Higher topological Hochschild homology of rings of integers in number fields

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For any strictly commutative ring spectrum $A$ and for any simplicial set $X$ one can define the simplicial commutative ring spectrum $A \otimes X$ as $(A \otimes X)_n = \bigwedge_{x \in X_n} A$. Similarly, if $M$ is an $A$-module spectrum and $X$ is a pointed simplicial set, we define $(M, A) \otimes X$ by placing $M$ at the basepoint of $X$ and $A$ at all other simplices of $X$. Important examples of this construction are

- topological Hochschild homology of $A$ with coefficients in $M$, $\text{THH}(A, M)$, given by $(M, A) \otimes S^1$,
- higher topological Hochschild homology of order $n$ of $A$ with coefficients in $M$, $\text{THH}^n(A, M) = (M, A) \otimes S^{n-n}$, $n \geq 1$,
- torus homology where we tensor with $(S^1)^n = \mathbb{T}^n$.

Ordinary topological Hochschild homology is the target of a trace map from algebraic K-theory. This trace map factors via topological cyclic homology, $\text{TC}$, and the latter is often a very good approximation of algebraic K-theory.

If we consider iterated algebraic K-theory, then we can use an iteration of the trace map and obtain torus homology as the natural target of such a trace map. Using the standard cell structure of an $n$-dimensional torus gives us a method of calculating torus homology from higher topological Hochschild homology.

An important class of examples are rings of integers in number fields. As a starting point we consider higher $\text{THH}$ of the integers with coefficients in the residue field $\mathbb{F}_p$, $\text{THH}^n(\mathbb{Z}, \mathbb{F}_p)$. Bökstedt \cite{2} calculated

$\text{THH}_*(\mathbb{Z}, \mathbb{F}_p) \cong \mathbb{F}_p[x_{2p}] \otimes \mathbb{F}_p \Lambda_{\mathbb{F}_p}(x_{2p-1})$.

There is a well-known description of iterated Tor-algebras due to Cartan \cite{4}: If we start with a polynomial algebra over $\mathbb{F}_p$ generated by an element $w$ of even degree, then we call this algebra $B^1_{\mathbb{F}_p} (w)$. Iteratively, we define

$B^{n+1}_{\mathbb{F}_p} (w) := \text{Tor}^{B^n_{\mathbb{F}_p} (w)}(\mathbb{F}_p, \mathbb{F}_p)$

for all $n$. The case $n = 2$ immediately gives

$B^2_{\mathbb{F}_p} (w) = \text{Tor}^{B^1_{\mathbb{F}_p} (w)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon w)$

where the degree of $\varepsilon w$ is one higher than the degree of $w$.

$B^3_{\mathbb{F}_p} (w) = \text{Tor}^{B^2_{\mathbb{F}_p} (w)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(q^0 \varepsilon w)$

where the latter denotes a divided power algebra. As the base field is of characteristic $p$ this algebra splits into a tensor product of truncated polynomial algebras

$\Gamma_{\mathbb{F}_p}(q^0 \varepsilon w) \cong \bigotimes_k \mathbb{F}_p[q^k \varepsilon w]/(q^k \varepsilon w)^p$;
here $\varrho^k \varepsilon w$ corresponds to the $p^k$th divided power of $\varrho^0 \varepsilon w$. For each of the tensor factors we obtain again a periodic resolution and we get
\begin{equation*}
B_{2p}^4(w) = \text{Tor} B_{2p}^4(w)(\mathbb{F}_p, \mathbb{F}_p) \cong \bigotimes_k \Gamma_{\mathbb{F}_p}(\varrho^0 \varrho^k \varepsilon w) \otimes \Lambda_{\mathbb{F}_p}(\varrho \varepsilon w).
\end{equation*}

From here on the iteration process yields terms of a form that already occurred before.

**Theorem 1** [Dundas-Lindenstrauss-R]

For all $n \geq 1$ and for all primes $p$:
\begin{equation*}
\text{THH}^{[n]}(\mathbb{Z}, \mathbb{F}_p) \cong B_{2p}^n(x_{2p}) \otimes_{\mathbb{F}_p} B_{2p}^{n+1}(y_{2p-2}).
\end{equation*}

A crucial ingredient in the proof is the following

**Lemma**

Let $C$ be a commutative augmented $H\mathbb{F}_p$-algebra and assume that there is an isomorphism of graded commutative $\mathbb{F}_p$-algebras $\pi_* C \cong \Lambda_{\mathbb{F}_p}(x)$ where $|x| = m > 0$. Then there is a zigzag of equivalences of commutative augmented $H\mathbb{F}_p$-algebras between $C$ and $H\mathbb{F}_p \vee \Sigma^m H\mathbb{F}_p$.

This Lemma was suggested by Mike Mandell. Our proof uses a Postnikov argument in the world of commutative $H\mathbb{F}_p$-algebras. With the help of this result we can split off the bottom Postnikov piece of $\text{THH}(\mathbb{Z}, \mathbb{F}_p)$ and obtain an iterated homotopy pushout diagram for $\text{THH}^{[2]}(\mathbb{Z}, \mathbb{F}_p)$:

\begin{equation*}
\begin{array}{c}
\text{THH}(\mathbb{Z}, \mathbb{F}_p) \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
H\mathbb{F}_p \\
\downarrow \quad \downarrow \quad \downarrow
\end{array}
\begin{array}{c}
H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\text{THH}^{[2]}(\mathbb{Z}, \mathbb{F}_p) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\begin{array}{c}
E \\
\downarrow
\end{array}
\end{equation*}

A Tor-spectral sequence calculations yields that $E$ has the exterior algebra $\Lambda_{\mathbb{F}_p}(z_{2p+1})$ as $\pi_*(E)$ and hence we know that $E \sim H\mathbb{F}_p \vee \Sigma^{2p+1} H\mathbb{F}_p$. The map $f$ factors via the augmentation and unit and this yields with another Tor-spectral sequence calculation that
\begin{equation*}
\pi_* \text{THH}^{[2]}(\mathbb{Z}, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(z_{2p+1}) \otimes_{\mathbb{F}_p} \Gamma_{\mathbb{F}_p}(a_{2p}).
\end{equation*}

For the iteration of this argument we use that we can express higher $\text{THH}$ via an iterated bar construction. For instance $\text{THH}^{[3]}(\mathbb{Z}, \mathbb{F}_p)$ is equivalent to the diagonal of the bisimplicial commutative augmented $H\mathbb{F}_p$-algebra $B(H\mathbb{F}_p, B(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, E), H\mathbb{F}_p)$ and as the module structure of $E$ over $H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p$ reduces to the $H\mathbb{F}_p$-module structure this bar construction splits as a bisimplicial commutative augmented $H\mathbb{F}_p$-algebra into
\begin{equation*}
B(H\mathbb{F}_p, B(H\mathbb{F}_p, H\mathbb{F}_p \vee \Sigma^{2p-1} H\mathbb{F}_p, H\mathbb{F}_p), H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} B(H\mathbb{F}_p, E, H\mathbb{F}_p)
\end{equation*}
where $E$ denotes the constant simplicial commutative augmented $HF_p$-algebra on $E$. For higher $n$ there is a similar splitting and we get that $THH^{[n+1]}(\mathbb{Z}, F_p)$ is equivalent to the diagonal of an $n$-fold reduced iterated bar construction on $HF_p \vee \Sigma^{2p-1}HF_p$ smashed with an $(n-1)$-fold iterated bar construction on $E$. The square zero extensions $E$ and $HF_p \vee \Sigma^{2p-1}HF_p$ can be modelled as the Eilenberg Mac Lane spectra on a simplicial commutative algebra and this allows for a comparison of the above iterated bar constructions with iterated algebraic bar constructions on exterior algebras. The homology groups of such bar constructions were determined in [1] and this gives the proof of Theorem 1.

Let $O$ denote the ring of integers in a number field and let $P$ be a non-trivial prime ideal in $O$ with residue field $O/P = F_q$ where $q = p^r$ for some prime $p$. Higher $THH$ detects ramification:

**Theorem 2** [Dundas-Lindenstrauss-R]

For all $n \geq 1$:

$$\text{THH}^{[n]}(O, O/P) \cong B^n_{F_q}(x) \otimes_{F_q} B^{n+1}_{F_q}(y)$$

where

(i) $|x| = 2$ and $|y| = 0$ if $A$ is ramified over $\mathbb{Z}$ at $P$, and

(ii) $|x| = 2p$ and $|y| = 2p - 2$, if $A$ is unramified over $\mathbb{Z}$ at $P$.

Lindenstrauss and Madsen determined the topological Hochschild homology groups of rings of integers in [6]. In the unramified case we show that we have an isomorphism

$$\text{THH}^{[n]}(O, O/P) \cong THH^{[n]}(\mathbb{Z}, F_p) \otimes_{F_p} F_q.$$ 

This uses the Lindenstrauss-Madsen result and an iterative spectral sequence argument. In the ramified case the important input is that the first Hochschild homology group (and therefore also the first $THH$-group) is isomorphic to $F_q$. This fact ensures that the differentials in the Brun spectral sequence [3, p. 30]

$$\text{THH}_*(O, O/P, \text{Tor}_*(O, O/P)) \Rightarrow \text{THH}_*(O, O/P)$$

have to vanish and we obtain that

$$\text{THH}_*(O, O/P) \cong F_q[u] \otimes_{F_q} \Lambda_{F_q}(\tau)$$

with $|u| = 2$ and $|\tau| = 1$. From this point on the argument is the same as in the case of the rational integers.

**References**


