable in the above way. There are

$$\prod_{j=2}^n \frac{\binom{n}{j}}{j!}$$

elements in  $\text{Sh}_i(n)/\Sigma_i$  and

$$\prod_{j=2}^{n-1} \frac{\binom{n}{j}}{(j-1)!} \cdot \binom{ni-1}{n-1}$$

elements in the product of  $\text{Sh}_{i-1}(n)/\Sigma_{i-1}$  and  $\text{Sh}(n(i-1), n)$ . As these numbers are equal, we obtain that the two sets are in bijection and  $d_0(\gamma_i(c))$  can be expressed as the product of  $\gamma_{i-1}(c)$  and  $d_0(c)$ .

The boundary criterium for the divided power structure can be seen as follows: if  $c$  is of the form  $d_0(b)$  for some  $b \in N\Gamma(C_*)_{n+1}$ , then  $\gamma_i(c)$  is

$$N(\mu) \left( \sum_{\sigma \in \text{Sh}_i(n)/\Sigma_i} \varepsilon(\sigma) s_{\sigma([ni-1] \setminus \underline{1})}(d_0(b)) \otimes \dots \otimes s_{\sigma([ni-1] \setminus \underline{n})}(d_0(b)) \right).$$

The simplicial identity  $s_{j-1}d_0 = d_0s_j$  for all  $j > 0$  allows us to move the  $d_0$ -terms in front by increasing the indices of the degeneracy maps. Therefore  $\gamma_i(d_0(b))$  is equal to an expression of the form  $N(\mu)N(d_0 \otimes \dots \otimes d_0)(x)$  for some suitable  $x$ . As  $N(\mu)N(d_0 \otimes \dots \otimes d_0)$  is equal to  $d_0 \circ N(\mu)$  we obtain the desired result.

For  $f: C_* \rightarrow D_*$  a morphism of commutative monoids in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$  we have to show that  $N\Gamma(f)$  is a morphism of divided power chain algebras, *i.e.*, that it is a multiplicative chain map that preserves units and divided powers. As  $f$  is a chain map,  $\Gamma(f)$  is a map of simplicial  $R$ -modules and  $N\Gamma(f)$  is a chain map. If  $j_{C_*}$  and  $j_{D_*}$  are the units for  $C_*$  and  $D_*$ , we have  $f \circ j_{C_*} = j_{D_*}$  and this implies

$$N\Gamma(f) \circ N\Gamma(j_{C_*}) \circ \varphi_{N(\underline{R})}^{-1} = N\Gamma(j_{D_*}) \circ \varphi_{N(\underline{R})}^{-1}$$

and thus the unit condition holds.

In order to establish that  $N\Gamma(f)$  is multiplicative we have to show that the back face in the diagram

$$\begin{array}{ccccc} N\Gamma(C_*) \otimes N\Gamma(C_*) & \xrightarrow{\quad \mu \quad} & & \xrightarrow{\quad} & N\Gamma(C_*) \\ & \searrow \text{sh} & & \nearrow \varphi^{-1} \circ \tilde{\mu} = N(\tilde{\mu}) & \downarrow N\Gamma(f) \\ & & N(\Gamma(C_*) \hat{\otimes} \Gamma(C_*)) & & \\ & & \downarrow N(\Gamma(f) \hat{\otimes} \Gamma(f)) & & \\ N\Gamma(D_*) \otimes N\Gamma(D_*) & \xrightarrow{\quad \mu \quad} & & \xrightarrow{\quad} & N\Gamma(D_*) \\ & \searrow \text{sh} & & \nearrow \varphi^{-1} \circ \tilde{\mu} = N(\tilde{\mu}) & \\ & & N(\Gamma(D_*) \hat{\otimes} \Gamma(D_*)) & & \end{array}$$

commutes. The top and bottom triangle commute by definition, the left front square commutes because the shuffle map is natural and the right front square commutes because we know from Theorem 6.4 that  $\Gamma(f)$  is multiplicative.

The fact that  $f$  preserves divided powers can be seen directly: using naturality of  $\varphi$  and the multiplicativity of  $f$  with respect to  $\tilde{\otimes}$ , the only thing we have to verify is the compatibility of  $f$  with the variant of the shuffle map. But this map is a sum of tensors of degeneracy maps and as  $\Gamma(f)$  respects the simplicial structure, the claim follows.  $\square$

**REMARK 7.6.** We can transfer the model structure on simplicial commutative  $R$ -algebras as in [Qui67, II, Theorem 4] to the category  $\text{Com}_{\tilde{\otimes}}$  of commutative monoids in  $(\text{Ch}_{\geq 0}^R, \tilde{\otimes}, N(\underline{R}), \tilde{\tau})$  by declaring that a map  $f: C_* \rightarrow D_*$  is a weak equivalence, fibration resp. cofibration in  $\text{Com}_{\tilde{\otimes}}$  if and only if  $\Gamma(f)$  is a weak equivalence, fibration resp. cofibration in the model structure on commutative simplicial  $R$ -algebras.

**REMARK 7.7.** It is not straightforward to check whether a chain complex is a commutative monoid with respect to  $\tilde{\otimes}$  and it would be desirable to have a direct characterization of such commutative monoids in terms of divided power structures and their higher versions. At the moment, we are not able to provide such a description; in particular we cannot characterize the subcategory of differential graded commutative algebras that corresponds to the category  $\text{Com}_{\tilde{\otimes}}$  in such a manner.

## 8. Bar constructions and Hochschild complex

One well-known example of a divided power chain algebra is the normalization of a bar construction of a commutative  $R$ -algebra (see for instance [C54, Exposé no 7] and [BK94, §3]).

Let  $A$  be a commutative  $R$ -algebra. The bar construction of  $A$  is the simplicial commutative  $R$ -algebra,  $\mathcal{B}_\bullet(A)$ , with

$$\mathcal{B}_n(A) = A^{\otimes(n+2)}.$$

The simplicial structure maps are given by inserting the multiplicative unit  $1 \in R$  for degeneracies and by multiplication for face maps. As  $A$  is commutative, we can multiply componentwise

$$\begin{aligned} \mathcal{B}_n(A) \otimes \mathcal{B}_n(A) &\rightarrow \mathcal{B}_n(A), \\ (a_0 \otimes \dots \otimes a_{n+1}) \otimes (a'_0 \otimes \dots \otimes a'_{n+1}) &\mapsto a_0 a'_0 \otimes \dots \otimes a_{n+1} a'_{n+1}. \end{aligned}$$

From Theorems 6.4 and 7.5 it follows that the normalization  $B_*(A) := N(\mathcal{B}_\bullet(A))$  is a divided power chain algebra.

The Hochschild complex of the commutative  $R$ -algebra  $A$  is defined as

$$C_*(A) = A \otimes_{A \otimes A} B_*(A)$$

where the  $A$ -bimodule structure on  $B_n(A)$  is induced by

$$(a \otimes \tilde{a})(a_0 \otimes \dots \otimes a_{n+1}) := a a_0 \otimes \dots \otimes a_{n+1} \tilde{a}.$$

If  $A$  is flat over  $R$ , the homology of this complex is  $\text{Tor}_*^{A \otimes A}(A, A)$ . As  $B_*(A)$  is acyclic and surjects onto  $C_*(A)$ , the Hochschild complex inherits a structure of a divided power chain algebra from  $B_*(A)$ . Cartan showed [C54, Exposé 7],



that the bar construction of strict differential graded commutative algebras has a divided power structure. Condition (b) of Definition 7.4 is in general satisfied on the iterated bar construction  $\mathcal{B}_*^{(n)}$ ,  $n \geq 2$  [C54, p. 7, Exposé 7], so that each  $\mathcal{B}_*^{(n)}$ ,  $n \geq 2$  is a divided power chain algebra.

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