Detecting and describing ramification for structured ring spectra

Birgit Richter, eCHT research seminar, April 22 2021

Joint work with Eva Höning

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Then $\mathbb{Z}[i] \supset (2) = (1+i)^2$ and 2 is the characteristic of the residue field $\mathbb{F}_2$, so (2) is wildy ramified.

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The fixed point condition is always satisfied in this situation, so the condition for being unramified is

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- Examples, examples, examples.

Definition [Rognes 2008]: A map $A \to B$ of commutative ring spectra is a G-Galois extension for a finite group G, if certain cofibrancy conditions are satisfied, if G acts on B from the left through commutative A-algebra maps and if the following two conditions are satisfied:

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Here, h is right adjoint to the composite map

$$B \wedge_A B \wedge G_+ \longrightarrow B \wedge_A B \longrightarrow B\,,$$

induced by the G-action and the multiplication on B.

Example 1 If A is the Eilenberg-MacLane spectrum $H\mathcal{O}_K$ and $B = H\mathcal{O}_L$ for a G-Galois extension $K \subset L$, then $H\mathcal{O}_K \to H\mathcal{O}_L$ is a G-Galois extension of commutative ring spectra iff $\mathcal{O}_K \subset \mathcal{O}_L$ is a G-Galois extension of commutative rings.

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Rognes [2008]: This turns $KO \to KU$ into a C_2-Galois extension.

Note that on homotopy groups we get

$$\pi_*(KO) = \mathbb{Z}[\eta, y, \omega^{\pm 1}]/(2\eta, \eta^3, \eta y, y^2 - 4\omega) \xrightarrow{\pi_*(c)} \mathbb{Z}[u^{\pm 1}] = \pi_*(KU)$$

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- $TMF[1/n] \to TMF(n)$ is $GL_2(\mathbb{Z}/n\mathbb{Z})$-Galois [MM-2015].

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So, in particular, connective covers of Galois extensions are rarely Galois extensions – these will be our main examples.

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Hurewicz theorem for topological André-Quillen homology

[Basterra 1999]: Let $\varphi\colon A \to B$ be an n-equivalence, where A and B are connective and $n \geq 1$. Then $TAQ(B|A)$ is n-connected and there is a map of A-modules $f\colon C\varphi \to TAQ(B|A)$ for which $f_*\colon \pi_{n+1} C\varphi \cong \pi_{n+1} TAQ(B|A)$.

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We *do* have ramification, but we don't see yet, whether it's tame or wild.

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If $B = H\mathcal{O}_L$, then the spectral sequence collapses and $\hat{H}^*(G; \mathcal{O}_L) \cong \pi_{-*}(H\mathcal{O}_L)^{tG}$.

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Beware! If $A \to B$ is a map between connective commutative ring spectra, then often $B^{hG} \not\simeq A$, but $A \to \tau_{\geq 0} B^{hG}$ might be an equivalence (e.g. $ko \simeq \tau_{\geq 0} ku^{hC_2}$).

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The proof of the second claim uses Stojanoska's calculation of $Tmf(2)^{t\Sigma_3}_{(3)} \simeq *$ via the Tate spectral sequence

$$E^2_{n.m} = \hat{H}^{-n}(\Sigma_3, \pi_m(Tmf(2)_{(3)})) \Longrightarrow \pi_{n+m}(Tmf(2)^{t\Sigma_3}_{(3)}).$$

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  but $tmf_1(3)_{(2)}^{tC_2} \not\simeq *$. But here, we don't know whether $tmf_1(3)_{(2)}$ is faithful as a $tmf_0(3)_{(2)}$-module.
  Lennart Meier: It is *not* dualizable.
- $TMF[1/n] \to TMF(n)$ is a $GL_2(\mathbb{Z}/n\mathbb{Z})$-Galois extension and the Tate spectrum $Tmf(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})}$ is contractible [Mathew-Meier, Stojanoska], but $tmf_{(3)} \to tmf(2)_{(3)}$ is wildly ramified.

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- or for $n = 2 \cdot 3 \cdot 7$ but not for $n = 2 \cdot 3 \cdot 11$.

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Thank you!