Models for spaces in $\mathcal{I}$-chain complexes

Birgit Richter

Purdue Topology Seminar, 27th of April 2022

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Thus, if you don't want to restrict to rational homotopy theory, then you need the full information of the E_∞-structure on the cochains!

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What about the other models? So what about differential graded cocommutative coalgebras and Lie-algebras?

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4. Behaviour of homotopy colimits

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The Day convolution product gives $\mathsf{Ch}^{\mathcal{I}}$ a symmetric monoidal structure. Explicitly, for two $\mathcal{I}$-chain complexes X_*, Y_*

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Definition: Commutative $\mathcal{I}$-chain algebras are commutative monoids in $\mathsf{Ch}^{\mathcal{I}}$.

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For any $\mathcal{I}$-chain complex X_*, the free commutative $\mathcal{I}$-chain algebra on X_* is

$$\mathrm{S}^{\mathcal{I}}(X_*) = \bigoplus_{n \geq 0} X_*^{\boxtimes n}/\Sigma_n.$$

The homotopy colimit, $\mathrm{hocolim}_{\mathcal{I}} X_*$, of an $\mathcal{I}$-chain complex X_* is the total complex associated to the bicomplex whose bidegree (p, q)-part is

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If C_* is a cocommutative comonoid in $\mathrm{Ch}^{\mathcal{I}}$, what can we say about $\mathrm{hocolim}_{\mathcal{I}} C_*$?
In general: not much, because $\mathrm{hocolim}_I$ is lax monoidal, but not lax *symmetric* (co)monoidal!

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Theorem There are reduced $X_* \in \mathrm{Ch}^{\mathcal{I}}$ (i.e., $X_*(0) = 0$) such that

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is *not* an isomorphism.

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If we consider $(\bar{F} \boxtimes \bar{F})(3)$ then this is the colimit over the category $\mathcal{I} \sqcup \mathcal{I} \to 3$ of $\bar{F}(\mathrm{p}) \otimes \bar{F}(\mathrm{q})$.

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So this element is invariant under the Σ_2-action, but it is not in the image of the norm map, unless 2 is invertible in k.

Definition An $\mathcal{I}$-chain complex X_* is *Tate trivial*, if the norm map

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For any chain complex C_*, for every m and for every $p \geq 1$ the norm $N_n = \sum_{\sigma \in \Sigma_n} \sigma \in \mathbb{Z}[\Sigma_n]$ induces an isomorphism of chain complexes

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This follows from the fact that $(F_p^{\mathcal{I}}(C_*))^{\boxtimes n} \cong F_{pn}^{\mathcal{I}}(C_*^{\otimes n})$ and that the Σ_n-action is free on $\mathcal{I}(\mathrm{pn}, \mathrm{m})$ as long as $p \geq 1$.

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Note that this implies that the free commutative monoid on $F_p^{\mathcal{I}}(C_*)$ is isomorphic to the free divided power algebra and the cofree cocommutative coalgebra generated on $F_p^{\mathcal{I}}(C_*)$.

For an $\mathcal{I}$-chain complex X_* we can consider the graded $\mathcal{I}$-chain module H_*X_* with

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Proposition Even if we work over a field, the Künneth map is in general *not* an isomorphism.
Again, we provide a concrete counterexample.

Consider a chain complex C_* over a field with a chosen zero cycle c_0 and let $\mathrm{Sym}^{\mathcal{I}}(C_*) \in \mathrm{Ch}^{\mathcal{I}}$ be defined as

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This results in $D^1 \oplus_{S^0} D^1$ which has nontrivial H_1.

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A typical example is $F_n^{\mathcal{I}}(C_*) = i_! F_n^{\Sigma}(C_*)$ with

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In this case:
Lemma: For all $n \geq 0$ and all chain complexes C_*:

$$\mathrm{hocolim}_{\mathcal{I}} F_n^{\mathcal{I}}(C_*) \simeq C_*.$$

Can we describe $\mathrm{hocolim}_{\mathcal{I}} i_! Z_*$ in general?

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The identity $1 \in C(1)$ is then defined to be id_1. □

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Note that by definition we get $\mathrm{hocolim}_{\mathcal{I}} i_!(C_*)_{p,q} = \bigoplus_{[f_q|\ldots|f_1] \in N\mathcal{I}_q} i_!(C_p)(sf_1) \cong \bigoplus_{[f_q|\ldots|f_1] \in N\mathcal{I}_q} k\{\mathcal{I}(i(-), sf_1)\} \otimes_{\Sigma} C_p$.

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This is isomorphic to

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$$\begin{aligned} &\bigoplus_{m \geq 0} O(m) \otimes_{\Sigma_m} (X_* \odot X_*)(\mathrm{m}) \\ = &\bigoplus_{m \geq 0} O(m) \otimes_{\Sigma_m} (\bigoplus_{p+q=m} k[\Sigma_m] \otimes_{k[\Sigma_p \times \Sigma_q]} X_*(\mathrm{p}) \otimes X_*(\mathrm{q})) \\ \cong &\bigoplus_{m \geq 0} \bigoplus_{p+q=m} O(m) \otimes_{k[\Sigma_p \times \Sigma_q]} X_*(\mathrm{p}) \otimes X_*(\mathrm{q}). \end{aligned}$$

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Suggestions?