Algebraic Topology, summer term 2010

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CHAPTER 1

Homology theory

1. Chain complexes

Definition 1.1. A *chain complex* is a sequence of abelian groups, $(C_n)_{n\in\mathbb{Z}}$, together with homomorphisms $d_n\colon C_n \to C_{n-1}$ for $n \in \mathbb{Z}$, such that $d_{n-1} \circ d_n = 0$.

Let R be an associative ring with unit 1_R. A *chain complex* of R-modules can analoguously be defined as a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ with R-linear maps $d_n\colon C_n \to C_{n-1}$ with $d_{n-1} \circ d_n = 0$.

Definition 1.2.
- The d_n are *differentials* or *boundary operators.*
- The $x \in C_n$ are called *n-chains.*
- Is $x \in C_n$ and $d_n x = 0$, then x is an *n-cycle.*

$$Z_n(C) := \{x \in C_n | d_n x = 0\}.$$

- If $x \in C_n$ is of the form $x = d_{n+1}y$ for some $y \in C_{n+1}$, then x is an *n-boundary.*

$$B_n(C) := Im(d_{n+1}) = \{d_{n+1}y, y \in C_{n+1}\}.$$

Note that the cycles and boundaries form subgroups of the chains. As $d_n \circ d_{n+1} = 0$, we know that the image of d_{n+1} is a subgroup of the kernel of d_n and thus

$$B_n(C) \subset Z_n(C).$$

We drop the subscript n from the boundary maps from now on. Often we just write C_* for the chain complex.

Definition 1.3. The abelian group $H_n(C) := Z_n(C)/B_n(C)$ is the *n-th homology group of the complex* C_*.

Notation: We denote by $[c]$ the equivalence class of a $c \in Z_n(C)$.
If $c, c' \in C_n$ satisfy that $c - c'$ is a boundary, then c is *homologous* to c'. That's an equivalence relation.

Examples:

1) Consider

$$C_n = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and let d_1 be the multiplication with $N \in \mathbb{N}$, then

$$H_n(C) = \begin{cases} \mathbb{Z}/N\mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2) Take $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ and

$$d_n = \begin{cases} \mathrm{id}_{\mathbb{Z}} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

What is the homology of this chain complex?

2') Consider $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}$ again, but let all boundary maps be trivial. What is the homology of this chain complex?

Definition 1.4. Let C_* and D_* be two chain complexes. A *chain map* $f\colon C_* \to D_*$ is a sequence of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n^D \circ f_n = f_{n-1} \circ d_n^C$ for all n, *i.e.*, the diagram

$$\begin{tikzcd} C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "d_n^D"] & D_{n-1} \end{tikzcd}$$

commutes for all n.

Such an f sends cycles to cycles and boundaries to boundaries. We therefore obtain an induced map

$$H_n(f)\colon H_n(C) \to H_n(D)$$

via $H_n(f)_*[c] = [f_n c]$.

There is a chain map from the chain complex mentioned in Example 1) to the chain complex D_* that is concentrated in degree zero and has $D_0 = \mathbb{Z}/N\mathbb{Z}$. Note, that $(f_0)_*$ is an isomorphism on zeroth homology groups.

Are there chain maps between the complexes from Examples 2) and 2')?

Lemma 1.5. *If $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ are two chain maps, then $H_n(g) \circ H_n(f) = H_n(g \circ f)$ for all n.*

When do two chain maps induce the same map on homology?

Definition 1.6. A *chain homotopy* H between two chain maps $f, g\colon C_* \to D_*$ is a sequence of maps $(H_n)_{n\in\mathbb{Z}}$ with $H_n\colon C_n \to D_{n+1}$ such that for all n

$$d_{n+1}^D \circ H_n + H_{n-1} \circ d_n^C = f_n - g_n.$$

$$\begin{tikzcd}[column sep=large] \cdots \arrow[r, "d_{n+2}^C"] & C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[dl, "H_{n+1}"'] \arrow[d, bend right, "f_{n+1}"'] \arrow[d, bend left, "g_{n+1}"] & C_n \arrow[r, "d_n^C"] \arrow[dl, "H_n"'] \arrow[d, bend right, "f_n"'] \arrow[d, bend left, "g_n"] & C_{n-1} \arrow[r, "d_{n-1}^C"] \arrow[dl, "H_{n-1}"'] \arrow[d, bend right, "f_{n-1}"'] \arrow[d, bend left, "g_{n-1}"] & \cdots \\ \cdots \arrow[r, "d_{n+2}^D"] & D_{n+1} \arrow[r, "d_{n+1}^D"] & D_n \arrow[r, "d_n^D"] & D_{n-1} \arrow[r, "d_{n-1}^D"] & \cdots \end{tikzcd}$$

If such an H exists, then f and g are *homotopic*: $f \simeq g$.

We will later see geometrically defined examples of chain homotopies.

Proposition 1.7. (a) *Being homotopic is an equivalence relation.*
(b) *If f and g are homotopic, then $H_n(f) = H_n(g)$ for all n.*

Proof. If H is a homotopy from f to g, then $-H$ is a homotopy from g to f. Each f is homotopic to itself with $H = 0$. If f is homotopic to g via H and g is homotopic to h via K, then f is homotopic to h via $H + K$. □

Definition 1.8. Let $f\colon C_* \to D_*$ be a chain map. We call f a *chain homotopy equivalence*, if there is a chain map $g\colon D_* \to C_*$ such that $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$. The chain complexes C_* and D_* are then *chain homotopically equivalent*.

Note, that such chain complexes have isomorphic homology. However, chain complexes with isomorphic homology do not have to be chain homotopically equivalent. (Can you find a counterexample?)

Definition 1.9. If C_* and C'_* are chain complexes, then their *direct sum*, $C_* \oplus C'_*$, is the chain complex with

$$(C_* \oplus C'_*)_n = C_n \oplus C'_n = C_n \times C'_n$$

with differential $d = d_\oplus$ given by

$$d_\oplus(c, c') = (dc, dc').$$

2. Singular homology

Let $v_0, \dots, v_n$ be $n+1$ points in $\mathbb{R}^{n+1}$. Consider the convex hull

$$K(v_0, \dots, v_n) := \{\sum_{i=0}^{n} t_i v_i | \sum t_i = 1, t_i \geqslant 0\}.$$

Definition 2.1. If the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent, then $K(v_0, \dots, v_n)$ is the *simplex generated by* $v_0, \dots, v_n$. We denote such a simplex by $\mathrm{simp}(v_0, \dots, v_n)$.

Example. The standard topological n-simplex is $\Delta^n := \mathrm{simp}(e_0, \dots, e_n)$. Here, e_i is the vector in $\mathbb{R}^{n+1}$ that has a 1 in coordinate $i+1$ and is zero in all other coordinates. The first examples are: Δ^0 is the point e_0, Δ^1 is the line segment between e_0 and e_1, Δ^2 is a triangle in $\mathbb{R}^3$ and Δ^3 is homeomorphic to a tetrahedron.

The coordinate description of the n-simplex is

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, t_i \geqslant 0\}.$$

We consider Δ^n as $\Delta^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} \subset \dots$.

The boundary of Δ^1 consists of two copies of Δ^0, the boundary of Δ^2 consists of three copies of Δ^1. In general, the boundary of Δ^n consists of $n+1$ copies of Δ^{n-1}.

We need the following *face maps* for $0 \leqslant i \leqslant n$

$$d_i = d_i^{n-1} \colon \Delta^{n-1} \hookrightarrow \Delta^n; (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

The image of d_i^{n-1} in Δ^n is the face that is opposite to e_i. It is the simplex generated by $e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n$. Draw the examples of the faces in Δ^1 and Δ^2!

Lemma 2.2. *Concerning the composition of face maps, the following rule holds:*

$$d_i^{n-1} \circ d_j^{n-2} = d_j^{n-1} d_{i-1}^{n-2}, \quad 0 \leqslant j < i \leqslant n.$$

Example: face maps for Δ^0 and composition into Δ^2: $d_2 \circ d_0 = d_0 \circ d_1$.

Proof. Both expressions yield

$$d_i^{n-1} \circ d_j^{n-2}(t_0, \dots, t_{n-2}) = (t_0, \dots, t_{j-1}, 0, \dots, 0, t_{i-1}, \dots, t_{n-2}) = d_j^{n-1} d_{i-1}^{n-2}(t_0, \dots, t_{n-2}).$$

$\square$

Let X be an arbitrary topological space, $X \neq \varnothing$.

Definition 2.3. A *singular n-simplex in* X is a continuous map $\alpha \colon \Delta^n \to X$.

Note, that α just has to be continuous, not smooth or anything!

Definition 2.4. Let $S_n(X)$ be the free abelian group generated by all singular n-simplices in X. We call $S_n(X)$ the *n-th singular chain module of* X.

Elements of $S_n(X)$ are finite sums $\sum_{i \in I} \lambda_i \alpha_i$ with $\lambda_i = 0$ for almost all $i \in I$ and $\alpha_i \colon \Delta^n \to X$.

For all $n \geqslant 0$ there are non-trivial elements in $S_n(X)$, because we assumed that $X \neq \varnothing$: we can always take an $x_0 \in X$ and the constant map $\kappa_{x_0} \colon \Delta^n \to X$ as α. By convention, we define $S_n(\varnothing) = 0$ for all $n \geqslant 0$.

If we want to define maps from $S_n(X)$ to some abelian group then it suffices to define such a map on generators.

Example. What is $S_0(X)$? A continuous $\alpha \colon \Delta^0 \to X$ is determined by its value $\alpha(e_0) =: x_\alpha \in X$, which is a point in X. A singular 0-simplex $\sum_{i \in I} \lambda_i \alpha_i$ can thus be identified with the formal sum of points $\sum_{i \in I} \lambda_i x_{\alpha_i}$.

Definition 2.5. We define $\partial_i \colon S_n(X) \to S_{n-1}(X)$ on generators

$$\partial_i(\alpha) = \alpha \circ d_i^{n-1}$$

and call it *the i-th face of* α.

On $S_n(X)$ we therefore get $\partial_i(\sum_j \lambda_j \alpha_j) = \sum_j \lambda_j (\alpha_j \circ d_i^{n-1})$.

LEMMA 2.6. *The face maps on $S_n(X)$ obey*

$$\partial_j \circ \partial_i = \partial_{i-1} \circ \partial_j, \quad 0 \leqslant j < i \leqslant n.$$

PROOF. The proof follows from the one of Lemma 2.2. $\square$

DEFINITION 2.7. We define the boundary operator on singular chains as $\partial\colon S_n(X) \to S_{n-1}(X)$, $\partial = \sum_{i=0}^{n}(-1)^i\partial_i$.

LEMMA 2.8. *The map ∂ actually is a boundary operator,* i.e., $\partial \circ \partial = 0$.

PROOF. We calculate

$$\begin{aligned}
\partial \circ \partial = (\sum_{j=0}^{n-1}(-1)^j\partial_j) \circ (\sum_{i=0}^{n}(-1)^i\partial_i) &= \sum\sum(-1)^{i+j}\partial_j \circ \partial_i \\
&= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_j \circ \partial_i + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i \\
&= \sum_{0\leqslant j<i\leqslant n}(-1)^{i+j}\partial_{i-1} \circ \partial_j + \sum_{0\leqslant i\leqslant j\leqslant n-1}(-1)^{i+j}\partial_j \circ \partial_i = 0.
\end{aligned}$$

$\square$

We therefore obtain the *singular chain complex*, $S_*(X)$,

$$\ldots \to S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_1(X) \xrightarrow{\partial} S_0(X) \to 0.$$

We abbreviate $Z_n(S_*(X))$ by $Z_n(X)$, $B_n(S_*(X))$ by $B_n(X)$ and $H_n(S_*(X))$ by $H_n(X)$.

DEFINITION 2.9. For a space X, $H_n(X)$ is the *n-th singular homology group of X*.

Note that $Z_0(X) = S_0(X)$.

As an example of a 1-cycle consider a 1-chain $c = \alpha + \beta + \gamma$ where $\alpha, \beta, \gamma\colon \Delta^1 \to X$ such that $\alpha(e_1) = \beta(e_0)$, $\beta(e_1) = \gamma(e_0)$ and $\gamma(e_1) = \alpha e_0$ and calculate that $\partial c = 0$.

We need to understand how continuous maps of topological spaces interact with singular chains and singular homology.

Let $f\colon X \to Y$ be a continuous map.

DEFINITION 2.10. The map $f_n = S_n(f)\colon S_n(X) \to S_n(Y)$ is defined on generators $\alpha\colon \Delta^n \to X$ as

$$f_n(\alpha) = f \circ \alpha : \Delta^n \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

LEMMA 2.11. *For any continuous $f\colon X \to Y$ we have*

$$\begin{tikzcd}
S_n(X) \arrow[r, "f_n"] \arrow[d, "\partial^X"'] & S_n(Y) \arrow[d, "\partial^Y"] \\
S_{n-1}(X) \arrow[r, "f_{n-1}"] & S_{n-1}(Y),
\end{tikzcd}$$

i.e., *$(f_n)_n$ is a chain map and hence induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

PROOF. By definition

$$\partial^Y(f_n(\alpha)) = \sum_{i=0}^{n}(-1)^i(f\circ\alpha)\circ d_i = \sum_{i=0}^{n}(-1)^i f\circ(\alpha\circ d_i) = f_{n-1}(\partial^X\alpha).$$

$\square$

Of course, the identity map on X induces the identity map on $H_n(X)$ for all $n \geqslant 0$ and if we have a composition of continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

then $S_n(g \circ f) = S_n(g) \circ S_n(f)$ and $H_n(g \circ f) = H_n(g) \circ H_n(f)$. In categorical language, this says precisely that $S_n(-)$ and $H_n(-)$ are functors from the category of topological spaces and continuous maps into the category of abelian groups. Taking all $S_n(-)$ together turns $S_*(-)$ into a functor from topological spaces and continuous maps into the category of chain complexes with chain maps as morphisms.

One implication of Lemma 2.11 is that homeomorphic spaces have isomorphic homology groups:

$$X \cong Y \Rightarrow H_n(X) \cong H_n(Y) \text{ for all } n \geqslant 0.$$

Our first (not too exciting) calculation is the following:

Proposition 2.12. *The homology groups of a one-point space* pt *are trivial but in degree zero,*

$$H_n(\mathrm{pt}) \cong \begin{cases} 0, & \text{if } n > 0 \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. For every $n \geqslant 0$ there is precisely one continuous map $\alpha\colon \Delta^n \to \mathrm{pt}$, namely the constant map. We denote this map by κ_n. Then the boundary of κ_n is

$$\partial \kappa_n = \sum_{i=0}^{n} (-1)^i \kappa_n \circ d_i = \sum_{i=0}^{n} (-1)^i \kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

For all n we have $S_n(\mathrm{pt}) \cong \mathbb{Z}$ generated by κ_n and therefore the singular chain complex looks as follows:

$$\ldots \xrightarrow{\partial=0} \mathbb{Z} \xrightarrow{\partial=\mathrm{id}_\mathbb{Z}} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z}.$$

□

3. H_0 and H_1

Before we calculate anything, we define a map.

Proposition 3.1. *For any topological space X there is a homomorphism $\varepsilon\colon H_0(X) \to \mathbb{Z}$ with $\varepsilon \neq 0$ for $X \neq \varnothing$.*

Proof. We said that $S_0(\varnothing)$ is zero, so $H_0(\varnothing) = 0$ and in this case we define ε to be the zero map.

If $X \neq \varnothing$, then we define $\varepsilon(\alpha) = 1$ for any $\alpha\colon \Delta^0 \to X$, thus $\varepsilon(\sum_{i \in I} \lambda_i \alpha_i) = \sum_{i \in I} \lambda_i$ on $S_0(X)$. As only finitely many λ_i are non-trivial, this is in fact a finite sum.

We have to show that this map is well-defined on homology, *i.e.*, that it vanishes on boundaries. Let $S_0(X) \ni c = \partial b$ be a boundary and write $b = \sum_{i \in I} \nu_i \beta_i$ with $\beta_i\colon \Delta^1 \to X$. Then we get

$$\partial b = \partial \sum_{i \in I} \nu_i \beta_i = \sum_{i \in I} \nu_i (\beta_i \circ d_0 - \beta_i \circ d_1) = \sum_{i \in I} \nu_i \beta_i \circ d_0 - \sum_{i \in I} \nu_i \beta_i \circ d_1$$

and hence

$$\varepsilon(c) = \varepsilon(\partial b) = \sum_{i \in I} \nu_i - \sum_{i \in I} \nu_i = 0.$$

□

If $X \neq \varnothing$, then any $\alpha\colon \Delta^0 \to X$ can be identified with its image point, so the map ε on $S_0(X)$ counts points in X with multiplicities.

Proposition 3.2. *If X is a path-connected, non-empty space, then $\varepsilon\colon H_0(X) \cong \mathbb{Z}$.*

Proof. As X is non-empty, there is a point $x \in X$ and the constant map κ_x with value x is an element in $S_0(X)$ with $\varepsilon(\kappa_x) = 1$. Therefore ε is surjective. For any other point $y \in X$ there is a continuous path $\omega\colon [0,1] \to X$ with $\omega(0) = x$ and $\omega(1) = y$. We define $\alpha_\omega\colon \Delta^1 \to X$ as

$$\alpha_\omega(t_0, t_1) = \omega(1 - t_0).$$

Then

$$\partial(\alpha_\omega) = \partial_0(\alpha_\omega) - \partial_1(\alpha_\omega) = \alpha_\omega(e_1) - \alpha_\omega(e_0) = \alpha_\omega(0,1) - \alpha_\omega(1,0) = \kappa_y - \kappa_x,$$

and the two generators κ_x, κ_y are homologous. This shows that ε is injective. □

Corollary 3.3. *If X is of the form $X = \bigsqcup_{i \in I} X_i$ such that the X_i are non-empty and path-connected, then*

$$H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}.$$

In this case, the zeroth homology group of X is the free abelian group generated by the path-components.

Proof. The singular chain complex of X splits as the direct sum of chain complexes of the X_i:

$$S_n(X) \cong \bigoplus_{i \in I} S_n(X_i)$$

for all n. Boundary summands ∂_i stay in a component, in particular,

$$\partial\colon S_1(X) \cong \bigoplus_{i \in I} S_1(X_i) \to \bigoplus_{i \in I} S_0(X_i) \cong S_0(X)$$

is the direct sum of the boundary operators $\partial\colon S_1(X_i) \to S_0(X_i)$ and the claim follows. □

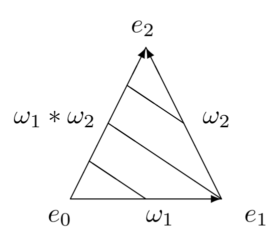
Next, we want to relate H_1 to the fundamental group. Let X be path-connected and $x \in X$.

Lemma 3.4. *Let $\omega_1, \omega_2, \omega$ be paths in X.*

(a) *Constant paths are null-homologous.*
(b) *If $\omega_1(1) = \omega_2(0)$, then $\omega_1 * \omega_2 - \omega_1 - \omega_2$ is a boundary. Here $\omega_1 * \omega_2$ is the concatenation of ω_1 followed by ω_2.*
(c) *If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0,1\}$, then ω_1 and ω_2 are homologous as singular 1-chains.*
(d) *Any 1-chain of the form $\bar{\omega} * \omega$ is a boundary. Here, $\bar{\omega}(t) := \omega(1-t)$.*

Proof. For a), consider the constant singular 2-simplex $\alpha(t_0, t_1, t_2) = x$ and c_x, the constant path on x. Then $\partial\alpha = c_x - c_x + c_x = c_x$.

For b), we define a singular 2-simplex $\beta\colon \Delta^2 \to X$ as follows.



We define β on the boundary components of Δ^2 as indicated and prolong it constantly along the sloped inner lines. Then

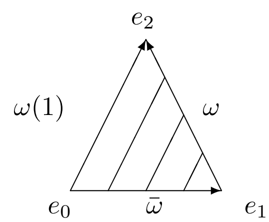
$$\partial\beta = \beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2 = \omega_2 - \omega_1 * \omega_2 + \omega_1.$$

For c): Let $H\colon [0,1] \times [0,1] \to X$ a homotopy from ω_1 to ω_2. As we have that $H(0,t) = \omega_1(0) = \omega_2(0)$, we can factor H over the quotient $[0,1] \times [0,1]/\{0\} \times [0,1] \cong \Delta^2$ with induced map $h\colon \Delta^2 \to X$. Then

$$\partial h = h \circ d_0 - h \circ d_1 + h \circ d_2.$$

The first summand is null-homologous, because it's constant (with value $\omega_1(1) = \omega_2(1)$), the second one is ω_2 and the last is ω_1, thus $\omega_1 - \omega_2$ is null-homologous.

For d): Consider $\gamma\colon \Delta^2 \to X$ as indicated below.



□

Definition 3.5. Let $h\colon \pi_1(X,x) \to H_1(X)$ be the map, that sends the homotopy class of a closed path ω, $[\omega]_{\pi_1}$, to its homology class $[\omega] = [\omega]_{H_1}$. This map is called *Hurewicz-homomorphism.*

Lemma 3.4 ensures that h is well-defined and

$$h([\omega_1][\omega_2]) = h([\omega_1 * \omega_2]) = [\omega_1] + [\omega_2] = h([\omega_1]) + h([\omega_2])$$

thus h is a homomorphism.

Note that for a closed path ω we have that $[\bar{\omega}] = -[\omega]$ in $H_1(X)$.

Definition 3.6. Let G be an arbitrary group, then its *abelianization*, G_{ab} is $G/[G,G]$.

Recall that $[G,G]$ is the commutator subgroup of G. That is the smallest subgroup of G containing all commutators $ghg^{-1}h^{-1}, g,h \in G$.

Proposition 3.7. *The Hurewicz homomorphism factors over the abelianization of* $\pi_1(X,x)$ *and induces an isomorphism*

$$\pi_1(X,x)_{\mathrm{ab}} \cong H_1(X)$$

for all path-connected X*.*

$$\begin{tikzcd}
\pi_1(X,x) \arrow[rr, "h"] \arrow[d, "p"'] & & H_1(X) \\
\pi_1(X,x)_{\mathrm{ab}} = \pi_1(X,x)/[\pi_1(X,x),\pi_1(X,x)] \arrow[urr, "\cong", "h_{\mathrm{ab}}"'] & &
\end{tikzcd}$$

Proof. We will construct an inverse to h_{ab}. For any $y \in X$ we choose a path u_y from x to y. For $y = x$ we take u_x to be the constant path on x. Let α be an arbitrary singular 1-simplex and $y_i = \alpha(e_i)$. Define $\phi\colon S_1(X) \to \pi_1(X,x)_{\mathrm{ab}}$ on generators as $\phi(\alpha) = [u_{y_0} * \alpha * \bar{u}_{y_1}]$ and extend ϕ linearly to all of $S_1(X)$, keeping in mind that the composition in π_1 is written multiplicatively.

We have to show that ϕ is trivial on boundaries, so let $\beta\colon \Delta^2 \to X$. Then

$$\phi(\partial\beta) = \phi(\beta \circ d_0 - \beta \circ d_1 + \beta \circ d_2) = \phi(\beta \circ d_0)\phi(\beta \circ d_1)^{-1}\phi(\beta \circ d_2).$$

Abbreviating $\beta \circ d_i$ with α_i we get as a result

$$[u_{y_1} * \alpha_0 * \bar{u}_{y_2}][u_{y_2} * \alpha_1 * \bar{u}_{y_0}]^{-1}[u_{y_0} * \alpha_2 * \bar{u}_{y_1}] = [u_{y_0} * \alpha_2 * \bar{u}_{y_1} * u_{y_1} * \alpha_0 * \bar{u}_{y_2} * u_{y_2} * \bar{\alpha_1} * \bar{u}_{y_0}].$$

Here, we've used that the image of ϕ is abelian. We can reduce $\bar{u}_{y_1} * u_{y_1}$ and $\bar{u}_{y_2} * u_{y_2}$ and are left with $[u_{y_0} * \alpha_2 * \alpha_0 * \bar{\alpha_1} * \bar{u}_{y_0}$ but $\alpha_2 * \alpha_0 * \bar{\alpha_1}$ is the closed path tracing the boundary of β and therefore it is null-homotopic in X. Thus $\phi(\partial\beta) = 0$ and ϕ passes to a map

$$\phi\colon H_1(X) \to \pi_1(X,x)_{\mathrm{ab}}.$$

The composition $\phi \circ h_{\mathrm{ab}}$ evaluated on the class of a closed path ω gives

$$\phi \circ h_{\mathrm{ab}}[\omega]_{\pi_1} = \phi[\omega]_{H_1} = [u_x * \omega * \bar{u}_x]_{\pi_1}.$$

But we chose u_x to be constant, thus $\phi \circ h_{\mathrm{ab}} = \mathrm{id}$.

If $c = \sum \lambda_i \alpha_i$ is a cycle, then $h_{\mathrm{ab}} \circ \phi(c)$ is of the form $[c + D_{\partial c}]$ where the $D_{\partial c}]$ part comes from the contributions of the u_{y_i}. The fact that $\partial(c) = 0$, implies that the summands in $D_{\partial c}$ cancel off and thus $h_{\mathrm{ab}} \circ \phi = \mathrm{id}_{H_1(X)}$. □

Note, that abelianization doesn't change anything for abelian groups, *i.e.*, whenever we have an abelian fundamental group, we know that $H_1(X) \cong \pi_1(X, x)$.

Corollary 3.8. *Knowledge of π_1 gives*

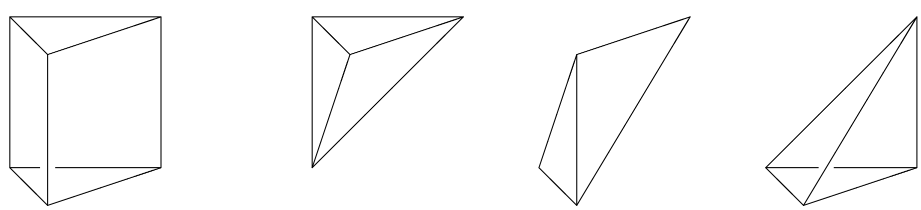
$$H_1(\mathbb{S}^n) = 0, \text{ for } n > 1, \quad H_1(\mathbb{S}^1) \cong \mathbb{Z},$$
$$H_1(\underbrace{\mathbb{S}^1 \times \ldots \times \mathbb{S}^1}_{n}) \cong \mathbb{Z}^n,$$
$$H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong (\mathbb{Z} * \mathbb{Z})_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$H_1(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & \text{for } n > 1. \end{cases}$$

4. Homotopy invariance

We want to show that two continuous maps that are homotopic induce identical maps on the level of homology groups.

Heuristics: If $\alpha \colon \Delta^n \to X$ is a singular n-simplex and if f, g are homotopic maps from X to Y, then the homotopy from $f \circ \alpha$ to $g \circ \alpha$ starts on $\Delta^n \times [0,1]$. We want to translate this geometric homotopy into a chain homotopy on the singular chain complex. To that end we have to cut the prism $\Delta^n \times [0,1]$ into $(n+1)$-simplices. In low dimensions this is easy:

$\Delta^0 \times [0,1]$ is homeomorphic to Δ^1, $\Delta^1 \times [0,1] \cong [0,1]^2$ and this can be cut into two copies of Δ^2 and $\Delta^2 \times [0,1]$ is a 3-dimensional prism and that can be glued together from three tetrahedrons, *e.g.*, like



As you might guess now, we use $n+1$ copies of Δ^{n+1} to build $\Delta^n \times [0,1]$.

Definition 4.1. For $i = 0, \ldots, n$ define $p_i \colon \Delta^{n+1} \to \Delta^n \times [0,1]$ as

$$p_i(t_0, \ldots, t_{n+1}) = ((t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}), t_{i+1} + \ldots + t_{n+1}) \in \Delta^n \times [0,1].$$

On the standard basis vectors e_k we obtain

$$p_i(e_k) = \begin{cases} (e_k, 0), & \text{for } 0 \leqslant k \leqslant i, \\ (e_{k-1}, 1), & \text{for } k > i. \end{cases}$$

We obtain chain maps $P_i \colon S_n(X) \to S_{n+1}(X \times [0,1])$ via $P_i(\alpha) = (\alpha \times \mathrm{id}) \circ p_i$:

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times [0,1] \xrightarrow{\alpha \times \mathrm{id}} X.$$

For $k = 0, 1$ let $j_k \colon X \to X \times [0,1]$ be the inclusion $x \mapsto (x, k)$.

Lemma 4.2. *The maps P_i satisfy the following relations*

(a) $\partial_0 \circ P_0 = S_n(j_1)$,

(b) $\partial_{n+1} \circ P_n = S_n(j_0)$,

(c) $\partial_i \circ P_i = \partial_i \circ P_{i-1}$ *for* $1 \leqslant i \leqslant n$.

(d)
$$\partial_j \circ P_i = \begin{cases} P_i \circ \partial_{j-1}, & \text{for } i \leqslant j-2 \\ P_{i-1} \circ \partial_j, & \text{for } i \geqslant j+1. \end{cases}$$

Proof. For the first two points, we note that on Δ^n we have

$$p_0 \circ d_0(t_0, \dots, t_n) = p_0(0, t_0, \dots, t_n) = ((t_0, \dots, t_n), \sum t_i) = ((t_0, \dots, t_n), 1) = j_1(t_0, \dots, t_n)$$

and

$$p_n \circ d_{n+1}(t_0, \dots, t_n) = p_0(t_0, \dots, t_n, 0) = ((t_0, \dots, t_n), 0) = j_0(t_0, \dots, t_n).$$

For c), one checks that $p_i \circ d_i = p_{i-1} \circ d_i$ on Δ^n: both give $((t_0, \dots, t_n), \sum_{j=i}^n t_j)$ on $(t_0, \dots, t_n)$.

For d) in the case $i \geqslant j+1$, consider the following diagram

$$\begin{tikzcd}
& \Delta^{n+1} \arrow[r, "p_i"] & \Delta^n \times [0,1] \arrow[dd, equal] \\
\Delta^n \arrow[ur, "d_j"] \arrow[dr, "p_{i-1}"'] & & \\
& \Delta^{n-1} \times [0,1] \arrow[r, "d_j \times \mathrm{id}"] & \Delta^n \times [0,1]
\end{tikzcd}$$

Checking coordinates one sees that this diagram commutes. The remaining case follows from a similar observation. □

Definition 4.3. We define $P\colon S_n(X) \to S_{n+1}(X \times [0,1])$ as $P = \sum_{i=0}^n (-1)^i P_i$.

Lemma 4.4. *The map P is a chain homotopy between $(S_n(j_0))_n$ and $(S_n(j_1))_n$,* i.e., $\partial \circ P + P \circ \partial = S_n(j_1) - S_n(j_0)$.

Proof. We take an $\alpha\colon \Delta^n \to X$ and calculate

$$\partial P\alpha + P\partial\alpha = \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \partial_j P_i \alpha + \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} P_i \partial_j \alpha.$$

If we single out the terms involving the pairs of indices $(0,0)$ and $(n, n+1)$ in the first sum, we are left with

$$S_n(j_1) - S_n(j_0) + \sum_{(i,j) \neq (0,0),(n,n+1)} (-1)^{i+j} \partial_j P_i \alpha + \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} P_i \partial_j \alpha.$$

Using Lemma 4.2 we see that only the first two summands survive. □

So, finally we can prove the main result of this section:

Theorem 4.5. *(Homotopy invariance)*
If $f, g\colon X \to Y$ are homotopic maps, then they induce the same map on homology.

Proof. Let $H\colon X \times [0,1] \to Y$ be a homotopy from f to g, *i.e.*, $H \circ j_0 = f$ and $H \circ j_1 = g$. Set $K_n := S_{n+1}(H) \circ P$. We claim that $(K_n)_n$ is a chain homotopy between $(S_n(f))_n$ and $(S_n(g))_n$. Note that H induces a chain map $(S_n(H))_n$. Therefore we get

$$\begin{aligned}
\partial \circ S_{n+1}(H) \circ P + S_n(H) \circ P \circ \partial &= S_n(H) \circ \partial \circ P + S_n(H) \circ P \circ \partial \\
&= S_n(H) \circ (\partial \circ P + P \circ \partial) \\
&= S_n(H) \circ (S_n(j_1) - S_n(j_0)) = S_n(H \circ j_1) - S_n(H \circ j_0) \\
&= S_n(g) - S_n(f).
\end{aligned}$$

Hence these two maps are chain homotopic and $H_n(g) = H_n(f)$ for all n. $\square$

COROLLARY 4.6. *If two spaces X, Y are homotopy equivalent, then $H_*(X) \cong H_*(Y)$. In particular, if X is contractible, then*

$$H_*(X) \cong \begin{cases} \mathbb{Z}, & \text{for } * = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Examples. As $\mathbb{R}^n$ is contractible for all n, the above corollary gives that its homology is trivial but in degree zero where it consists of the integers.

As the Möbius strip is homotopy equivalent to $\mathbb{S}^1$, we know that their homology groups are isomorphic.

If you know about vector bundles: the zero section of a vector bundle induces a homotopy equivalence between the base and the total space, hence these two have isomorphic homology groups.

5. The long exact sequence in homology

A typical situation is that there is a subspace A of a topological space X and you might know something about A or X and want to calculate the homology of the other space using that partial information.

But before we can move on to topological applications we need some techniques about chain complexes. We need to know that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.

DEFINITION 5.1. Let A, B, C be abelian groups and

$$A \xrightarrow{f} B \xrightarrow{g} C$$

a sequence of homomorphisms. Then this sequence is *exact*, if the image of f is the kernel of g.

DEFINITION 5.2. Is

$$\ldots \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i-1}} \ldots$$

is a sequence of homomorphisms of abelian groups (indexed over the integers), then this sequence is called *(long) exact*, if it is exact at every A_i, *i.e.*, the image of f_{i+1} is the kernel of f_i for all i.

A sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

Examples. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is a short exact sequence.

If $\iota\colon U \to A$ is a monomorphism, then $0 \longrightarrow U \xrightarrow{\iota} A$ is exact. Similarly, an epimorphism $\varrho\colon B \to Q$ gives rise to an exact sequence $B \xrightarrow{\varrho} Q \longrightarrow 0$ and an isomorphism $\phi\colon A \cong A'$ sits in an exact sequence $0 \longrightarrow A \xrightarrow{\phi} A' \longrightarrow 0$.

A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact iff f is injective, the image of f is the kernel of g and g is an epimorphism. Another equivalent description to view a sequence as above as a chain complex with vanishing homology groups.

DEFINITION 5.3. If A_*, B_*, C_* are chain complexes and $f_*\colon A_* \to B_*$, $g\colon B_* \to C_*$ are chain maps, then we call the sequence

$$A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_*$$

exact, if the image of f_n is the kernel of g_n for all $n \in \mathbb{Z}$.

Thus such an exact sequence of chain complexes is a double ladder

$$\begin{tikzcd}
\vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] \\
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
A_n \arrow[r, "f_n"] \arrow[d, "d"] & B_n \arrow[r, "g_n"] \arrow[d, "d"] & C_n \arrow[d, "d"] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "d"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

in which every row is exact.

Example. Let p be a prime, then

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, "p"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "p^2"] & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathbb{Z}/p\mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\mathbb{Z}/p\mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r, "\pi"] \arrow[d] & \mathbb{Z}/p\mathbb{Z} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

has exact rows and columns, in particular it is an exact sequence of chain complexes. Here, π denotes varying canonical projection maps.

Proposition 5.4. *If* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *is a short exact sequence of chain complexes, then there exists a homomorphism* $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ *for all* $n \in \mathbb{Z}$ *which is natural,* i.e., *if*

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "f"] \arrow[d, "\alpha"] & B_* \arrow[r, "g"] \arrow[d, "\beta"] & C_* \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "f'"] & B'_* \arrow[r, "g'"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram in which the rows are exact then $H_{n-1}(\alpha) \circ \delta = \delta \circ H_n(\gamma)$,

$$\begin{tikzcd}
H_n(C_*) \arrow[r, "\delta"] \arrow[d, "H_n(\gamma)"'] & H_{n-1}(A_*) \arrow[d, "H_{n-1}(\alpha)"] \\
H_n(C'_*) \arrow[r, "\delta"] & H_{n-1}(A'_*)
\end{tikzcd}$$

The method of proof is an instance of a *diagram chase.* The homomorphism δ is called *connecting homomorphism.*

Proof. We show the existence of a δ first and then prove that the constructed map satisfies the naturality condition.

a) Definition of δ:

Is $c \in C_n$ with $d(c) = 0$, then we choose a $b \in B_n$ with $g_n b = c$. This is possible because g_n is surjective. We know that $dg_n b = dc = 0 = g_{n-1} db$ thus db is in the kernel of g_{n-1}, hence it is in the image of f_{n-1}. Thus there is an $a \in A_{n-1}$ with $f_{n-1}a = db$. We have that $f_{n-2}da = df_{n-1}a = ddb = 0$ and as f_{n-2} is injective, this shows that a is a cycle.

We define $\delta[c] := [a]$.

$$\begin{tikzcd} & B_n \ni b \arrow[r, mapsto, "g_n"] & c \in C_n \\ A_{n-1} \ni a \arrow[r, mapsto, "f_{n-1}"] & db \in B_{n-1} & \end{tikzcd}$$

In order to check that δ is well-defined, we assume that there are b and b' with $g_n b = g_n b' = c$. Then $g_n(b - b') = 0$ and thus there is an $\tilde{a} \in A_n$ with $f_n\tilde{a} = b - b'$. Define a' as $a - d\tilde{a}$. Then

$$f_{n-1}a' = f_{n-1}a - f_{n-1}d\tilde{a} = db - db + db' = db'$$

because $f_{n-1}d\tilde{a} = db - db'$. As f_{n-1} is injective, we get that a' is uniquely determined with this property. As a is homologous to a' we get that $[a] = [a'] = \delta[c]$, thus the latter is independent of the choice of b.

In addition, we have to make sure that the value stays the same if we add a boundary term to c, *i.e.*, take $c' = c + d\tilde{c}$ for some $\tilde{c} \in C_{n+1}$. Choose preimages of $c, \tilde{c}$ under g_n and g_{n+1}, *i.e.*, b and $\tilde{b}$ with $g_n b = c$ and $g_{n+1}\tilde{b} = \tilde{c}$. Then the element $b' = b + d\tilde{b}$ has boundary $db' = db$ and thus both choices will result in the same a.

Therefore $\delta\colon H_n(C_*) \to H_{n-1}(A_*)$ is well-defined.

b) We have to show that δ is natural with respect to maps of short exact sequences.

Let $c \in Z_n(C_*)$, then $\delta[c] = [a]$ for a $b \in B_n$ with $g_n b = c$ and an $a \in A_{n-1}$ with $f_{n-1}a = db$. Therefore, $H_{n-1}(\alpha)(\delta[c]) = [\alpha_{n-1}(a)]$.

On the other hand, we have

$$f'_{n-1}(\alpha_{n-1}a) = \beta_{n-1}(f_{n-1}a) = \beta_{n-1}(db) = d\beta_n b$$

and

$$g'_n(\beta_n b) = \gamma_n g_n b = \gamma_n c$$

and we can conclude that by the construction of δ

$$\delta[\gamma_n(c)] = [\alpha_{n-1}(a)]$$

and this shows $\delta \circ H_n(\gamma) = H_{n-1}(\alpha) \circ \delta$. □

With this auxiliary result at hand we can now prove the main result in this section:

Proposition 5.5. *For any short exact sequence*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

of chain complexes we obtain a long exact sequence of homology groups

$$\ldots \xrightarrow{\delta} H_n(A_*) \xrightarrow{H_n(f)} H_n(B_*) \xrightarrow{H_n(g)} H_n(C_*) \xrightarrow{\delta} H_{n-1}(A_*) \xrightarrow{H_{n-1}(f)} \ldots$$

Proof. a) Exactness at the spot $H_n(B_*)$:

We have $H_n(g) \circ H_n(f)[a] = [g_n(f_n(a))] = 0$ because the composition of g_n and f_n is zero. This proves that the image of $H_n(f)$ is contained in the kernel of $H_n(g)$.

For the converse, let $[b] \in H_n(B_*)$ with $[g_n b] = 0$. Then there is a $c \in C_{n+1}$ with $dc = g_n b$. As g_{n+1} is surjective, we find a $b' \in B_{n+1}$ with $g_{n+1}b' = c$. Hence

$$g_n(b - db') = g_n b - dg_{n+1}b' = dc - dc = 0.$$

Exactness gives an $a \in A_n$ with $f_n a = b - db'$ and therefore $f_n a$ is homologous to b and $H_n(f)[a] = [b]$ thus the kernel of $H_n(g)$ is contained in the image of $H_n(f)$.

b) Exactness at the spot $H_n(C_*)$:

Let $b \in H_n(B_*)$, then $\delta[g_n b] = 0$ because b is a cycle, so 0 is the only preimage under f_{n-1} of $db = 0$. Therefore the image of $H_n(g)$ is contained in the kernel of δ.

Now assume that $\delta[c] = 0$, thus in the construction of δ, the a is a boundary, $a = da'$. Then for a preimage of c under g_n, b we have by definition of a

$$d(b - f_n a') = db - df_n a' = db - f_n a = 0.$$

Thus $b - f_n a'$ is a cycle and $g_n(b - f_n a') = g_n b - g_n f_n a' = g_n b - 0 = g_n b = c$, so we found a preimage for $[c]$ and the kernel of δ is contained in the image of $H_n(g)$.

c) Exactness at $H_{n-1}(A_*)$:

Let c be a cycle in $Z_n(C_*)$. Again, we choose a preimage b of c under g_n and an a with $f_{n-1}(a) = db$. Then $H_{n-1}(f)\delta[c] = [f_{n-1}(a)] = [db] = 0$. Thus the image of δ is contained in the kernel of $H_{n-1}(f)$.

If $a \in Z_{n-1}(A_*)$ with $H_{n-1}(f)[a] = 0$. Then $f_{n-1}a = db$ for some $b \in B_n$. Take $c = g_n b$. Then by definition $\delta[c] = [a]$. $\square$

6. The long exact sequence of a pair of spaces

Let X be a topological space and $A \subset X$ a subspace of X. Consider the inclusion map $i\colon A \to X$, $i(a) = a$. We obtain an induced map $S_n(i)\colon S_n(A) \to S_n(X)$, but we know that the inclusion of spaces doesn't have to yield a monomorphism on homology groups. For instance, we can include $A = \mathbb{S}^1$ into $X = \mathbb{D}^2$.

We consider pairs of spaces (X, A).

Definition 6.1. The *relative chain complex of* (X, A) is

$$S_*(X, A) := S_*(X)/S_*(A).$$

Alternatively, $S_n(X, A)$ is isomorphic to the free abelian group generated by all n-simplices $\beta\colon \Delta^n \to X$ whose image is not completely contained in A, *i.e.*, $\beta(\Delta^n) \cap (X \backslash A) \neq \varnothing$.

Definition 6.2.

- Elements in $S_n(X, A)$ are called *relative chains in* (X, A)
- Cycles in $S_n(X, A)$ are chains c with $\partial^X(c)$ whose generators have image in A. These are *relative cycles.*
- Boundaries in $S_n(X, A)$ are chains c in X such that $c = \partial^X b + a$ where a is a chain in A.

The following facts are immediate from the definition:

(a) $S_n(X, \varnothing) \cong S_n(X)$.
(b) $S_n(X, X) = 0$.
(c) $S_n(X \sqcup X', X') \cong S_n(X)$.

Definition 6.3. The *relative homology groups of* (X, A) are

$$H_n(X, A) := H_n(S_*(X, A)).$$

Theorem 6.4. *For any pair of topological spaces $A \subset X$ we obtain a long exact sequence*

$$\ldots \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(i)} H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

For a map of spaces $f\colon X \to Y$ with $f(A) \subset B \subset Y$, we get an induced map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r, "\delta"] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(f|_A)"] & H_n(X) \arrow[r] \arrow[d, "H_n(f)"] & H_n(X, A) \arrow[r, "\delta"] \arrow[d, "H_n(f)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(i)"] \arrow[d, "H_{n-1}(f|_A)"] & \cdots \\
\cdots \arrow[r, "\delta"'] & H_n(B) \arrow[r, "H_n(i)"'] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r, "\delta"'] & H_{n-1}(B) \arrow[r, "H_{n-1}(i)"'] & \cdots
\end{tikzcd}$$

A map $f\colon X \to Y$ with $f(A) \subset B$ is denoted by $f\colon (X, A) \to (Y, B)$.

Proof. By definition of $S_*(X,A)$ the sequence

$$0 \longrightarrow S_*(A) \xrightarrow{S_*(i)} S_*(X) \xrightarrow{\pi} S_*(X,A) \longrightarrow 0$$

is an exact sequence of chain complexes and by Proposition 5.5 we obtain the first claim.

For a map f as above the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S_n(A) \arrow[r, "S_n(i)"] \arrow[d, "S_n(f|_A)"] & S_n(X) \arrow[r, "\pi"] \arrow[d, "S_n(f)"] & S_n(X,A) \arrow[r] \arrow[d, "S_n(f)/S_n(f|A)"] & 0 \\
0 \arrow[r] & S_n(B) \arrow[r, "S_n(i)"] & S_n(Y) \arrow[r, "\pi"] & S_n(Y,B) \arrow[r] & 0
\end{tikzcd}$$

commutes. □

Example. Let $A = \mathbb{S}^{n-1}$ and $X = \mathbb{D}^n$, then we know that $H_j(i)$ is not injective for $j > 0$. From the long exact sequence we get that $\delta\colon H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$ for $j > 1$ and $n > 1$.

Proposition 6.5. *If $i\colon A \hookrightarrow X$ is a weak retract,* i.e., *if there is an $r\colon X \to A$ with $r \circ i \simeq \mathrm{id}_A$, then*

$$H_n(X) \cong H_n(A) \oplus H_n(X,A), \quad 0 \leqslant n.$$

Proof. From the assumption we get that $H_n(r) \circ H_n(i) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}$ for all n and hence $H_n(i)$ is injective for all n. This implies that $0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X)$ is exact. Injectivity of $H_{n-1}(i)$ yields that the image of $\delta\colon H_n(X,A) \to H_{n-1}(A)$ is trivial. Therefore we get short exact sequences

$$0 \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \to 0$$

for all n. As $H_n(r)$ is a left-inverse for $H_n(i)$ we obtain a splitting

$$H_n(X) \cong H_n(A) \oplus H_n(X,A)$$

because we map $[c] \in H_n(X)$ to $([rc], \pi_*[c])$ with inverse

$$H_n(A) \oplus H_n(X,A) \ni ([a],[b]) \mapsto H_n(i)[a] + [a'] - H_n(i \circ r)[a'] \in H_n(X)$$

for any $[a'] \in H_n(X)$ with $\pi_*[a'] = [b]$. □

Proposition 6.6. *For any $\varnothing \neq A \subset X$ such that $A \subset X$ is a deformation retract, then*

$$H_n(i)\colon H_n(A) \cong H_n(X), \quad H_n(X,A) \cong 0, \quad 0 \leqslant n.$$

Proof. Recall, that $i\colon A \hookrightarrow X$ is a *deformation retract*, if there is a homotopy $R\colon X \times [0,1] \to X$ such that

(a) $R(x,0) = x$ for all $x \in X$,
(b) $R(x,1) \in A$ for all $x \in X$, and
(c) $R(a,1) = a$ for all $a \in A$.

In particular, R is a homotopy from id_X to $i \circ r$ where $r = R(-,1)\colon X \to A$. Condition (c) can be rewritten as $r \circ i = \mathrm{id}_A$, *i.e.*, r is a retraction, and thus A and X are homotopically equivalent and $H_n(i)$ is an isomorphism for all $n \geqslant 0$. □

Definition 6.7. If X has two subspaces $A, B \subset X$, then (X,A,B) is called a *triple*, if $B \subset A \subset X$.

Any triple gives rise to three pairs of spaces (X,A), (X,B) and (A,B) and accordingly we have three long exact sequence in homology. But there is another one.

Proposition 6.8. *For any triple (X,A,B) there is a natural long exact sequence*

$$\ldots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \xrightarrow{\delta} H_{n-1}(A,B) \longrightarrow \ldots$$

This sequence is part of the following braided commutative diagram displaying four long exact sequences

$$\begin{tikzcd}
\cdots \arrow[dr] & & & & & & \cdots \\
& H_{n+1}(X,A) \arrow[dr] \arrow[rr, bend left] & & H_n(A,B) \arrow[dr] \arrow[rr, bend left] & & H_{n-1}(B) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & H_n(A) \arrow[ur] \arrow[dr] & & H_n(X,B) \arrow[ur] \arrow[dr] & & \cdots \\
& H_n(B) \arrow[ur] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X,A) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & & & & & \cdots
\end{tikzcd}$$

In particular, the connecting homomorphism $\delta\colon H_n(X,A) \to H_{n-1}(A,B)$ *is the composite* $\delta = \pi_*^{(A,B)} \circ \delta^{(X,A)}$.

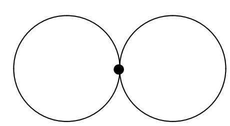
Proof. Consider the sequence

$$0 \longrightarrow S_n(A)/S_n(B) \longrightarrow S_n(X)/S_n(B) \longrightarrow S_n(X)/S_n(A) \longrightarrow 0.$$

This sequence is exact, because $S_n(B) \subset S_n(A) \subset S_n(X)$. $\square$

7. Excision

The aim is to simplify relative homology groups. Let $A \subset X$ be a subspace. Then it is easy to see that $H_*(X,A)$ is not isomorphic to $H_*(X \backslash A)$: Consider the figure eight as X and A as the point connecting the two copies of $\mathbb{S}^1$, then $H_0(X,A)$ is trivial, but $H_0(X \backslash A) \cong \mathbb{Z} \oplus \mathbb{Z}$.



So if we want to simplify $H_*(X,A)$ by excising something, then we have to be more careful. The first step towards that is to make singular simplices 'smaller'. The technique is called barycentric subdivision and is a tool that's frequently used.

First, we construct cones. Let $v \in \Delta^p$ and let $\alpha\colon \Delta^n \to \Delta^p$ be a singular n-simplex in Δ^p.

Definition 7.1. The *cone of α with respect to v* is $K_v(\alpha)\colon \Delta^{n+1} \to \Delta^p$,

$$(t_0, \ldots, t_{n+1}) \mapsto \begin{cases} (1-t_{n+1})\alpha(\frac{t_0}{1-t_{n+1}}, \ldots, \frac{t_n}{1-t_{n+1}}) + t_{n+1}v, & t_{n+1} < 1 \\ v, & t_{n+1} = 1. \end{cases}$$

This map is well-defined and continuous. On the standard basis vectors K_v gives $K_v(e_i) = \alpha(e_i)$ for $0 \leqslant i \leqslant n$ but $K_v(e_{n+1}) = v$. Extending K_v linearly gives a map

$$K_v\colon S_n(\Delta^p) \to S_{n+1}(\Delta^p).$$

Lemma 7.2. *The map K_v satisfies*

- $\partial K_v(c) = \varepsilon(c).\kappa_v - c$ *for* $c \in S_0(\Delta^p)$, $\kappa_v(e_0) = v$ *and* ε *the augmentation.*
- *For* $n > 0$ *we have that* $\partial \circ K_v - K_v \circ \partial = (-1)^{n+1}\mathrm{id}$.

Proof. For a singular 0-simplex $\alpha\colon \Delta^0 \to \Delta^p$ we know that $\varepsilon(\alpha) = 1$ and we calculate

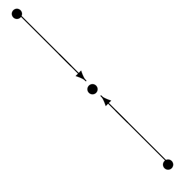
$$\partial K_v(\alpha)(e_0) = K_v(\alpha) \circ d_0(e_0) - K_v(\alpha) \circ d_1(e_0) = K_v(\alpha)(e_1) - K_v(\alpha)(e_0) = v - \alpha(e_0).$$

For $n > 0$ we have to calculate $\partial_i K_v(\alpha)$ and it is straightforward to see that $\partial_{n+1} K(\alpha) = \alpha$ and $\partial_i(K_v(\alpha)) = K_v(\partial_i \alpha)$ for all $i < n+1$. $\square$

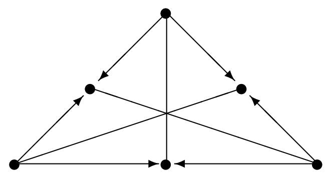
Definition 7.3. For $\alpha\colon \Delta^n \to \Delta^p$ let $v(\alpha) = v := \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$. The *barycentric subdivision* $B\colon S_n(\Delta_p) \to S_n(\Delta_p)$ is defined inductively as $B(\alpha) = \alpha$ for $\alpha \in S_0(\Delta_p)$ and $B(\alpha) = (-1)^n K_v(B(\partial\alpha))$ for $n > 0$.

For $n \geqslant 1$ this yields $B(\alpha) = \sum_{i=0}^n (-1)^{n+i} K_v(B(\partial_i \alpha))$.

If we take $n = p$ and $\alpha = \mathrm{id}_{\Delta^n}$, then for small n this looks as follows: You cannot subdivide a point any further. For $n = 1$ we get



And for $n = 2$ we get (up to tilting)



Lemma 7.4. *The barycentric subdivision is a chain map.*

Proof. We have to show that $\partial B = B\partial$. If α is a singular zero chain, then $\partial B\alpha = \partial\alpha = 0$ and $B\partial\alpha = B(0) = 0$.

Let $n = 1$. Then

$$\partial B\alpha = -\partial K_v B(\partial_0\alpha) + \partial K_v B(\partial_1 \alpha).$$

But the boundary terms are zero chains and there B is the identity so we get

$$-\partial K_v(\partial_0\alpha) + \partial K_v(\partial_1\alpha) = -\kappa_v + \partial_0\alpha + \kappa_v - \partial_1\alpha = \partial\alpha = B\partial\alpha.$$

(Note, that the v is $v(\alpha)$, not a $v(\partial_i\alpha)$.)

We prove the claim inductively, so let $\alpha \in S_n(\Delta^p)$. Then

$$\begin{aligned}\partial B\alpha =&(-1)^n \partial K_v(B\partial\alpha)\\ =&(-1)^n((-1)^n B\partial\alpha + K_v\partial B\partial\alpha)\\ =&B\partial\alpha + (-1)^n K_v B\partial\partial\alpha = B\partial\alpha.\end{aligned}$$

Here, the first equality is by definition, the second one follows by Lemma 7.2 and then we use the induction hypothesis and the fact that $\partial\partial = 0$. $\square$

Our aim is to show that B doesn't change anything on the level of homology groups and to that end we prove that it is chain homotopic to the identity.

We construct $\psi_n\colon S_n(\Delta^p) \to S_{n+1}(\Delta^p)$ again inductively as

$$\psi_0(\alpha) := 0, \quad \psi_n(\alpha) := (-1)^{n+1} K_v(B\alpha - \alpha - \psi_{n-1}\partial\alpha)$$

with $v = \frac{1}{n+1}\sum_{i=0}^n \alpha(e_i)$.

Lemma 7.5. *The sequence $(\psi_n)_n$ is a chain homotopy from B to the identity.*

Proof. For $n = 0$ we have $\partial\psi_0 = 0$ and this agrees with $B - \mathrm{id}$ in that degree.

For $n = 1$, we get

$$\partial\psi_1 + \psi_0\partial = \partial\psi_1 = \partial(K_v B - K_v - K_v\psi_0\partial) = \partial K_v B - \partial K_v.$$

with Lemma 7.2 we can transform the latter to $B + K_v\partial B - \partial K_v$ and as B is a chain map, this is $B + K_v B\partial - \partial K_v$. In chain degree one $B\partial$ agrees with ∂, thus this reduces to

$$B + K_v\partial - \partial K_v = B - (\partial K_v - K_v\partial) = B - \mathrm{id}.$$

So, finally we can do the inductive step:

$$\begin{aligned}
\partial\psi_n =& (-1)^{n+1}\partial K_v(B - \mathrm{id} - \psi_{n-1}\partial) \\
=& (-1)^{n+1}\partial K_v B - (-1)^{n+1}\partial K_v - (-1)^{n+1}\partial K_v\psi_{n-1}\partial \\
=& (-1)^{n+1}((-1)^{n+1}B + K_v\partial B) \\
&\qquad - (-1)^{n+1}((-1)^{n+1}\mathrm{id} + K_v\partial) \\
&\qquad - (-1)^{n+1}((-1)^{n+1}\psi_{n-1}\partial + K_v\partial\psi_{n-1}\partial \\
=& B - \mathrm{id} - \psi_{n-1}\partial + \text{remaining terms}
\end{aligned}$$

The equation

$$K_v\partial\psi_{n-1}\partial + K_v\psi_{n-2}\partial^2 = K_v B\partial - K_v\partial$$

from the inductive assumption ensures that these terms give zero. □

Definition 7.6. A singular n-simplex $\alpha\colon \Delta^n \to \Delta^p$ is called *affine*, if

$$\alpha(\sum_{i=0}^n t_i e_i) = \sum_{i=0}^n t_i\alpha(e_i).$$

We abbreviate $\alpha(e_i)$ with v_i, so $\alpha(\sum_{i=0}^n t_i e_i) = \sum_{i=0}^n t_i v_i$ and we call the v_i's the *vertices of* α.

Definition 7.7. Let A be a subset of a metric space (X, d). The *diameter of* A is

$$\sup\{d(x,y) | x, y \in A\}$$

and we denote it by $\mathrm{diam}(A)$.

Accordingly, the *diameter of an affine n-simplex α in Δ^p* is the diameter of its image, and we abbreviate that with $\mathrm{diam}(\alpha)$.

Lemma 7.8. *For any affine α with vertices v_i, each simplex in the chain $B\alpha$ has diameter $\leqslant \frac{n}{n+1}\mathrm{diam}(\alpha)$.*

Either you believe this lemma, or you prove it, or you check Bredon, Proof of Lemma 13.7 (p. 226).

Each simplex in $B\alpha$ is again affine; this allows us to iterate the application of B and get smaller and smaller diameter. Thus, the k-fold iteration, $B^k(\alpha)$, has diameter at most $\left(\frac{n}{n+1}\right)^k \mathrm{diam}(\alpha)$.

In the following we use the easy but powerful trick to express α as

$$\alpha = \alpha \circ \mathrm{id}_{\Delta^n} = S_n(\alpha)(\mathrm{id}_{\Delta^n}).$$

This allows us to use the barycentric subdivision for general spaces.

Definition 7.9. (a) We define $B^X\colon S_n(X) \to S_n(X)$ as

$$B^X(\alpha) := S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}).$$

(b) Similarly, $\psi^X\colon S_n(X) \to S_{n+1}(X)$ is

$$\psi_n^X(\alpha) := S_{n+1}(\alpha) \circ \psi_n(\mathrm{id}_{\Delta^n}).$$

Lemma 7.10. *The maps B^X are natural in X and are homotopic to the identity on $S_n(X)$.*

Proof. Let $f\colon X \to Y$ be a continuous map. We have

$$\begin{aligned}
S_n(f)B^X(\alpha) =& S_n(f) \circ S_n(\alpha) \circ B(\mathrm{id}_{\Delta^n}) \\
=& S_n(f \circ \alpha) \circ B(\mathrm{id}_{\Delta^n}) \\
=& B^Y(f \circ \alpha).
\end{aligned}$$

The calculation for $\partial\psi^X + \psi^X\partial = B^X - \mathrm{id}_{S_n(X)}$ should be routine by now. □

Now we consider singular n-chains that are spanned by 'small' singular n-simplices.

Definition 7.11. Let $\mathfrak{U} = \{U_i, i \in I\}$ be an open covering of X. Then $S_n^{\mathfrak{U}}(X)$ is the free abelian group generated by all $\alpha\colon \Delta^n \to X$ such that the image of Δ^n under α is contained in one of the $U_i \in \mathfrak{U}$.

Note that $S_n^{\mathfrak{U}}(X)$ is an abelian subgroup of $S_n(X)$. As we will see now, these chains suffice to detect everything in singular homology.

Lemma 7.12. *Every chain in $S_n(X)$ is homologous to a chain in $S_n^{\mathfrak{U}}(X)$.*

Proof. Let $\alpha = \sum_{j=1}^m \lambda_j \alpha_j \in S_n(X)$ and let L_j for $1 \leqslant j \leqslant m$ be the Lebesgue numbers for the coverings $\{\alpha_j^{-1}(U_i), i \in I\}$ of Δ^n. Choose a k, such that $\left(\frac{n}{n+1}\right)^k \leqslant L_1, \ldots, L_m$. Then $B^k\alpha_1$ up to $B^k\alpha_m$ are all in $S_n^{\mathfrak{U}}(X)$. Therefore

$$B^k(\alpha) = \sum_{j=1}^m \lambda_j B^k(\alpha_j) =: \alpha' \in S_n^{\mathfrak{U}}(X).$$

As B is homotopic to the identity we have

$$\alpha \sim B\alpha \sim \ldots \sim B^k\alpha = \alpha'.$$

$\square$

With this we get the main result of this section:

Theorem 7.13. *Let $W \subset A \subset X$ such that $\bar{W} \subset \mathring{A}$. Then the inclusion $i\colon (X\backslash W, A\backslash W) \hookrightarrow (X, A)$ induces an isomorphism*

$$H_n(i)\colon H_n(X\backslash W, A\backslash W) \cong H_n(X, A)$$

for all $n \geqslant 0$.

Proof. We first prove that $H_n(i)$ is surjective, so let $c \in S_n(X, A)$ be a relative cycle, *i.e.*, let $\partial c \in S_{n-1}(A)$. There is a k such that $c' := B^k c$ is a chain in $S_n^{\mathfrak{U}}(X)$ for the open covering $\mathfrak{U} = \{\mathring{A}, X\backslash\bar{W}\} =: \{U, V\}$. We decompose c' as $c' = c^U + c^V$ with c^U and c^V being elements in the corresponding chain complex. (This decomposition is not unique.)

We know that the boundary of c' is $\partial c' = \partial B^k c = B^k \partial c$ and by assumption this is a chain in $S_{n-1}(A)$. But $\partial c' = \partial c^U + \partial c^V$ with $\partial c^U \in S_{n-1}(U) \subset S_{n-1}(A)$. Thus, $\partial c^V \in S_{n-1}(A)$, in fact, $\partial c^V \in S_{n-1}(A\backslash W)$ and therefore c^V is a relative cycle in $S_n(X\backslash W, A\backslash W)$. This shows that $H_n(i)[c^V] = [c] \in H_n(X, A)$ because $[c] = [c^U + c^V] = [c^V]$ in $H_n(X, A)$.

The injectivity of $H_n(i)$ is shown as follows. Assume that there is a $c \in S_n(X\backslash W)$ with $\partial c \in S_{n-1}(A\backslash W)$ and assume $H_n(i)[c] = 0$, *i.e.*, c is of the form $c = \partial b + a'$ with $b \in S_{n+1}(X)$ and $a' \in S_n(A)$ and write b as $b^U + b^V$ with $b^U \in S_{n+1}(U) \subset S_{n+1}(A)$ and $b^V \in S_{n+1}(V) \subset S_{n+1}(X\backslash W)$. Then

$$c = \partial b^U + \partial b^V + a'.$$

But ∂b^U and a' are elements in $S_n(A\backslash W)$ and hence $c = \partial b^V \in S_n(X\backslash W, A\backslash W)$. $\square$

8. Mayer-Vietoris sequence

We consider the following situation: there are subspaces $X_1, X_2 \subset X$ such that X_1 and X_2 are open in X and such that $X = X_1 \cup X_2$. We consider the open covering $\mathfrak{U} = \{X_1, X_2\}$. We need the following maps:

$$\begin{tikzcd} & X_1 \arrow[dr, "\kappa_1"] & \\ X_1 \cap X_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X. \\ & X_2 \arrow[ur, "\kappa_2"'] & \end{tikzcd}$$

Note that by definition, the sequence

$$0 \longrightarrow S_*(X_1 \cap X_2) \xrightarrow{(i_1,i_2)} S_*(X_1) \oplus S_*(X_2) \longrightarrow S_*^{\mathfrak{U}}(X) \longrightarrow 0 \tag{8.1}$$

is exact. Here, the second map is

$$(\alpha_1, \alpha_2) \mapsto \kappa_1(\alpha_1) - \kappa_2(\alpha_2).$$

THEOREM 8.1. *(The Mayer-Vietoris sequence)*
There is a long exact sequence

$$\ldots \xrightarrow{\delta} H_n(X_1 \cap X_2) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \xrightarrow{\delta} H_{n-1}(X_1 \cap X_2) \longrightarrow \ldots$$

PROOF. The proof follows from Lemma 7.12, because $H_n^{\mathfrak{U}}(X) \cong H_n(X)$. □

As an application, we calculate the homology groups of spheres. Let $X = \mathbb{S}^m$ and let $X^{\pm} := \mathbb{S}^m \setminus \{\mp e_{m+1}\}$. The subspaces X^+ and X^- are contractible and therefore $H_*(X^{\pm}) = 0$ for all positive $*$.

The Mayer-Vietoris sequence is as follows

$$\ldots \xrightarrow{\delta} H_n(X^+ \cap X^-) \longrightarrow H_n(X^+) \oplus H_n(X^-) \longrightarrow H_n(\mathbb{S}^m) \xrightarrow{\delta} H_{n-1}(X^+ \cap X^-) \longrightarrow \ldots$$

For positive n we can deduce

$$H_n(\mathbb{S}^m) \cong H_{n-1}(X^+ \cap X^-) \cong H_{n-1}(\mathbb{S}^{m-1}).$$

The first map is the connecting homomorphism and the second map is $H_{n-1}(i)\colon H_{n-1}(\mathbb{S}^{m-1}) \to H_{n-1}(X^+ \cap X^-)$ where i is the inclusion of $\mathbb{S}^{m-1}$ into $X^+ \cap X^-$ and this inclusion is a homotopy equivalence. Thus define $D := H_{n-1}(i)^{-1} \circ \delta$. This D is an isomorphism for all $n \geqslant 2$.

We have to controll what is going on in small degrees and dimensions.

In order to compute $H_1(\mathbb{S}^m)$ for $m > 1$, we have to understand the map

$$\mathbb{Z} \cong H_0(X^+ \cap X^-) \to H_0(X_1) \oplus H_0(X_2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let 1 be a base point of $X^+ \cap X^-$. Then the map on H_0 is

$$[1] \mapsto ([1], [1]).$$

This map is injective and therefore the connecting homomorphism $\delta\colon H_1(\mathbb{S}^m) \to H_0(X^+ \cap X^-)$ is trivial and we obtain that

$$H_1(\mathbb{S}^m) \cong 0, \quad m > 1.$$

(Of course, we knew this from the Hurewicz isomorphism.)

Next, we consider the case of $n = 1 = m$. In this case the intersection $X^+ \cap X^-$ splits into two components. We choose a $P_+ \in X^+$ and a $P_- \in X^-$. Then, for $H_0(i_1, i_2)$ we have

$$H_0(X^+) \oplus H_0(X^-) \ni (H_0(i_1)[P_+], H_0(i_2)[P_-]) \sim ([e_2], [-e_2]).$$

Thus $[P_+] \mapsto ([e_2], 0)$ and $[P_-] \mapsto (0, [-e_2])$ and the difference $[P_+] - [P_-] \in H_0(X^+ \cap X^-)$ generates the kernel of $H_0(\kappa_1) - H_0(\kappa_2)$:

$$(H_0(\kappa_1) - H_0(\kappa_2))([e_2], [e_2]) = 0.$$

Consider the exact sequence

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} H_0(X^+ \cap X^-) \xrightarrow{(H_0(i_1), H_0(i_2))} H_0(X^+) \oplus H_0(X^-) \longrightarrow H_0\mathbb{S}^1$$

which gives

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}.$$

where $[P_+] - [P_-] \mapsto ([e_2], [e_2]) \mapsto 0$. The image of $(H_0(i_1), H_0(i_2))$ is isomorphic to the kernel of the difference of $H_0(\kappa_1)$ and $H_0(\kappa_2)$ and this is isomorphic to the free abelian group generated by $([e_2], [e_2])$ which is $\mathbb{Z}$.

Therefore

$$0 \longrightarrow H_1\mathbb{S}^1 \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

is short exact and $H_1\mathbb{S}^1 \cong \mathbb{Z}$. (This we knew from the Hurewicz isomorphism as well.)

For $0 < n < m$ we get

$$H_n\mathbb{S}^m \xrightarrow{\cong} H_{n-1}\mathbb{S}^{m-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^{m-n+1}) \cong \pi_1(\mathbb{S}^{m-n+1}).$$

and the latter is trivial.

Similarly, for $0 < m < n$ we have

$$H_n\mathbb{S}^m \xrightarrow{\cong} H_{n-1}\mathbb{S}^{m-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_{n-m+1}(\mathbb{S}^1) \cong 0.$$

The last claim follows directly by another simple Mayer-Vietoris argument.

The remaining case $0 < m = n$ gives something non-trivial

$$H_n\mathbb{S}^n \xrightarrow{\cong} H_{n-1}\mathbb{S}^{n-1} \xrightarrow{\cong} \ldots \xrightarrow{\cong} H_1(\mathbb{S}^1) \cong \mathbb{Z}.$$

We can summarize the result as follows.

Proposition 8.2.

$$H_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = m = 0, \\ \mathbb{Z}, & n = 0, m > 0, \\ \mathbb{Z}, & n = m > 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 8.3. Let $\mu_0 := [P_+] - [P_-] \in H_0(X^+ \cap X^-) \cong H_0(\mathbb{S}^0)$ and let $\mu_1 \in H_1(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1)$ be given by the degree one map (aka the class of the identity on $\mathbb{S}^1$, aka the class of the loop $t \mapsto e^{2\pi i t}$).

Define the higher μ_n via $D\mu_n = \mu_{n-1}$. Then μ_n is called *the fundamental class in* $H_n(\mathbb{S}^n)$.

In order to obtain a relative version of the Mayer-Vietoris sequence, we need a tool from homological algebra.

Lemma 8.4. *(The five-lemma)*
Let

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \end{tikzcd}$$

be a commutative diagram of exact sequences. If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Proof. Again, we are chasing diagrams.

In order to prove that f_3 is injective, assume that there is an $a \in A_3$ with $f_3 a = 0$. Then $\beta_3 f_3 a = f_4 \alpha_3 a = 0$, as well. But f_4 is injective, thus $\alpha_3 a = 0$. Exactness of the top row gives, that there is an $a' \in A_2$ with $\alpha_2 a' = a$. This implies

$$f_3 \alpha_2 a' = f_3 a = 0 = \beta_2 f_2 a'.$$

Exactness of the bottom row gives us a $b \in B_1$ with $\beta_1 b = f_2 a'$, but f_1 is an isomorphism so we can lift b to $a_1 \in A_1$ with $f_1 a_1 = b$.

Thus $f_2 \alpha_1 a_1 = \beta_1 b = f_2 a'$ and as f_2 is injective, this implies that $\alpha_1 a_1 = a'$. So finally we get that $a = \alpha_2 a' = \alpha_2 \alpha_1 a_1$, but the latter is zero, thus $a = 0$.

For the surjectivity of f_3 assume $b \in B_3$ is given. Move b over to B_4 via β_3 and set $a := f_4^{-1} \beta_3 b$. (Note here, that if $\beta_3 b = 0$ we actually get a shortcut: Then there is a $b_2 \in B_2$ with $\beta_2 b_2 = b$ and thus an $a_2 \in A_2$ with $f_2 a_2 = b_2$. Then $f_3 \alpha_2 a_2 = f_2 b_2 = b$.)

Consider $f_5 \alpha_4 a$. This is equal to $\beta_4 \beta_3 b$ and hence trivial. Therefore $\alpha_4 a = 0$ and thus there is an $a' \in A_3$ with $\alpha_3 a' = a$. Then $b - f_3 a'$ is in the kernel of β_3 because

$$\beta_3(b - f_3 a') = \beta_3 b - f_4 \alpha_3 a' = \beta_3 b - f_4 a = 0.$$

Hence we get a $b_2 \in B_2$ with $\beta_2 b_2 = b - f_3 a'$. Define a_2 as $f_2^{-1}(b_2)$, so $a' + \alpha_2 a_2$ is in A_3 and

$$f_3(a' + \alpha_2 a_2) = f_3 a' + \beta_2 f_2 a_2 = f_3 a' + \beta_2 b_2 = f_3 a' + b - f_3 a' = b.$$

$\square$

We now consider a relative situation, so let X be a topological space with $A, B \subset X$ open in $A \cup B$ and set $\mathfrak{U} := \{A, B\}$. This is an open covering of $A \cup B$. The following diagram of exact sequences combines absolute chains with relative ones:

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & S_n(A \cup B) \arrow[d] \\
0 \arrow[r] & S_n(A \cap B) \arrow[r] \arrow[d] & S_n(A) \oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] \arrow[urr, "\varphi"] & 0 \arrow[d] & S_n(X) \arrow[d] \\
0 \arrow[r] & S_n(X) \arrow[r, "\Delta"] \arrow[d] & S_n(X) \oplus S_n(X) \arrow[r, "\mathrm{diff}"] \arrow[d] & S_n(X) \arrow[r] \arrow[d] \arrow[urr, equal] & 0 \arrow[d] & S_n(X, A \cup B) \arrow[d] \\
0 \arrow[r] & S_n(X, A \cap B) \arrow[r] \arrow[d] & S_n(X, A) \oplus S_n(X, B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] \arrow[urr, "\psi"] & 0 & 0 \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

Here, ψ is induced by the inclusion $\varphi\colon S_n^{\mathfrak{U}}(A \cup B) \to S_n(A \cup B)$, Δ denotes the diagonal map and diff the difference map. It is clear that the first two rows are exact. That the third row is exact follows by a version of the nine-lemma or a direct diagram chase.

Consider the two right-most non-trivial columns in this diagram. Each gives a long exact sequence in homology and we focus on five terms.

$$\begin{tikzcd}
H_n(S_*^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\varphi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_*(X)/S_*^{\mathfrak{U}}(A \cup B)) \arrow[r, "\delta"] \arrow[d, "H_n(\psi)"'] & H_{n-1}(S_*^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_{n-1}(\varphi)"'] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r, "\delta"] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

Then by the five-lemma, as $H_n(\varphi)$ and $H_{n-1}(\varphi)$ are isomorphisms, so is $H_n(\psi)$. This observation together with the bottom non-trivial exact row proves the following.

Theorem 8.5. *(Relative Mayer-Vietoris sequence)*
If $A, B \subset X$ are open in $A \cup B$, then the following sequence is exact:

$$\ldots \xrightarrow{\delta} H_n(X, A \cap B) \longrightarrow H_n(X, A) \oplus H_n(X, B) \longrightarrow H_n(X, A \cup B) \xrightarrow{\delta} \quad \ldots$$

9. Reduced homology and suspension

For any path-connected space we have that the zeroth homology is isomorphic to the integers, so somehow this copy of $\mathbb{Z}$ is superfluous information and we want to get rid of it in a civilized manner. Let P denote the one-point topological space. Then for any space X there is a continuous map $\varepsilon\colon X \to P$.

Definition 9.1. We define $\widetilde{H}_n(X) := \ker(H_n(\varepsilon)\colon H_n(X) \to H_n(P))$ and call it the *reduced n-th homology group of the space X*.

- Note that $\widetilde{H}_n(X) \cong H_n(X)$ for all positive n.
- If X is path-connected, then $\widetilde{H}_0(X) = 0$.
- For any choice of a base point $x \in X$ we get

$$\widetilde{H}_n(X) \oplus H_n(\{x\}) \cong H_n(X)$$

because $H_n(P) \cong H_n(\{x\})$ and the composition

$$\{x\} \hookrightarrow X \to \{x\}$$

is the identity. Therefore, $\widetilde{H}_n(X) \cong H_n(X, \{x\})$ because the retraction $r\colon X \to \{x\}$ splits the exact sequence

$$\dots H_n(\{x\}) \to H_n(X) \to H_n(X, \{x\}) \to \dots$$

- We can prolong the singular chain complex $S_*(X)$ and consider $\widetilde{S}_*(X)$:

$$\dots \to S_1(X) \to S_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

where $\varepsilon(\alpha) = 1$ for every singular 0-simplex α. This is precisely the augmentation we considered before. Then for all $n \geqslant 0$,

$$\widetilde{H}_*(X) \cong H_*(\widetilde{S}_*(X)).$$

As every continuous map $f\colon X \to Y$ induces a chain map $S_*(f)\colon S_*(X) \to S_*(Y)$ and as $\varepsilon^Y \circ S_0(f) = \varepsilon^X$ we obtain the following result.

Lemma 9.2. *The assignment $X \mapsto H_*(\widetilde{S}_*(X))$ is a functor,* i.e., *for a continuous $f\colon X \to Y$ we get an induced map $H_*(\widetilde{S}_*(f))\colon H_*(\widetilde{S}_*(X)) \to H_*(\widetilde{S}_*(Y))$ such that the identity on X induces the identity and composition of maps is respected.*

Similarly, $\widetilde{H}_*(-)$ is a functor.

Definition 9.3. For $\varnothing \neq A \subset X$ we define

$$\widetilde{H}_n(X, A) := H_n(X, A).$$

As we identified reduced homology groups with relative homology groups we obtain a reduced version of the Mayer-Vietoris sequence. A similar remark applies to the long exact sequence for a pair of spaces.

Proposition 9.4. *For each pair of spaces, there is a long exact sequence*

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

and a reduced Mayer-Vietoris sequence.

Examples.

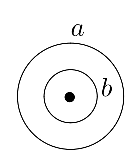
1) Recall that we can express $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:

$$\mathbb{R}P^2 \cong \mathbb{S}^2/\pm\mathrm{id} \cong \mathbb{D}^2/z \sim -z \text{ for } z \in \mathbb{S}^1.$$

We use the latter definition and set $X = \mathbb{R}P^2$, $A = X \backslash \{[0,0]\}$ (which is an open Möbius strip and hence homotopically equivalent to $\mathbb{S}^1$) and $B = \mathring{\mathbb{D}}^2$. Then

$$A \cap B = \mathring{\mathbb{D}}^2 \backslash \{[0,0]\} \simeq \mathbb{S}^1.$$

Thus we know that $H_1(A) \cong 0$, $H_1(B) \cong \mathbb{Z}$ and $H_2 A = H_2 B = 0$. We choose generators for $H_1(A)$ and $H_1(A \cap B)$ as follows.



Let a be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1,0)$. Let b be the loop that runs along the inner circle in mathematical positive direction. Then the inclusion $i_{A\cap B}\colon A\cap B\to A$ induces

$$H_1(i_A)[b] = 2[a].$$

This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is

$$H_2A\oplus H_2B = 0\to \widetilde{H}_2(X)\to \widetilde{H}_1(A\cap B)\cong\mathbb{Z}\to\widetilde{H}_1(A)\cong\mathbb{Z}\to\widetilde{H}_1(X)\to\widetilde{H}_0(A\cap B)=0.$$

On the two copies of the integers, the map is given as above and thus we obtain:

$$\begin{aligned}
&H_2(\mathbb{R}P^2)\cong \ker(2\cdot\colon\mathbb{Z}\to\mathbb{Z}) = 0,\\
&H_1(\mathbb{R}P^2)\cong \operatorname{coker}(2\cdot\colon\mathbb{Z}\to\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z},\\
&H_0(\mathbb{R}P^2)\cong\mathbb{Z}.
\end{aligned}$$

The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

2) We can now calculate the homology groups of bouquets of spaces in terms of the homology groups of the single spaces, at least in good cases. Let $(X_i)_{i\in I}$ be a family of topological spaces with chosen basepoints $x_i\in X_i$. Consider

$$X=\bigvee_{i\in I}X_i.$$

If the inclusion of x_i into X_i is pathological, then we cannot apply the Mayer-Vietoris sequence. However, we get the following:

Proposition 9.5. *If there are neighbourhoods U_i of $x_i\in X_i$ together with a deformation of U_i to $\{x_i\}$, then we have for any finite $E\subset I$*

$$\widetilde{H}_n(\bigvee_{i\in E}X_i)\cong\bigoplus_{i\in E}\widetilde{H}_n(X_i).$$

In the situation above we say that the X_i are *well-pointed* with respect to x_i.

Proof. First we consider the case of two bouquet summands. We have $X_1\vee U_2\cup U_1\vee X_2$ as an open covering of $X_1\vee X_2$. The Mayer-Vietoris sequence then gives that $H_n(X)\cong H_n(X_1\vee U_2)\oplus H_n(U_1\vee X_2)$ for $n>0$. For H_0 we get the exact sequence

$$0\to\widetilde{H}_0(X_1\vee U_2)\oplus\widetilde{H}_0(U_1\vee X_2)\to H_0(X)\to 0.$$

By induction we obtain the case of finitely many bouquet summands. □

We also get

$$\widetilde{H}_n(\bigvee_{i\in I}X_i)\cong\bigoplus_{i\in I}\widetilde{H}_n(X_i)$$

but for this one needs a colimit argument. We postpone that for a while.

We can extend such results to the full relative case. Let $A\subset X$ be a closed subspace and assume that A is a deformation retract of an open neighbourhood $U\subset A$. Let $\pi\colon X\to X/A$ be the canonical projection and $b=\{A\}$ the image of A. Then X/A is well-pointed with respect to b.

Proposition 9.6. *In the situation above*

$$H_n(X,A)\cong\widetilde{H}_n(X/A),\quad 0\leqslant n.$$

Proof. The canonical projection, π, induces a homeomorphism $(X\backslash A, U\backslash A)\cong(X/A\backslash\{b\},\pi(U)\backslash\{b\})$. Consider the following diagram:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong"] \arrow[d, "H_n(\pi)"] & H_n(X,U) & H_n(X\backslash A, U\backslash A) \arrow[l, "\cong"'] \arrow[d, "\cong"', "H_n(\pi)"] \\
H_n(X/A,b) \arrow[r, "\cong"] & H_n(X/A,\pi(U)) & H_n(X/A\backslash\{b\},\pi(U)\backslash\{b\}) \arrow[l, "\cong"']
\end{tikzcd}$$

The upper and lower left arrows are isomorphisms because A is a deformation retract of U, the isomorphism in the upper right is a consequence of excision, because $A = \bar{A} \subset U$ and the lower right one follows from excision as well. $\square$

Theorem 9.7. *(Suspension isomorphism) If $A \subset X$ is as above, then*

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \quad \textit{for all } n > 0.$$

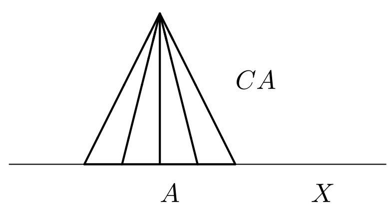
Proof. Consider the inclusion of pairs $(X, A) \subset (CX, CA) \subset (\Sigma X, \Sigma A)$ and the resulting triple $(CX, X \cup CA, CA)$. We obtain the corresponding long exact sequence on homology groups

$$\ldots \longrightarrow H_n(CX, CA) \longrightarrow H_n(CX, CA \cup X) \xrightarrow{\delta} \tilde{H}_{n-1}(X \cup CA, CA) \longrightarrow \ldots$$

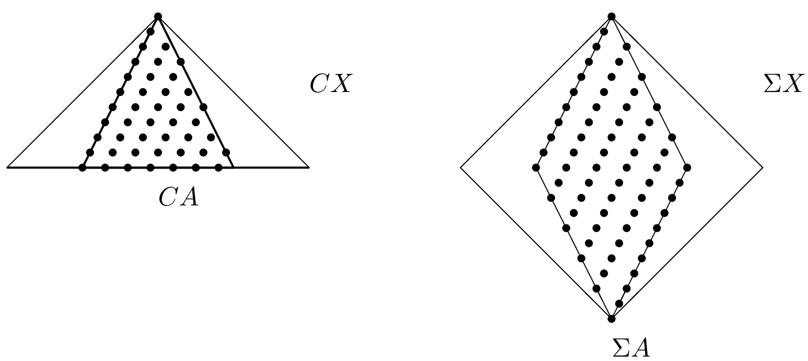
By Proposition 9.6 we get that $\tilde{H}_n(CX, CA \cup X) \cong \tilde{H}_n(CX/CA \cup X)$ and $\tilde{H}_{n-1}(X \cup CA, CA) \cong \tilde{H}_{n-1}(X \cup CA/CA)$ and the latter is isomorphic to $\tilde{H}_{n-1}(X/A) \cong \tilde{H}_{n-1}(X, A)$. Similarly, as $CX/CA \cup X \simeq \Sigma X/\Sigma A$, we get

$$\tilde{H}_n(CX, CA \cup X) \cong \tilde{H}_n(CX/CA \cup X) \cong \tilde{H}_n(\Sigma X/\Sigma A) \cong H_n(\Sigma X, \Sigma A).$$

$X \cup CA/CA \cong X/A$:



$CX/CA \cup X \cong \Sigma X/\Sigma A$:



$\square$

Note, that the corresponding statement is terribly wrong for homotopy groups. We have $\Sigma\mathbb{S}^2 \cong \mathbb{S}^3$, but $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, whereas $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, so homotopy groups (unlike homology groups) don't satisfy such an easy form of a suspension isomorphism. There is a Freundenthal suspension theorem for homotopy groups, but that's more complicated.

10. Mapping degree

Recall that we defined fundamental classes $\mu_n \in \tilde{H}_n(\mathbb{S}^n)$ for all $n \geqslant 0$. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be any continuous map.

Definition 10.1. The map f induces a homomorphism

$$\tilde{H}_n(f)\colon \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n)$$

and therefore we get

$$\tilde{H}_n(f)\mu_n = \mathrm{grad}(f)\mu_n$$

with $\mathrm{grad}(f) \in \mathbb{Z}$. We call this integer *the degree of* f.

In the case $n = 1$ we can relate this notion of a mapping degree to the one defined via the fundamental group of the 1-sphere: if we represent the generator of $\pi_1(\mathbb{S}^1, 1)$ as the class given by the loop

$$\omega\colon [0,1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t},$$

then the abelianized Hurewicz, $h_{\mathrm{ab}}\colon \pi_1(\mathbb{S}^1, 1) \to H_1(\mathbb{S}^1)$, sends the class of ω precisely to μ_1 and therefore the naturality of h_{ab}

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1,1) & \xrightarrow{\pi_1(f)} & \pi_1(\mathbb{S}^1,1) \\ {\scriptstyle h_{\mathrm{ab}}}\downarrow & & \downarrow{\scriptstyle h_{\mathrm{ab}}} \\ H_1(\mathbb{S}^1) & \xrightarrow{H_1(f)} & H_1(\mathbb{S}^1) \end{array}$$

shows that

$$\mathrm{grad}(f)\mu_1 = H_1(f)\mu_1 = h_{\mathrm{ab}}(\pi_1(f)[w]) = h_{\mathrm{ab}}(k[w]) = k\mu_1.$$

where k is the degree of f defined via the fundamental group. Thus both notions coincide for $n = 1$.

As we know that the connecting homomorphism induces an isomorphism between $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ and $\tilde{H}_{n-1}(\mathbb{S}^{n-1})$, we can consider degrees of maps $f\colon (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ by defining $\bar{\mu}_n := \delta^{-1}\mu_n$. Then $H_n(f)(\bar{\mu}_n) := \mathrm{grad}(f)\bar{\mu}_n$ gives a well-defined integer $\mathrm{grad}(f) \in \mathbb{Z}$.

The degree of self-maps of $\mathbb{S}^n$ satisfies the following properties:

Proposition 10.2. (a) *If f is homotopic to g, then* $\mathrm{grad}(f) = \mathrm{grad}(g)$.
(b) *The degree of the identity on $\mathbb{S}^n$ is one.*
(c) *The degree is multiplicative,* i.e., $\mathrm{grad}(g \circ f) = \mathrm{grad}(g)\mathrm{grad}(f)$.
(d) *If f is not surjective, then* $\mathrm{grad}(f) = 0$.

Proof. The first three properties follow directly from the definition of the degree. If f is not surjective, then it is homotopic to a constant map and this has degree zero. □

It is true that the group of (pointed) homotopy classes of self-maps of $\mathbb{S}^n$ is isomorphic to $\mathbb{Z}$ and thus the first property can be upgraded to an 'if and only if', but we won't prove that here.

Recall that $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. If $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then $\Sigma(f)\colon \Sigma\mathbb{S}^n \to \Sigma\mathbb{S}^n$ is given as $\Sigma\mathbb{S}^n \ni [x,t] \mapsto [f(x), t]$.

Lemma 10.3. *Suspensions leave the degree invariant,* i.e., *for $f\colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\mathrm{grad}(\Sigma(f)) = \mathrm{grad}(f).$$

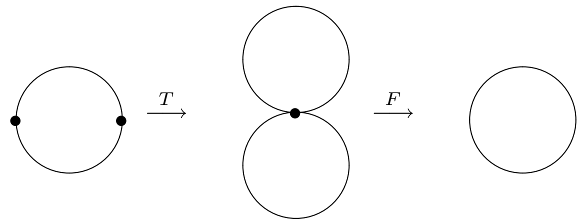
In particular, for every $k \in \mathbb{Z}$ there is an $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with $\mathrm{grad}(f) = k$.

Proof. The suspension isomorphism of Theorem 9.7 is induced by a connecting homomorphism. Using the isomorphism $H_{n+1}(\mathbb{S}^{n+1}) \cong H_{n+1}(\Sigma\mathbb{S}^n)$, the connecting homomorphism sends $\mu_{n+1} \in H_{n+1}(\mathbb{S}^{n+1})$ to $\pm\mu_n \in \tilde{H}_n(\mathbb{S}^n)$. But then the commutativity of

$$\begin{array}{ccccccc} H_{n+1}(\mathbb{S}^{n+1}) & \xrightarrow{\cong} & H_{n+1}(\Sigma\mathbb{S}^n) & \xrightarrow{H_{n+1}(\Sigma f)} & H_{n+1}(\Sigma\mathbb{S}^n) & \xleftarrow{\cong} & H_{n+1}(\mathbb{S}^{n+1}) \\ & & {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta} & & \\ & & \tilde{H}_n(\mathbb{S}^n) & \xrightarrow{H_n(f)} & \tilde{H}_n(\mathbb{S}^n) & & \end{array}$$

ensures that $\pm\mathrm{grad}(f)\mu_n = \pm\mathrm{grad}(\Sigma f)\mu_n$ with the same sign. □

For the degree of a self-map of $\mathbb{S}^1$ one has an additivity relation. We can generalize this to higher dimensions. Consider the *pinch map* $T\colon \mathbb{S}^n \to \mathbb{S}^n/\mathbb{S}^{n-1} \simeq \mathbb{S}^n \vee \mathbb{S}^n$ and the *fold map* $F\colon \mathbb{S}^n \vee \mathbb{S}^n \to \mathbb{S}^n$. Here, F is induced by the identity of $\mathbb{S}^n$.



Proposition 10.4. *For $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ we have*

$$\mathrm{grad}(F \circ (f \vee g) \circ T) = \mathrm{grad}(f) + \mathrm{grad}(g).$$

Proof. The map $H_n(T)$ sends μ_n to $(\mu_n, \mu_n) \in \tilde{H}_n\mathbb{S}^n \oplus \tilde{H}_n\mathbb{S}^n \cong \tilde{H}_n(\mathbb{S}^n \vee \mathbb{S}^n)$. Under this isomorphism, the map $H_n(f \vee g)$ corresponds to $(\mu_n, \mu_n) \mapsto (\tilde{H}_n(f)\mu_n, \tilde{H}_n(g)\mu_n)$ and this yields $(\mathrm{grad}(f)\mu_n, \mathrm{grad}(g)\mu_n)$ which under the fold map is sent to the sum. $\square$

We use the mapping degree to show some geometric properties of self-maps of spheres.

Proposition 10.5. *Let $f^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map*

$$(x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n).$$

Then $f^{(n)}$ has degree -1.

Proof. We prove the claim by induction. μ_0 was the difference class $[+1] - [-1]$, and

$$f^{(0)}([+1] - [-1]) = [-1] - [+1] = -\mu_0.$$

We defined μ_n in such a way that $D\mu_n = \mu_{n-1}$. Therefore

$$H_n(f^{(n)})\mu_n = H_n(f^{(n)})D^{-1}\mu_{n-1} = D^{-1}H_{n-1}(f^{(n-1)})\mu_{n-1} = D^{-1}(-\mu_{n-1}) = -\mu_n.$$

$\square$

Corollary 10.6. *The antipodal map $A\colon \mathbb{S}^n \to \mathbb{S}^n$, that sends $x \in \mathbb{S}^n$ to $-x$ has degree $(-1)^{n+1}$.*

Proof. Let $f_i^{(n)}\colon \mathbb{S}^n \to \mathbb{S}^n$ be the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$. As in Proposition 10.5 one shows that the degree of $f_i^{(n)}$ is -1. As $A = f_n^{(n)} \circ \ldots \circ f_0^{(n)}$, the claim follows. $\square$

In particular, the antipodal map cannot be homotopic to the identity as long as n is even!

Proposition 10.7. *For $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ with $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$,*

$$\mathrm{grad}(f) = (-1)^{n+1}\mathrm{grad}(g).$$

Proof. If $f(x) \neq -g(x)$, then $f(x)$ and $-g(x)$ span a two-dimensional subspace and

$$H(x,t) = \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

connects f to $-g = A \circ g$. For $f(x) = -g(x)$ we have that $H(x,t)$ is $f(x)$ for all t, thus in any case f is homotopic to $A \circ g$. $\square$

Corollary 10.8. *For any $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with* $\mathrm{grad}(f) = 0$ *there is an $x_+ \in \mathbb{S}^n$ with $f(x_+) = x_+$ and an x_- with $f(x_-) = -x_-$.*

Proof. If $f(x) \neq x$ for all x, then $\mathrm{grad}(f) = \mathrm{grad}(A) \neq 0$. If $f(x) \neq -x$ for all x, then $\mathrm{grad}(f) = (-1)^{n+1}\mathrm{grad}(A) \neq 0$. $\square$

Corollary 10.9. *If n is even, then there is an $x \in \mathbb{S}^n$ with $f(x) = x$ or $f(x) = -x$.*

Finally, we can say the following about hairstyles of hedge-hogs of arbitrary even dimension:

PROPOSITION 10.10. *Any tangential vector field on $\mathbb{S}^{2k}$ is trivial in at least one point.*

PROOF. Recall that we can describe the tangent space at a point $x \in \mathbb{S}^{2k}$ as

$$T_x(\mathbb{S}^{2k}) = \{y \in \mathbb{R}^{2k+1} | \langle x, y \rangle = 0\}.$$

Assume V is a tangential vector field which does not vanish, *i.e.*, $V(x) \neq 0$ for all $x \in \mathbb{S}^{2k}$ and $V(x) \in T_x(\mathbb{S}^{2k})$ for all x.

Define $f(x) := \frac{V(x)}{||V(x)||}$. Assume $f(x) = x$, hence $V(x) = ||V(x)||x$. But this means that $V(x)$ points into the direction of x and thus it cannot be tangential. Similarly, $f(x) = -x$ yields the same contradiction. Thus such a V cannot exist. $\square$

11. CW complexes

DEFINITION 11.1. Let X be a topological space. Then X is called an *n-cell*, if X is homeomorphic to $\mathbb{R}^n$. The number n is then the *dimension of the cell*.

Examples. Every point is a zero cell and $\mathring{D}^n \cong \mathbb{R}^n \cong \mathbb{S}^n \backslash N$ are n-cells.
Note that an n-cell cannot be an m-cell for $n \neq m$, because $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$.

DEFINITION 11.2. A *cell decomposition of a space* X is a decomposition of Y into subspaces, each of which is a cell of some dimension, *i.e.*,

$$Y = \bigsqcup_{i \in I} Y_i, \quad Y_i \cong \mathbb{R}^{n_i}.$$

Here, this decomposition is meant as a set, not as a topological space.

Examples. A 3-dimensional cube has a cell decomposition into 8 points, 12 open edges, and 6 open faces.
The standard 2-simplex can be decomposed into 4 zero-cells, 6 1-cells and 4 2-cells.
The n-dimensional sphere (for $n > 0$) has a cell decomposition into the north pole and its complement.

DEFINITION 11.3. A topological hausdorff space X together with a cell decomposition is called a *CW complex*, if it satisfies the following conditions:

(a) For every n-cell $\sigma \subset X$ there is a continuous map $\Phi_\sigma \colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \colon \mathring{\mathbb{D}}^n \xrightarrow{\cong} \sigma$$

and Φ_σ maps $\mathbb{S}^{n-1}$ to the union of cells of dimension at most $n-1$.

(b) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with only finitely many cells of X.

(c) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells σ in X.

- The map Φ_σ as in (a) is called the *characteristic map of the cell σ*. Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called *attaching map*.
- Property (b) is the *closure finite* condition: the closure of every cell is contained in finitely many cells. That's the 'C' in CW.
- Property (c) tells us that X has the *weak topology*. That's the 'W'.
- If X is a CW complex with only finitely many cells, then we call X *finite*.

DEFINITION 11.4.
- We set $X^n := \bigcup_{\sigma \subset X, \dim(\sigma) \leqslant n} \sigma$ and call it *the n-skeleton of X*.
- If we have $X = X^n$, but $X^{n-1} \subsetneq X$, then we say that X is n-dimensional, *i.e.*, $\dim(X) = n$.
- A subset $Y \subset X$ of a CW complex X is called a *subcomplex* (sub-CW complex), if it has a cell decomposition by cells in X and if for any cell $\sigma \subset Y$ we also have $\bar{\sigma} \subset Y$.
- For any subcomplex $Y \subset X$, (X, Y) is a *CW-pair*.

Note, that any subcomplex of a CW complex is again a CW complex: the characteristic maps Φ_σ for Y are the same as for X. We obtain that Y is closed in X because of the second requirement and this guarantees that Y has the weak topology. If $\bar{\sigma} \subset X$ and $\sigma \subset Y$, then $\bar{\sigma} \subset Y$. As Y is closed, this says that Y satisfies condition (b) of a CW complex.

Examples The unit interval $[0,1]$ has a CW structure with two zero cells and one 1-cell. But for instance the decomposition $\sigma_0^0 = \{0\}$, $\sigma_k^0 = \{\frac{1}{k}\}, k > 0$ and $\sigma_k^1 = (\frac{1}{k+1}, \frac{1}{k})$ does not give a CW structure on $[0,1]$. Consider the following $A \subset [0,1]$

$$A := \left\{ \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right) | k \in \mathbb{N} \right\}.$$

Then $A \cap \sigma_k^1$ is precisely the point $\frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})$ and this is closed, but A isn't.

We want to understand the topology of CW complexes. Note that cells don't have to be open in X: if X is a CW complex and σ is an n-cell, then σ is open in the n-skeleton of X, X^n and X^n is closed in X.

Of course, as a set we have $X = \bigcup_{n\geqslant 0} X^n$. From the condition that A is closed in X if and only if the intersection of A with $\bar{\sigma}$ is closed for any cell σ we see that A is closed in X if and only if $A \cap X^n$ is closed for all $n \geqslant 0$. This is an instance of a *direct limit topology on* X and this is denoted by

$$X = \varinjlim X^n.$$

Such a direct limit has the following universal property: for any system of maps $(f_n\colon X^n \to Z)_{n\geqslant 0}$ such that $f_{n+1}|X^n = f_n$ there is a uniquely determined continuous map $f\colon X \to Z$ such that $f|X^n = f_n$.

Note that CW structures on a fixed topological space are not unique. For instance you can consider $\mathbb{S}^2$ with the CW structure coming from the cell decomposition $\mathbb{S}^2 = \mathbb{S}^2\backslash N \sqcup N$. Then the zero skeleton of $\mathbb{S}^2$ only consists of the north pole N and this agrees with the 1-skeleton, but the 2-skeleton is equal to $\mathbb{S}^2$.

But of course there are many other CW structures. Take your favorite dice, *i.e.*, a tetrahedron, a cube, an octahedron, a dodecahedron, an icosahedron or something less regular like a rhombic dodecahedron. Imagine these dice are hollow and project them to $\mathbb{S}^2$. Then you get different CW structures on $\mathbb{S}^2$ that way.

Definition 11.5. Let X and Y be CW complexes. A continuous map $f\colon X \to Y$ is called *cellular*, if $f(X^n) \subset Y^n$ for all $n \geqslant 0$.

The category of CW complexes together with cellular maps is rather flexible. Most of the classical constructions don't lead out of it, but one has to be careful with respect to products:

Proposition 11.6. *If X and Y are CW complexes then $X \times Y$ is a CW complex if one of the factors is locally compact.*

Proof. As products of cells are cells, $X \times Y$ inherits a cell decomposition from its factors. We need to ensure that $X \times Y$ carries the weak topology. For this we prove a slightly more general auxiliary fact: if X, Y and Z are topological spaces satisfying the Hausdorff condition and if $\pi\colon X \to Y$ gives Y the quotient topology, then if Z is locally compact, then

$$\pi \times \mathrm{id}\colon X \times Z \to Y \times Z$$

gives $Y \times Z$ the quotient topology. For this we show that $Y \times Z$ has the universal property of a quotient space, so if $g\colon Y \times Z \to W$ is a map of sets and assume that the composition $g \circ (\pi \times \mathrm{id})$ is continuous. As Z is locally compact and as all spaces are hausdorff, there is a homeomorphism

$$C(X \times Z, W) \cong C(X, C(Z, W))$$

of topological spaces. (Here for two spaces U, V, $C(U, V)$ is the set of all continuous maps from U to V and the topology of $C(U, V)$ is generated (under finite intersections and arbitrary unions) by the sets $V(K, O) := \{f \in C(U, V) | f(K) \subset O\}$ for compact $K \subset U$ and open $O \subset V$.)

Under this adjunction $g \circ (\pi \times \mathrm{id})$ corresponds to the composite

$$\tilde{g}\colon X \xrightarrow{\pi} Y \xrightarrow{\bar{g}} C(Z, W).$$

As $\tilde{g}$ is continuous and as Y carries the quotient topology, we get that $\bar{g}$ is continuous.

With the help of this result we consider the characteristic maps of X and Y,

$$\Phi_\sigma \colon \mathring{\mathbb{D}}^n \to X, \sigma \text{ a cell in } X$$
$$\Psi_\tau \colon \mathring{\mathbb{D}}^m \to Y, \tau \text{ a cell in } Y.$$

Then we can use these maps to write $X \times Y$ as a target of a map

$$\Phi \times \Psi \colon (\bigsqcup_\sigma \mathring{\mathbb{D}}^n) \times (\bigsqcup_\tau \mathring{\mathbb{D}}^m) \to X \times Y.$$

We have to show that $X \times Y$ carries the quotient topology with respect to this map. We know that each $\mathring{\mathbb{D}}^n$ is locally compact, thus so is the disjoint union of open discs. By assumption Y is locally compact and therefore $\mathrm{id}_{\bigsqcup \mathring{\mathbb{D}}^n} \times \Psi$ gives $(\bigsqcup \mathring{\mathbb{D}}^n) \times Y$ the quotient topology and $\Phi \times \mathrm{id}_Y$ induces the quotient topology on $X \times Y$. $\square$

Another important fact (that I won't prove) is that CW complexes are always paracompact, so you find partitions of unity.

Lemma 11.7. *If D is a subset of a CW complex X and D intersects each cell in at most one point, then D is discrete.*

Proof. Let S be an arbitrary subset of D. We show that S is closed. We know that $S \cap \bar{\sigma}$ is finite, because $\bar{\sigma}$ is covered by finitely many cells. Therefore $S \cap \bar{\sigma}$ is closed in $\bar{\sigma}$, because X is hausdorff (and therefore T_1). But then the weak topology guarantees that S is closed. $\square$

Corollary 11.8. *Let X be a CW complex.*

(a) *Every compact subset $K \subset X$ is contained in a finite union of cells.*
(b) *The space X is compact if and only if it is a finite CW complex.*
(c) *The space X is locally compact if and only if it is locally finite,* i.e., *every point has a neighborhood that is contained in finitely many cells.*

Proof. It is easy to see that (a) implies (b) and that implies (c). Thus we only prove (a): consider the intersections $K \cap \sigma$ and choose a point p_σ in every non-empty intersection. Then $D := \{p_\sigma | \sigma \text{ a cell in } X\}$ is discrete, but also compact and therefore finite. $\square$

Corollary 11.9. *If $f \colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton.*

For the proof just note that $f(K)$ is compact in X.

Proposition 11.10. *Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.*

Proof. For $r \colon \mathbb{D}^n \times [0,1] \to \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times [0,1]$ we can choose the standard retraction of a cylinder onto its bottom and sides.

As $X^n \times [0,1]$ is built out of $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times [0,1]$ by gluing in copies of $\mathbb{D}^n \times [0,1]$ along $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times [0,1]$ we get that $X^n \times [0,1]$ is a deformation retract of $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times [0,1]$. We can parametrize the retracting homotopy in such a way that it takes place in the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. Using the direct limit topology on X, we obtain a deformation of $X \times I$ to $X \times \{0\} \cup A \times [0,1]$. $\square$

The property in Proposition 11.10 implies the so-called homotopy extension property, *(HEP): If $g \colon X \to Y$ is a map and $H \colon A \times [0,1] \to Y$ is a homotopy such that $H|_{A \times \{0\}} = g$, then there is an extension of H to $X \times [0,1]$, compatible with g and H. This identifies $A \to X$ as a so-called* cofibration.

Lemma 11.11. *For any subcomplex $A \subset X$ there is an open neighborhood U of A in X together with a strong deformation retraction to A. In particular, each skeleton X^n there is an open neighborhood U in X (and as well in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.*

Proof. Exercise. The second claim follows because skeleta are subcomplexes. $\square$

For any CW complex X the following facts are true. We state them without proof.

- *X is locally path-connected and locally contractible.*
- *X is semi-locally 1-connected.*

These conditions guarantee that every path connected CW complex possesses a universal covering space.

For any CW complex X we get for the skeleta:

(a)
$$X^n \backslash X^{n-1} = \bigsqcup_{\sigma \text{ an } n\text{-cell}} \sigma \cong \bigsqcup_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

(b)
$$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n.$$

Proof. The first claim follows directly from the definition of a CW complex. For the second claim note that the characteristic maps send the boundary $\partial \mathbb{D}^n$ to the $n-1$-skeleton and hence for every n-cell we get a copy of $\mathbb{S}^n$ in the quotient. $\square$

Example Consider the hollow cube W. Then $W^2/W^1 \cong \bigvee_{i=1}^6 \mathbb{S}^2$.

12. Cellular homology

In the following, X will always be a CW complex.

Lemma 12.1. *For all $q \neq n \geqslant 1$, $H_q(X^n, X^{n-1}) = 0$.*

Proof. Using the identification of relative homology and reduced homology of the quotient gives

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

$\square$

Lemma 12.2. *Consider the inclusion $i_n \colon X^n \to X$.*

(a) *The induced map $H_n(i_n) \colon H_n(X^n) \to H_n(X)$ is surjective.*
(b) *On the $(n+1)$-skeleton we get an isomorphism*

$$H_n(i_{n+1}) \colon H_n(X^{n+1}) \cong H_n(X).$$

Proof. We can factor i_n as

$$\begin{tikzcd}
X^n \arrow[rrrr, "i_n"] \arrow[d, "\alpha_1"] & & & & X \\
X^{n+1} \arrow[r, "\alpha_2"'] \arrow[urrrr, "i_{n+1}"] & X^{n+2} \arrow[r, "\alpha_3"'] \arrow[urrr, "i_{n+2}"] & X^{n+3} \arrow[r, "\alpha_4"'] \arrow[urr, "i_{n+3}"] & \cdots
\end{tikzcd}$$

The map $H_n(\alpha_1) \colon H_n(X^n) \to H_n(X^{n+1})$ is surjective, because $H_n(X^{n+1}, X^n) = 0$. For $i > 1$ we have the following piece of the long exact sequence of the pair (X^{n+i}, X^{n+i-1})

$$0 \cong H_{n+1}(X^{n+i}, X^{n+i-1}) \longrightarrow H_n(X^{n+i-1}) \xrightarrow{H_n(\alpha_i)} H_n(X^{n+i}) \longrightarrow H_n(X^{n+i}, X^{n+i-1}) \cong 0.$$

Therefore $H_n(\alpha_i)$ is an isomorphism in this range. $\square$

Corollary 12.3. *For CW complexes X, Y we have*

(a) *If the n-skeleta X^n and Y^n are homeomorphic, then $H_q(X) \cong H_q(Y)$, for all $q < n$.*
(b) *If X has no q-cells, then $H_q(X) \cong 0$.*
(c) *If q exceeds the dimension of X, then $H_q(X) \cong 0$.*

Proof. The first claim is a direct consequence of the lemma above. For (b) and (c): By assumption in (b) $X^{q-1} = X^q$, therefore we have $H_q(X^{q-1}) \cong H_q(X^q)$ and the latter surjects onto $H_q(X)$. We show that $H_n(X^r) \cong 0$ for $n > r$. To that end we use the chain of isomorphisms

$$H_n(X^r) \cong H_n(X^{r-1}) \cong \ldots \cong H_n(X^0)$$

which holds because the adjacent relative groups $H_n(X^i, X^{i-1})$ are trivial for $i < n$. $\square$

Again, X is a CW complex.

Definition 12.4. The *cellular chain complex* $C_*(X)$ consists of $C_n(X) := H_n(X^n, X^{n-1})$ with boundary operator

$$d\colon H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is the map induced by the projection map $S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2})$.

Note that $C_n(X)$ is a free abelian group with

$$C_n(X) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_n(\mathbb{S}^n) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}.$$

For $n < 0$, $C_n(X)$ is trivial. If X has only finitely many n-cells, then $C_n(X)$ is finitely generated. If X is a finite CW complex, then $C_*(X)$ is finitely generated as a chain complex, *i.e.*, $C_n(X)$ is only non-trivial in finitely many degrees n, and in these degrees, $C_n(X)$ is finitely generated.

Lemma 12.5. *The map d is a boundary operator.*

Proof. The composition d^2 is $\varrho \circ \delta \circ \varrho \circ \delta$, but $\delta \circ \varrho$ is a composition in an exact sequence. $\square$

Theorem 12.6. *(Comparison of cellular and singular homology) For every CW complex X, there is an isomorphism $\Upsilon\colon H_*(C_*(X), d) \cong H_*(X)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, equal] \arrow[dd, "d"] & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_{n+1}(X, X^n) \arrow[r, "\delta'"] & H_n(X^n) \arrow[r, "H_n(i_n)"] \arrow[dl, "\varrho"] & H_n(X) \\
C_n(X) \arrow[r, equal] \arrow[dd, "d"] & H_n(X^n, X^{n-1}) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_n(X, X^{n-1}) \arrow[r, "\delta'"] & H_{n-1}(X^{n-1}) \arrow[r, "H_{n-1}(i_{n-1})"] \arrow[dl, "\varrho"] & H_{n-1}(X) \\
C_{n-1}(X) \arrow[r, equal] \arrow[dd, "d"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "\lambda"] \arrow[dr, "\delta"] & & \\
& H_{n-1}(X, X^{n-1}) \arrow[r, "\delta'"] & H_{n-2}(X^{n-2}) \arrow[r, "H_{n-2}(i_{n-2})"] \arrow[dl, "\varrho"] & H_{n-2}(X) \\
\cdots & \cdots & &
\end{tikzcd}$$

- All occurring ϱ-maps are injective because $H_k(X^{k-1}) \cong 0$ for all k.
- For every $a \in H_n(X^n)$ $\varrho(a)$ is a cycle for d:
$$d\varrho(a) = \varrho\delta\varrho(a) = 0.$$
- Let $c \in C_n(X)$ be a d-cycle, thus $0 = dc = \varrho\delta c$ and as ϱ is injective we obtain $\delta c = 0$. Exactness yields that $c = \varrho(a)$ for an $a \in H_n(X^n)$. Hence,
$$H_n(X^n) \cong \ker(d\colon C_n(X) \to C_{n-1}(X)).$$
- We define $\Upsilon\colon \ker(d) \to H_n(X)$ as $\Upsilon[c] = H_n(i_n)(a)$ for $c = \varrho(a)$ and $H_n(i_n)\colon H_n(X^n) \to H_n(X)$.
- The map Υ is surjective because $H_n(i_n)$ is surjective.
- In the diagram, the triangles commute, *i.e.*, $\delta = \delta' \circ \lambda$.

- Consider the sequence

$$H_{n+1}(X^{n+1}) \twoheadrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, X^{n+1}) \longrightarrow H_n(X^{n+1}) \xrightarrow{\cong} H_n(X)$$

which tells us that $H_{n+1}(X, X^{n+1}) = 0$ and this in turn implies that λ is surjective.
- Using this we obtain

$$\mathrm{im}(\delta) = \mathrm{im}(\delta') = \ker(H_n(i_n)).$$

As $d = \varrho \circ \delta$, the map ϱ induces an isomorphism between the image of d and the image of δ.
- Taking all facts into account we get that ϱ induces an isomorphism

$$\frac{\ker(d\colon C_n(X) \to C_{n-1}(X))}{\mathrm{im}(d\colon C_{n+1}(X) \to C_n(X))} \cong \frac{H_n(X^n)}{\ker(H_n(i_n))}$$

But the sequence

$$0 \longrightarrow \ker H_n(i_n) \longrightarrow H_n(X^n) \longrightarrow \mathrm{im}(H_n(i_n)) \longrightarrow 0$$

is exact and therefore

$$H_n(X^n)/\ker(H_n(i_n)) \cong \mathrm{im} H_n(i_n) \cong H_n(X).$$

□

Examples Projective Spaces

Let K be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and let $K^* = K \backslash \{0\}$. We let K^* act on K^{n+1} via

$$K^* \times K^{n+1} \backslash \{0\} \to K^{n+1} \backslash \{0\}, \quad (\lambda, v) \mapsto \lambda v.$$

We define $KP^n = (K^{n+1} \backslash \{0\})/K^*$ and we denote the equivalence class of $(x_0, \ldots, x_n)$ by $KP^n \ni x = [x_0 : \ldots : x_n]$.

We define

$$X_i := \{[x_0 : \ldots : x_n] | x_i \neq 0, x_{i+1} = \ldots = x_n = 0\}$$

and consider the map

$$\xi_i\colon X_i \to K^i, \quad \xi_i[x_0 : \ldots : x_n] = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}).$$

As ξ_i is a homeomorphism, we see that X_i is a cell of dimension $i\dim_{\mathbb{R}}(K) =: m$. We can write KP^n as $X_0 \sqcup \ldots \sqcup X_n$ and we have characteristic maps $\Phi_i\colon \mathbb{D}^{mi} \to KP^n$ as

$$\Phi_i(y) = \Phi_i(y_0, \ldots, y_{i-1}) = [y_0 : \ldots : y_{i-1} : 1 - ||y|| : 0 : \ldots : 0]$$

with $X_i = \Phi_i(\mathring{\mathbb{D}}^{mi})$.

1) First we consider the case $K = \mathbb{C}$. Here, we have a cell in each even dimension $0, 2, 4, \ldots, 2n$ for $\mathbb{C}P^n$. Therefore the cellular chain complex is

$$C_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k = 2i, 0 \leqslant i \leqslant n, \\ 0 & k = 2i-1 \text{ or } k > 2n. \end{cases}$$

The boundary operator is zero in each degree and thus

$$H_*(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & * = 2i, 0 \leqslant * \leqslant 2n, \\ 0 & \text{otherwise.} \end{cases}$$

2) The case of the quaternions is similar. Here the cells are spread in degrees congruent to zero modulo four, thus

$$H_*(\mathbb{H}P^n) = \begin{cases} \mathbb{Z} & * = 4i, 0 \leqslant * \leqslant 4n, \\ 0 & \text{otherwise.} \end{cases}$$

3) Non-trivial boundary operators occur in the case of the real numbers. Here, we have a cell in each dimension up to n and thus the homology of $\mathbb{R}P^n$ is the homology of the chain complex

$$0 \to C_n \cong \mathbb{Z} \xrightarrow{d} C_{n-1} \cong \mathbb{Z} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \cong \mathbb{Z}.$$

For the computation of d we consider the maps

$$\varphi_i = \Phi_i|_{\mathbb{S}^{i-1}} : \mathbb{S}^{i-1} \to \mathbb{S}^{i-1}/\pm \mathrm{id}.$$

The preimage of a class $x \in \mathbb{S}^{i-1}/\pm\mathrm{id}$ is $\{\pm x\}$. We consider the composition

$$\begin{tikzcd}
\mathbb{S}^{i-1} \arrow[r, "\varphi_i"] \arrow[dr, "\bar{\varphi}_i"'] & \mathbb{S}^{i-1}/\pm\mathrm{id} = \mathbb{R}P^{i-1} \arrow[d, "\pi"] \\
& \mathbb{R}P^{i-1}/\mathbb{R}P^{i-2} \cong \mathbb{S}^{i-1}
\end{tikzcd}$$

By construction $\bar{\varphi}_i \circ A = \bar{\varphi}_i$ and thus

$$\mathrm{grad}(\bar{\varphi}_i) = \mathrm{grad}(\bar{\varphi}_i \circ A) = (-1)^i \mathrm{grad}(\bar{\varphi}_i)$$

and hence the degree of $\bar{\varphi}_i$ is trivial for odd i. The complement $\mathbb{S}^{i-1} \backslash \mathbb{S}^{i-2}$ has two components X_+, X_- and A exchanges these two components. The map $\bar{\varphi}_i$ sends X_+ and X_- to X_+. Therefore the degree of $\bar{\varphi}_i$ is

$$\mathrm{grad}(\bar{\varphi}_i) = \mathrm{grad}(F \circ (\mathrm{id} \vee A) \circ T) = \mathrm{grad}(\mathrm{id}) + \mathrm{grad}(A) = 1 + (-1)^i.$$

and d is either zero or two. Thus, depending on n we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k \leqslant n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

for n even.

For odd dimensions n we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n, k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

13. Homology with coefficients

Let G be an arbitrary abelian group.

Definition 13.1. The *singular chain complex of a topological space X with coefficients in G*, $S_*(X;G)$, has as elements in $S_n(X;G)$ finite sums of the form $\sum_{i=1}^N g_i\alpha_i$ with g_i in G and $\alpha_i \colon \Delta^n \to X$. Addition in $S_n(X;G)$ is given by

$$\sum_{i=1}^N g_i\alpha_i + \sum_{i=1}^N h_i\alpha_i = \sum_{i=1}^N (g_i + h_i)\alpha_i.$$

The *n-th (singular) homology group of X with coefficients in G* is

$$H_n(X;G) := H_n(S_*(X;G))$$

where the boundary operator $\partial \colon S_n(X;G) \to S_{n-1}(X;G)$ is given by

$$\partial(\sum_{i=1}^N g_i\alpha_i) = \sum_{j=0}^n (-1)^j (\sum_{i=1}^N g_i(\alpha_i \circ d_j)).$$

We use a similar definition for *cellular homology of a CW complex X with coefficients in G*. Recall, that $C_n(X) = H_n(X^n, X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \mathbb{Z}$.

Definition 13.2. We denote a $c \in C_n(X;G)$ as $c = \sum_{i=1}^N g_i\sigma_i \in \bigoplus_{\sigma \text{ an } n\text{-cell}} G$ and let the boundary operator $\tilde{d}$ be defined by $\tilde{d}c = \sum_{i=1}^N g_i d(\sigma_i)$ where $d \colon C_n(X) \to C_{n-1}(X)$ is the boundary in the cellular chain complex of X.

We can transfer Theorem 12.6 to the case of homology with coefficients:

$$H_n(X;G) \cong H_n(C_*(X;G), \tilde{d})$$

for every CW complex X and therefore we denote the latter by $H_n(X;G)$ as well.

Note, that $H_n(X;\mathbb{Z}) = H_n(X)$ for every space X.

Example If we consider the case $X = \mathbb{R}P^2$, then we see that coefficients really make a difference.

Recall that for $G = \mathbb{Z}$ we had that $H_0(\mathbb{R}P^2) \cong \mathbb{Z}$, $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\mathbb{R}P^2) = 0$. However, for $G = \mathbb{Z}/2\mathbb{Z}$ the outcome differs drastically. The cellular chain complex looks as follows:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and therefore $H_i(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $0 \leqslant i \leqslant 2$.

If we consider $H_*(\mathbb{R}P^2;\mathbb{Q})$ we obtain the cellular complex

$$0 \longrightarrow \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0$$

But here, multiplication by 2 is an isomorphism and we get $H_0(\mathbb{R}P^2;\mathbb{Q}) = \mathbb{Q}$, $H_1(\mathbb{R}P^2;\mathbb{Q}) = \mathbb{Q}/2\mathbb{Q} = 0$ and $H_2(\mathbb{R}P^2;\mathbb{Q}) = 0$.

14. Tensor products and universal coefficient theorem

The question we want to pursue in this section is, whether $H_*(X,G)$ is computable from $H_*(X)$ and G. The general answer is 'Yes', but we need some basics from algebra to see that.

Let A and B be abelian groups.

Definition 14.1. The *tensor product of A and B*, $A \otimes B$, is the quotient of the free abelian group generated by $A \times B$ by the subgroup generated by

(a) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$,
(b) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$

for $a_1, a_1, a \in A$ and $b_1, b_2, b \in B$.

We denote an equivalence class of (a,b) in $A \otimes B$ by $a \otimes b$.

Note, that relations (a) and (b) imply that $\lambda(a \otimes b) = (\lambda a) \otimes b = a \otimes (\lambda b)$ for any integer $\lambda \in \mathbb{Z}$ and $a \in A$, $b \in B$. Elements in $A \otimes B$ are finite sums of equivalence classes $\sum_{i=1}^n \lambda_i a_i \otimes b_i$.

- Of course, $A \otimes B$ is generated by $a \otimes b$ with $a \in A$, $b \in B$.
- The tensor product is symmetric up to isomorphism and the isomorphism $A \otimes B \cong B \otimes A$ is given by
$$\sum_{i=1}^n \lambda_i a_i \otimes b_i \mapsto \sum_{i=1}^n \lambda_i b_i \otimes a_i.$$
- It is associative up to isomorphism:
$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
for all abelian groups A, B, C.
- For homomorphisms $f\colon A \to A'$ and $g\colon B \to B'$ we get an induced homomorphism
$$f \otimes g\colon A \otimes B \to A' \otimes B'$$
which is given by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ on generators.
- The tensor product has the following universal property. For abelian groups A, B, C, the bilinear maps from $A \times B$ to C are in bijection with the linear maps from $A \otimes B$ to C.
- We've already seen tensor products: Note that $S_n(X) \otimes G$ is isomorphic to $S_n(X,G)$ and $C_n(X) \otimes G \cong C_n(X,G)$.

We collect the following properties of tensor products:

(a) For every abelian group A, we have
$$A \otimes \mathbb{Z} \cong A \cong \mathbb{Z} \otimes A.$$

(b) For every abelian group A, we have

$$A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/nA.$$

Here, note that $nA = \{na | a \in A\}$ makes sense in any abelian group. The isomorphism above is given by

$$a \otimes \bar{i} \mapsto \bar{ia}$$

where $\bar{i}$ denotes an equivalence class of $i \in \mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$ and $\bar{ia}$ the class of $ia \in A$ in A/nA.

(c) If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence, then in general,

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha \otimes \mathrm{id}} B \otimes D \xrightarrow{\beta \otimes \mathrm{id}} C \otimes \mathrm{id} \longrightarrow 0$$

is not exact for D abelian. For example,

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

is exact, but

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to 0$$

isn't, because $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \cong 0$.

Lemma 14.2. *For every abelian group D, $(-) \otimes D$ is right exact,* i.e., *if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence, then*

$$A \otimes D \xrightarrow{\alpha \otimes \mathrm{id}} B \otimes D \xrightarrow{\beta \otimes \mathrm{id}} C \otimes \mathrm{id} \longrightarrow 0$$

is exact. If the exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a split short exact sequence, then

$$0 \longrightarrow A \otimes D \xrightarrow{\alpha \otimes \mathrm{id}} B \otimes D \xrightarrow{\beta \otimes \mathrm{id}} C \otimes \mathrm{id} \longrightarrow 0$$

is exact.

Proof. Exercise. □

A consequence of the failure of the functor $(-) \otimes D$ to be exact on the left hand side has as a consequence that $H_n(X, G) = H_n(S_*(X) \otimes G)$ is not always isomorphic to $H_n(X) \otimes G = H_n(S_*(X)) \otimes G$.

Definition 14.3. Let A be an abelian group. A short exact sequence $0 \to R \longrightarrow F \longrightarrow A \to 0$ with F a free abelian group is called a *free resolution of A*.

Note that in the situation above, R is also free abelian, because it can be identified with a subgroup of F.

Example For every $n \geqslant 1$, the sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$.

Proposition 14.4. *Every abelian group possesses a free resolution.*

The resolution that we will construct in the proof is called the *standard resolution of A*.

Proof. Let F be the free abelian group generated by the elements of the underlying set of A. We denote by y_a the basis element in F corresponding to $a \in A$. Define a homomorphism

$$p \colon F \to A,\ p(\sum_{a \in A} \lambda_a y_a) = \sum_{a \in A} \lambda_a a.$$

Here, $\lambda_a \in \mathbb{Z}$ and this integer is non-trivial for only finitely many $a \in A$. By construction, p is an epimorphism. We set R to be the kernel of p and in that way obtain the desired free resolution of A. □

Definition 14.5. For two abelian groups A and B and for $0 \to R \xrightarrow{i} F \longrightarrow A \to 0$ the standard resolution of A we define

$$\mathrm{Tor}(A, B) := \ker(i \otimes \mathrm{id} \colon R \otimes B \to F \otimes B).$$

In general, $i \otimes \mathrm{id}$ doesn't have to be injective, thus $\mathrm{Tor}(A, B)$ won't be trivial. We will show that we can calculate $\mathrm{Tor}(A, B)$ via an arbitrary free resolution of A. To that end we prove the following result.

Proposition 14.6. *For every homomorphism $f\colon A \to B$ and for free resolutions $0 \to R \xrightarrow{i} F \longrightarrow A \to 0$ and $0 \to R' \xrightarrow{i'} F' \longrightarrow B \to 0$ we have:*

(a) *There are homomorphisms $g\colon F \to F'$ and $h\colon R \to R'$, such that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "h"] & F \arrow[r, "p"] \arrow[d, "g"] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & R' \arrow[r, "i'"] & F' \arrow[r, "p'"] & B \arrow[r] & 0 \end{tikzcd}$$

commutes.

If g', h' are other homomorphisms with this property, then there is an $\alpha\colon F \to R'$ with $i' \circ \alpha = g - g'$ and $\alpha \circ i = h - h'$.

(b) *For every abelian group D the map $h \otimes \mathrm{id}\colon R \otimes D \to R' \otimes D$ maps the kernel of $i \otimes \mathrm{id}$ to the kernel of $i' \otimes \mathrm{id}$ and the restriction $h \otimes \mathrm{id}|_{\ker(i\otimes \mathrm{id})}$ is independent of the choice of g and h. We denote this map by $\varphi(f, R \to F, R' \to F')$.*

(c) *For a homomorphism $f'\colon B \to C$ the map $\varphi(f' \circ f, R \to F, R'' \to F'')$ is equal to the composition $\varphi(f', R' \to F', R'' \to F'') \circ \varphi(f, R \to F, R' \to F')$.*

Note that we can view the α above as a chain homotopy between the chain maps g, h and g', h'.

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, bend right, "h'"'] \arrow[d, bend left, "h"] & F \arrow[r] \arrow[dl, "\alpha"'] \arrow[d, bend right, "g'"'] \arrow[d, bend left, "g"] & 0 \\ 0 \arrow[r] & R' \arrow[r, "i'"] & F' \arrow[r] & 0 \end{tikzcd}$$

Proof. For (a) let $\{x_i\}$ be a basis of F and choose $y_i \in F'$ with $p'(y_i) = fp(x_i)$. We define $g\colon F \to F'$ via $g(x_i) = y_i$. Thus $p' \circ g(x_i) = p'(y_i) = fp(x_i)$. For every $r \in R$ we obtain $p' \circ g(i(r)) = f \circ p \circ i(r) = 0$ and therefore $g(i(r))$ is contained in the kernel of p' which is equal to the image of i'. In order to define h we use the injectivity of i', thus $h(r)$ is the unique preimage of $g(i(r))$ under i'.

For h, h' and g, g' as in (a) we get for $x \in F$ that $g(x) - g'(x)$ is in the kernel of p' which is the image of i'. Define α as $(i')^{-1}(g - g')$. Then by construction $i'\alpha = g - g'$ and it is

$$i'(h - h') = (g - g')i = i'\alpha i.$$

As i' is injective, this yields $h - h' = \alpha i$.

For (b) we consider an element z in the kernel of $i \otimes \mathrm{id} \subset R \otimes D$. Then

$$(i' \otimes \mathrm{id}) \circ (h \otimes \mathrm{id})(z) = (g \otimes \mathrm{id}) \circ (i \otimes \mathrm{id})(z) = 0$$

and thus $(h \otimes \mathrm{id})(z)$ is in the kernel of $(i' \otimes \mathrm{id})$. If h' is any other map satisfying the properties, then

$$(h' \otimes \mathrm{id})(z) - (h \otimes \mathrm{id})(z) = ((h' - h) \otimes \mathrm{id})(z) = ((\alpha \circ i) \otimes \mathrm{id})(z) = (\alpha \otimes \mathrm{id})(i \otimes \mathrm{id})(z) = 0.$$

For (c) we note that the uniqueness in (b) implies (c). $\square$

Corollary 14.7. *For every free resolution $0 \to R' \xrightarrow{i'} F' \longrightarrow A \to 0$ we get a unique isomorphism*

$$\varphi(\mathrm{id}_A, R' \to F', R \to F)\colon \ker(i' \otimes \mathrm{id}) \to \mathrm{Tor}(A, D).$$

Thus we can calculate $\mathrm{Tor}(A, D)$ with every free resolution of A.

Examples

(a) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \{d \in D | nd = 0\}$ for all $n \geqslant 1$. That's why Tor is sometimes called torsion product. For the calculation we use the resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$. By definition and by Corollary 14.7 we have

$$\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, D) \cong \ker(n \otimes \mathrm{id}\colon \mathbb{Z} \otimes D \to \mathbb{Z} \otimes D).$$

As $\mathbb{Z} \otimes D \cong D$ and as $n \otimes \mathrm{id}$ induces the multiplication by n, we get the claim.

(b) From the first example we obtain $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$ because the n-torsion subgroup in $\mathbb{Z}/m\mathbb{Z}$ is $\mathbb{Z}/\gcd(m, n)\mathbb{Z}$.

(c) For A free abelian, $\mathrm{Tor}(A, D) \cong 0$ for arbitrary D. For this note that $0 \to 0 \to A \xrightarrow{\mathrm{id}} A \to 0$ is a free resolution of A and the kernel is a subgroup of $0 \otimes D = 0$ and hence trivial.

(d) For two abelian groups A_1, A_2, D there is an isomorphism

$$\mathrm{Tor}(A_1 \oplus A_2, D) \cong \mathrm{Tor}(A_1, D) \oplus \mathrm{Tor}(A_2, D).$$

Consider free resolutions

$$0 \to R_i \to F_i \to A_i \to 0, i = 1, 2.$$

Their direct sum

$$0 \to R_1 \oplus R_2 \to F_1 \oplus F_2 \to A_1 \oplus A_2 \to 0$$

is a free resolution of $A_1 \oplus A_2$ with

$$\ker((i_1 \oplus i_2) \otimes \mathrm{id}) = \ker(i_1 \otimes \mathrm{id}) \oplus \ker(i_2 \otimes \mathrm{id}).$$

We extend the definition of tensor products to chain complexes.

Definition 14.8. Are (C_*, d) and (C'_*, d') two chain complexes, then $(C_* \otimes C'_*, d_\otimes)$ is the chain complex with

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

and with $d_\otimes(c_p \otimes c'_q) = (dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q$.

Lemma 14.9. *The map $d_\otimes$ is a differential.*

Proof. The composition is

$$d_\otimes((dc_p) \otimes c'_q + (-1)^p c_p \otimes d'c'_q) = 0 + (-1)^{p-1}(dc_p) \otimes (d'c'_q) + (-1)^p(dc_p) \otimes (d'c'_q) + 0 = 0.$$

□

Example Let G be an abelian group, then let C_G be the chain complex with

$$(C_G)_n = \begin{cases} G, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Then for every chain complex (C_*, d)

$$(C_* \otimes C_G)_n = C_n \otimes G, \quad d_\otimes = d \otimes \mathrm{id}.$$

In particular, for every topological space X,

$$S_*(X) \otimes C_G \cong S_*(X) \otimes G = S_*(X, G).$$

Similarly, for a CW complex X we get $C_*(X; G) = C_*(X) \otimes C_G$.

For every pair of spaces (X, A) we set

$$S_*(X, A; G) := S_*(X, A) \otimes C_G.$$

A map $f\colon (C_*, d) \to (D_*, d_D)$ induces a map of chain complexes

$$f \otimes \mathrm{id}\colon C_* \otimes C'_* \to D_* \otimes C'_*.$$

In particular, for every continuous (cellular) map we get induced maps on singular (cellular) homology of spaces.

Note, that $H_*(\mathrm{pt}; G) \cong \begin{cases} G, & * = 0 \\ 0, & * \neq 0. \end{cases}$.

Definition 14.10. A chain complex C_* is called *free*, if C_n is a free abelian group for all $n \in \mathbb{Z}$.

Examples The complexes $S_*(X, A)$ and $C_*(X)$ are free.

THEOREM 14.11. *(Universal coefficient theorem (algebraic version)) Let C_* be a free chain complex and G an abelian group, then for all $n \in \mathbb{Z}$ we have a short exact sequence*

$$0 \to H_n(C_*) \otimes G \to H_n(C_* \otimes G) \to \operatorname{Tor}(H_{n-1}(C_*), G) \to 0,$$

in particular

$$H_n(C_* \otimes G) \cong H_n(C_*) \otimes G \oplus \operatorname{Tor}(H_{n-1}(C_*), G).$$

THEOREM 14.12. *(Universal coefficient theorem (topological version)) For every space X there is a split short exact sequence*

$$0 \to H_n(X) \otimes G) \to H_n(X; G) \to \operatorname{Tor}(H_{n-1}(X), G) \to 0,$$

and an isomorphism

$$H_n(X; G) \cong H_n(X) \otimes G \oplus \operatorname{Tor}(H_{n-1}(X), G).$$

Example For $X = \mathbb{R}P^2$ we obtain

$$H_n(\mathbb{R}P^2; G) \cong H_n(\mathbb{R}P^2) \otimes G \oplus \operatorname{Tor}(H_{n-1}(\mathbb{R}P^2), G)$$

thus

$$H_0(\mathbb{R}P^2; G) \cong H_0(\mathbb{R}P^2) \otimes G \oplus \operatorname{Tor}(H_{-1}(\mathbb{R}P^2), G) \cong G,$$
$$H_1(\mathbb{R}P^2; G) \cong H_1(\mathbb{R}P^2) \otimes G \oplus \operatorname{Tor}(H_0(\mathbb{R}P^2), G) \cong G/2G \oplus 0 \cong G/2G,$$

and

$$H_2(\mathbb{R}P^2; G) \cong H_2(\mathbb{R}P^2) \otimes G \oplus \operatorname{Tor}(H_1(\mathbb{R}P^2), G) \cong \operatorname{Tor}(\mathbb{Z}/2\mathbb{Z}, G).$$

The universal coefficient theorems are both corollaries of the following more general statement.

THEOREM 14.13. *(Künneth formula) For a free chain complex C_* and a chain complex C'_* we have the following split exact sequence for every integer n*

$$0 \longrightarrow \textstyle\bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \xrightarrow{\lambda} H_n(C_* \otimes C'_*) \longrightarrow \textstyle\bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(C_*), H_q(C'_*)) \longrightarrow 0,$$

i.e.,

$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \oplus \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(C_*), H_q(C'_*)).$$

The map $\lambda\colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \to H_n(C_* \otimes C'_*)$ in the theorem is given on the (p,q)-summand by

$$\lambda([c_p] \otimes [c'_q]) := [c_p \otimes c'_q]$$

for $c_p \in C_p$ and $c'_q \in C'_q$. By the definition of the tensor product of complexes, this map is well-defined.

LEMMA 14.14. *For any free chain complex C_* with trivial differential and an arbitrary chain complex, C'_*, λ is an isomorphism*

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*) \cong H_n(C_* \otimes C'_*).$$

PROOF. We abbreviate the subgroup of cycles in C'_q with Z'_q and the subgroup of boundaries in C'_q with B'_q and use analog abbreviations for C_*. By definition $0 \to Z'_q \longrightarrow C'_q \longrightarrow B'_{q-1} \to 0$ is a short exact sequence. By assumption Z_p is free because $Z_p = C_p$, in particular $Z_p \otimes (-)$ is exact and thus

$$0 \to Z_p \otimes Z'_q \longrightarrow Z_p \otimes C'_q \longrightarrow Z_p \otimes B'_{q-1} \to 0$$

is a short exact sequence and this implies that $Z_p \otimes Z'_q$ is the subgroup of cycles in $Z_p \otimes C'_q = C_p \otimes C'_q$. Summation over $p + q = n$ yields that the n-cycles in $C_* \otimes C'_*$ are

$$Z_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} Z_p \otimes Z'_q$$

and the n-boundaries are given by

$$B_n(C_* \otimes C'_*) = \bigoplus_{p+q=n} Z_p \otimes B'_q.$$

The sequence
$$0 \to B'_q \longrightarrow Z'_q \longrightarrow H_q(C'_*) \to 0$$
is exact by definition. Tensoring with Z_p and summing over $p+q=n$ then yields due to the freeness of Z_p that
$$0 \to \bigoplus_{p+q=n} Z_p \otimes B'_q \longrightarrow \bigoplus_{p+q=n} Z_p \otimes Z'_q \longrightarrow \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) \to 0$$
is exact. Our identification of $Z_n(C_* \otimes C'_*)$ and $B_n(C_* \otimes C'_*)$ yields that the right-most term is isomorphic to the n-th homology group of $C_* \otimes C'_*$ and therefore
$$H_n(C_* \otimes C'_*) \cong \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$
□

Proof of Theorem 14.13. We consider again the short exact sequence $0 \to Z_p \longrightarrow C_p \longrightarrow B_{p-1} \to 0$ and tensor it with C'_q and sum over $p+q=n$. As B_{p-1} is free, the original sequence is split and hence the resulting sequence is exact.

We define two chain complexes Z_* and D_* via
$$(Z_*)_p = Z_p, D_p = B_{p-1}.$$
Then Z_* and D_* are free chain complexes with trivial differential and the exact sequence
$$0 \to \bigoplus_{p+q=n} Z_p \otimes C'_q \longrightarrow \bigoplus_{p+q=n} C_p \otimes C'_q \longrightarrow \bigoplus_{p+q=n} B_{p-1} \otimes C'_q \to 0$$
can be interpreted as a short exact sequence of complexes and this gives a long exact sequence
$$\ldots \longrightarrow H_{n+1}(D_* \otimes C'_*) \xrightarrow{\delta_{n+1}} H_n(Z_* \otimes C'_*) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow H_n(D_* \otimes C'_*) \xrightarrow{\delta_n} H_{n-1}(Z_* \otimes C'_*) \to \ldots$$
Lemma 14.14 gives us a description of $H_*(D_* \otimes C'_*)$ and $H_*(Z_* \otimes C'_*)$ and therefore we can consider δ_{n+1} as a map
$$\delta_{n+1}\colon \bigoplus_{p+q=n+1} H_p(D_*) \otimes H_q(C'_*) = \bigoplus_{p+q=n+1} B_{p-1} \otimes H_q(C'_*) \xrightarrow{j\otimes \mathrm{id}} \bigoplus_{p+q=n} Z_p \otimes H_q(C'_*) = \bigoplus_{p+q=n} H_p(Z) \otimes H_q(C'_*)$$
with $j\colon B_p \hookrightarrow Z_p$. We can cut the long exact sequence in homology in short exact pieces and obtain that
$$0 \to \operatorname{coker}(\delta_{n+1}) \longrightarrow H_n(C_* \otimes C'_*) \longrightarrow \ker(\delta_n) \to 0$$
is exact. The cokernel of δ_{n+1} is isomorphic to $\bigoplus_{p+q=n} Z_p/B_p \otimes H_q(C'_*)$ because the tensor functor is right exact, thus
$$\operatorname{coker}(\delta_{n+1}) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$
As $0 \to B_p \longrightarrow Z_p \longrightarrow H_p(C_*) \to 0$ is a free resolution of $H_p(C_*)$ we obtain that
$$\mathrm{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(j \otimes \mathrm{id}\colon B_p \otimes H_q(C'_*) \to Z_p \otimes H_q(C'_*))$$
and therefore
$$\bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C_*), H_q(C'_*)) \cong \ker(\delta_n)$$
which proves the exactness of the Künneth sequence.

We will prove that the Künneth sequence is split in the case where both chain complexes, C_* and C'_*, are free. In that case the sequences
$$0 \to Z_p \to C_p \to B_{p-1} \to 0, \quad 0 \to Z'_q \to C'_q \to B'_{q-1} \to 0$$
are split and we denote by $r\colon C_p \to Z_p$ and $r'\colon C'_q \to Z'_q$ chosen retractions. Consider the two compositions
$$C_p \xrightarrow{r} Z_p \twoheadrightarrow H_p(C_*), \quad C'_q \xrightarrow{r'} Z'_q \twoheadrightarrow H_q(C'_*)$$

and view $H_*(C_*)$ and $H_*(C'_*)$ as chain complexes with trivial differential. Then these compositions yield a chain map

$$C_* \otimes C'_* \xrightarrow{r\otimes r'} H_*(C_*) \otimes H_*(C'_*)$$

which on homology is

$$H_n(C_* \otimes C'_*) \longrightarrow H_n(H_*(C_*) \otimes H_*(C'_*)) = \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

This map gives the desired splitting. $\square$

In the cases we are interested in (singular or cellular chains), the complexes will be free. Be careful! The splitting of the Künneth sequence is *not* natural. We have chosen a splitting of the short exact sequences in the proof and usually, there is no canonical choice possible.

15. The topological Künneth formula

What does the Künneth formula give for two topological spaces and their chain complexes? The Künneth sequence for $C_* = S_*(X)$ and $C'_* = S_*(Y)$ yields that

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(S_*(X) \otimes S_*(Y)) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

is exact. But what is $H_n(S_*(X) \otimes S_*(Y))$? In the following we will show that this group is actually isomorphic to $H_n(X \times Y)$, thus the Künneth Theorem has some geometric content! First of all, we define a map.

Lemma 15.1. *There is a homomorphism* $\times\colon S_p(X) \otimes S_q(Y) \longrightarrow S_{p+q}(X \times Y)$ *for all* $p, q \geqslant 0$ *with the following properties.*

(a) *For all points* $x_0 \in X$ *viewed as zero chains*

$$(x_0 \times \beta)(t_0, \ldots, t_q) = (x_0, \beta(t_0, \ldots, t_q))$$

for $\beta\colon \Delta^q \to Y$. *Analogously, for all* $y_0 \in Y$ *and* $\alpha\colon \Delta^p \to X$

$$(\alpha \times y_0)(t_0, \ldots, t_p) = (\alpha(t_0, \ldots, t_p), y_0).$$

(b) *The map* $\times$ *is natural in* X *and* Y, *so for* $f\colon X \to X'$ *and* $g\colon Y \to Y'$

$$S_{p+q}(f, g) \circ (\alpha \times \beta) = (S_p(f) \circ \alpha) \times (S_q(g) \circ \beta).$$

(c) *The Leibniz rule holds*

$$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

The map $\times$ is called the *homology cross product.*

Proof. For p or q equal to zero, we define $\times$ as dictated by property (a). Therefore we can assume that $p, q \geqslant 1$. The method of proof that we will apply here is called *method of acyclic models* – you'll see why. Let $X = \Delta^p$, $Y = \Delta^q$, $\alpha = \mathrm{id}_{\Delta^p}$, and $\beta = \mathrm{id}_{\Delta^q}$. If $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}$ were already defined, then property (c) would force

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \times \partial(\mathrm{id}_{\Delta^q}) =: R \in S_{p+q}(\Delta^p \times \Delta^q).$$

For this element R we get

$$\partial R = \partial^2(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^{p-1}\partial(\mathrm{id}_{\Delta^p}) \times \partial(\mathrm{id}_{\Delta^q}) + (-1)^p \partial(\mathrm{id}_{\Delta^p}) \times \partial(\mathrm{id}_{\Delta^q}) + (-1)^p \mathrm{id}_{\Delta^p} \times \partial^2(\mathrm{id}_{\Delta^q}) = 0$$

so R is a cycle. But $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ because $p + q - 1 \geqslant 1$ and $\Delta^p \times \Delta^q$ is contractible and therefore $S_*(\Delta^p \times \Delta^q)$ is acyclic. Thus R has to be a boundary, so there is a $c \in S_{p+q+1}(\Delta^p \times \Delta^q)$ with $\partial c = R$.

We define

$$\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} := c.$$

Now let X and Y be arbitrary spaces and $\alpha\colon \Delta^p \to X$, $\beta\colon \Delta^q \to Y$. Then $S_p(\alpha)(\mathrm{id}_{\Delta^p}) = \alpha$ and $S_q(\beta)(\mathrm{id}_{\Delta^q}) = \beta$ and therefore naturality dictates

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) = S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}).$$

By construction, this definition satisfies all desired properties. $\square$

Note that for spaces X, Y with trivial homology in positive degrees, the Künneth Theorem yields that $H_n(S_*(X) \otimes S_*(Y)) = 0$ for positive n.

LEMMA 15.2. *Are C_* and C'_* two chain complexes which are trivial in negative degrees and such that C_n is free abelian for all n and $H_n C'_* = 0$ for all positive n, then we have*

(a) *Any two chain maps $f, g\colon C_* \to C'_*$ with $f_0 = g_0$ are chain homotopic.*
(b) *Is $f_0\colon C_0 \to C'_0$ a homomorphism with $f_0(\partial C_1) \subset \partial C'_1$ then there is a chain map $f_*\colon C_* \to C'_*$ extending f_0.*

PROOF. We will define a map $H_n\colon C_n \to C'_{n+1}$ for all $n \geqslant 0$ with $\partial H_n + H_{n-1}\partial = f_n - g_n$ inductively. For $n = 0$ we can take zero because $f_0 = g_0$ by assumption. Assume that we have H_k for $k \leqslant n$. Let $\{x_i\}$ be a basis of the free abelian group C_n and define

$$y_i := f_n(x_i) - g_n(x_i) - H_{n-1}\partial(x_i) \in C'_n.$$

Then

$$\begin{aligned}\partial y_i =& \partial f_n(x_i) - \partial g_n(x_i) - \partial H_{n-1}\partial(x_i)\\ =& \partial f_n(x_i) - \partial g_n(x_i) - H_{n-2}\partial^2(x_i) - f_{n-1}\partial(x_i) + g_{n-1}\partial(x_i)\\ =& 0.\end{aligned}$$

But C'_* is acyclic by assumption and therefore y_i has to be a boundary and we define $H_n(x_i) = z_i$ if $\partial z_i = y_i$. Then

$$(\partial H_n + H_{n-1}\partial)(x_i) = y_i + H_{n-1}\partial(x_i) = f_n(x_i) - g_n(x_i).$$

For (b) we define $f_n\colon C_n \to C'_n$ inductively with $\partial f_n = f_{n-1}\partial$. Assume that $\{x_i\}$ is a basis of C_n. Then $f_{n-1}\partial(x_i)$ is a cycle and thus there is a y_i with $\partial y_i = f_{n-1}\partial(x_i)$ due to the acyclicity of C'_*. We define $f_n(x_i)$ as y_i. Then

$$\partial f_n(x_i) = \partial y_i = f_{n-1}\partial(x_i).$$

□

PROPOSITION 15.3. *Any two natural chain maps $f_{X,Y}, g_{X,Y}$ from $S_*(X) \otimes S_*(Y)$ to $S_*(X \times Y)$ which agree in degree zero and send the zero chain $x_0 \otimes y_0 \in (S_*(X) \otimes S_*(Y))_0 = S_0(X) \otimes S_0(Y)$ to $(x_0, y_0) \in S_0(X \times Y)$ are chain homotopic.*

PROOF. First we deal with the case $X = \Delta^p$ and $Y = \Delta^q$ for $p, q \geqslant 0$. If $f, g\colon S_*(\Delta^p) \otimes S_*(\Delta^q) \longrightarrow S_*(\Delta^p \times \Delta^q)$ are two chain maps then $S_*(\Delta^p) \otimes S_*(\Delta^q)$ is free abelian and $S_*(\Delta^p \times \Delta^q)$ is acyclic so we can apply Lemma 15.2 and get a chain homotopy $(H_n)_n$,

$$H_n\colon (S_*(\Delta^p) \otimes S_*(\Delta^q))_n \longrightarrow S_{n+1}(\Delta^p) \otimes S_*(\Delta^q)$$

with $\partial H_n + H_{n-1}\partial = f_n - g_n$.

Note that for arbitrary X and Y naturality implies

$$f_{X,Y} \circ (S_*(\alpha) \otimes S_*(\beta)) = S_*(\alpha, \beta) \circ f_{\Delta^p,\Delta^q}, \quad g_{X,Y} \circ (S_*(\alpha) \otimes S_*(\beta)) = S_*(\alpha, \beta) \circ g_{\Delta^p,\Delta^q}$$

for all $\alpha\colon \Delta^p \to X$, $\beta\colon \Delta^q \to Y$.

We define

$$H_n\colon (S_*(X) \otimes S_*(Y))_n \longrightarrow S_{n+1}(X \times Y)$$

as

$$H_n(\alpha \otimes \beta) = S_{n+1}(\alpha, \beta) \circ H_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}).$$

This is well-defined and by construction:

$$\begin{aligned}\partial H_n(\alpha \otimes \beta) &= \partial S_{n+1}(\alpha, \beta) \circ H_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})\\ &= S_{n+1}(\alpha, \beta)\partial H_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})\\ &= -S_{n+1}(\alpha, \beta) \circ (H_{n-1}\partial(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) + f_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) - g_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}))\\ &= f_n(\alpha \otimes \beta) - g_n(\alpha \otimes \beta) - H_{n-1}\partial(\alpha \otimes \beta).\end{aligned}$$

□

Proposition 15.4. (a) *There is a chain map $S_*(X \times Y) \longrightarrow S_*(X) \otimes S_*(Y)$ for all spaces X and Y such that this map is natural in X and Y and such that in degree zero this map sends (x_0, y_0) to $x_0 \otimes y_0$ for all $x_0 \in X$ and $y_0 \in Y$.*

(b) *Any two such maps are chain homotopic.*

Proof. Let $X = \Delta^n = Y$ for $n \geqslant 0$ and set $C_* = S_*(\Delta^n \times \Delta^n)$ and $C'_* = S_*(\Delta^n) \otimes S_*(\Delta^n)$. Set $f_0 \colon C_0 \to C'_0$ as dictated by condition (a). Then by Lemma 15.2 there is a chain map $(f_m)_m$, $f_m \colon S_m(\Delta^n \times \Delta^n) \to (S_*(\Delta^n) \otimes S_*(\Delta^n))_m$. For $\alpha \colon \Delta^n \to X \times Y$ we then define

$$\tilde{f}_n(\alpha) := (S_*(p_1 \circ \alpha)) \otimes S_*((p_2 \circ \alpha)) \circ f(\Delta_{\Delta^n}).$$

Here, $\Delta_{\Delta^n} \colon \Delta^n \longrightarrow \Delta^n \times \Delta^n$ is the diagonal map viewed as a singular simplex $\Delta_{\Delta^n} \in S_n(\Delta^n \times \Delta^n)$ and the p_i are the projection maps $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$, thus

$$\begin{array}{ccc} S_n(\Delta^n \times \Delta^n) & \xrightarrow{\quad f \quad} & (S_*(\Delta^n) \otimes S_*(\Delta^n))_n \\ & & \Big\downarrow {\scriptstyle S_*(\alpha) \otimes S_*(\alpha)} \\ & & (S_*(X \times Y) \otimes S_*(X \times Y))_n \\ & & \Big\downarrow {\scriptstyle S_*(p_1) \otimes S_*(p_2)} \\ & & (S_*(X) \otimes S_*(Y))_n. \end{array}$$

$\square$

Theorem 15.5. *(Eilenberg-Zilber) The homology cross product $\times \colon S_*(X) \otimes S_*(Y) \longrightarrow S_*(X \times Y)$ is a homotopy equivalence of chain complexes.*

Proof. Let f be any natural chain map $S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ with $f_0(x_0, y_0) = x_0 \otimes y_0$ for any pair of points. Then

$$f \circ (- \times -) \colon S_*(X) \otimes S_*(Y) \to S_*(X) \otimes S_*(Y)$$

and this composition sends $x_0 \otimes y_0$ to itself. Using Lemma 15.2 for $X = \Delta^p$ and $Y = \Delta^q$ and then extending by naturality again, we get that the identity and $f \circ (- \times -)$ are homotopic. Similarly we get that, the composition $(- \times -) \circ f$ is homotopic to the identity. $\square$

Corollary 15.6. *(topological Künneth formula) For any pair of spaces X and Y the following sequence is split short exact*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

The sequence is natural in X and Y but the splitting is not.

Examples For the n-torus $T^n = (\mathbb{S}^1)^n$ we get

$$H_i(T^n) \cong \mathbb{Z}^{\binom{n}{i}}$$

For a space of the form $X \times \mathbb{S}^n$ we obtain

$$H_q(X \times \mathbb{S}^n) \cong H_q(X) \oplus H_{q-n}(X).$$

There is also a relative version of the Künneth formula. The homology cross product in its relative form is a map

$$\times \colon H_p(X, A) \otimes H_q(Y, B) \longrightarrow H_{p+q}(X \times Y, A \times Y \cup X \times B).$$

In particular for A and B a point we get a reduced Künneth formular which yields

$$\tilde{H}_p(X) \otimes \tilde{H}_q(Y) \longrightarrow \tilde{H}_{p+q}(X \times Y, X \vee Y).$$

If the points are strong deformation retracts of open neighborhoods, then the latter is isomorphic to $\tilde{H}_{p+q}(X \wedge Y)$.

CHAPTER 2

Singular cohomology

1. Definition of singular cohomology

Definition 1.1. A *cochain complex of abelian groups* is a sequence $(C^n)_{n\in\mathbb{Z}}$ of abelian groups C^n together with homomorphisms $\delta\colon C^n \to C^{n+1}$ with $\delta^2 = 0$. The map δ is called *coboundary operator*. The group

$$H^n(C^*) = \frac{\ker(\delta\colon C^n \to C^{n+1})}{\operatorname{im}(\delta\colon C^{n-1} \to C^n)}$$

is the *n-th cohomology group of* C^*.

Definition 1.2. For a topological space X we call $S^n(X) := \operatorname{Hom}(S_n(X), \mathbb{Z})$ the n-th singular cochain group of X and $\delta = \operatorname{Hom}(\partial, \mathbb{Z})$ is the corresponding coboundary operator.

For $\alpha\colon \Delta^{n+1} \to X$ and $\varphi\colon S_n(X) \to \mathbb{Z}$, $\delta(\varphi)(\alpha) = \varphi(\partial\alpha)$.

$$\begin{tikzcd}
S_n(X) \arrow[r, "\varphi"] & \mathbb{Z} \\
S_{n+1}(X) \arrow[u, "\partial"] \arrow[ur, "\delta\varphi"'] &
\end{tikzcd}$$

The composition $\delta^2(\varphi)(\beta)$ is $(\delta\varphi)(\partial\beta) = \varphi(\partial^2\beta) = 0$ for $\beta\colon \Delta^{n+2} \to X$.

Definition 1.3. Let G be an abelian group, then

$$S^n(X; G) := \operatorname{Hom}(S_n(X), G)$$

the *cochain group of* X *with coefficients in* G.

$$H_n(X; G) = \frac{\ker(\delta\colon S^n(X;G) \to S^{n+1}(X;G))}{\operatorname{im}(\delta\colon S^{n-1}(X;G) \to S^n(X;G))}$$

is the *n-th cohomology group of* X *with coefficients in* G.

Note that $S^n(-;G)$ and $H^n(-;G)$ are contravariant functors from the category of topological spaces and continuous maps to the category of abelian groups. For a continuous map $f\colon X \to Y$ we denote $S_*(f)$ by f_*. Then $S^*(f) = f^*\colon S^*(Y;G) \to S^*(X;G)$: For $\varphi \in S^n(Y;G)$ and $\alpha \in S_*(X)$,

$$f^*(\varphi)(\alpha) = \varphi(f_*\alpha) \in G.$$

Definition 1.4. For two cochain complexes (C^*, δ) and $(\tilde{C}^*, \tilde{\delta})$ a *map of cochain complexes* from C^* to $\tilde{C}^*$ is a sequence of homomorphisms $f^n\colon C^n \to \tilde{C}^n$ with $f^{n+1} \circ \delta = \tilde{\delta} \circ f^n$.

$$\begin{tikzcd}
C^{n+1} \arrow[r, "f^{n+1}"] & \tilde{C}^{n+1} \\
C^n \arrow[u, "\delta"] \arrow[r, "f^n"] & \tilde{C}^n. \arrow[u, "\tilde{\delta}"']
\end{tikzcd}$$

Maps of cochain complexes induce maps on cohomology. For example, every continuous map $f\colon X \to Y$ induces a map of cochain complexes $S^*(Y;G) \to S^*(X;G)$.

DEFINITION 1.5. • For two abelian groups A and G, $\varphi \in \mathrm{Hom}(A, G)$, $a \in A$ we define the *Kronecker pairing*

$$\langle -, - \rangle \colon \mathrm{Hom}(A, G) \otimes A \longrightarrow G, \quad \langle \varphi, a \rangle = \varphi(a) \in G.$$

- For a homomorphism $f \colon B \to A$, $f^*(\varphi) \in \mathrm{Hom}(B, G)$ and $b \in B$ we have
$$\langle f^*\varphi, b \rangle = \langle \varphi, fb \rangle = \varphi \circ f(b).$$
- For a chain complex C_* and $C^n = \mathrm{Hom}(C_n, G)$ we define
$$\langle -, - \rangle \colon C^n \otimes C_n \to G, \varphi \otimes a \mapsto \langle \varphi, a \rangle = \varphi(a).$$
- In particular, for $A = S_n(X)$ we get a Kronecker pairing
$$\langle -, - \rangle \colon S^n(X; G) \times S_n(X) \to G.$$
- For $\partial \colon S_{n+1}(X) \to S_n(X)$ and $a \in S_{n+1}(X)$ we get
$$\langle \delta\varphi, a \rangle = \langle \varphi, \partial a \rangle = \varphi(\partial(a)).$$

LEMMA 1.6. *The Kronecker pairing $\langle -, - \rangle \colon C^n \otimes C_n \to G$ is well-defined on the level of cohomology and homology,* i.e., *we obtained an induced map*

$$\langle -, - \rangle \colon H^n(C^*) \otimes H_n(C_*) \to G.$$

PROOF. Let φ be a cocycle, then

$$\langle \varphi, a + \partial b \rangle = \langle \varphi, a \rangle + \langle \varphi, \partial b \rangle = \langle \varphi, a \rangle + \langle \delta\varphi, b \rangle = \langle \varphi, a \rangle.$$

Therefore $\langle \varphi, - \rangle$ is well-defined on $H_n(C_*)$.

Assume that $\varphi = \delta\psi$ and a is a cycle. Then we get

$$\langle \varphi, a \rangle = \langle \delta\psi, a \rangle = \langle \psi, \partial a \rangle = 0.$$

□

Changing perspective, we get

$$\kappa \colon H^n(C^*) \longrightarrow \mathrm{Hom}(H_n(C_*), G)$$

via $\kappa[\varphi][a] := \langle \varphi, a \rangle$. How much does the map κ see?

Example Consider $X = \mathbb{R}P^2$ and $C_* = S_*(X)$ and as G we choose $\mathbb{Z}$. Thus

$$\mathrm{Hom}(H_1(\mathbb{R}P^2), \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0.$$

But we will see soon that $H^1(\mathbb{R}P^2)$ is not trivial. So in this case, the cohomology of the space $\mathbb{R}P^2$ is not dual to the homology.

2. Universal coefficient theorem for cohomology

Dual to Tor, we consider a corresponding construction for the functor $\mathrm{Hom}(-, -)$ instead of $(-) \otimes (-)$. For a short exact sequence

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

the sequence

$$0 \to \mathrm{Hom}(C, G) \longrightarrow \mathrm{Hom}(B, G) \longrightarrow \mathrm{Hom}(A, G) \to 0$$

doesn't have to be exact. A problem can arise with respect to the surjectivity at the end.

As an example, consider $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$ for a natural number $n > 1$. Then the sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{n} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

is exact but multiplication by n isn't surjective, so we cannot prolong this sequence to the right with a zero.

DEFINITION 2.1. For a free resolution $0 \to R \xrightarrow{i} F \longrightarrow A \to 0$ we call the cokernel of $\mathrm{Hom}(i, G) \colon \mathrm{Hom}(F, G) \to \mathrm{Hom}(R, G)$ $\mathrm{Ext}(A, G)$.

Here, Ext comes from 'extension', because one can describe $\mathrm{Ext}(A, G)$ in terms of extensions of abelian groups.

- As for Tor it is true that $\mathrm{Ext}(A,G)$ is independent of the free resolution of A.
- Note that $\mathrm{Ext}(A,G)$ is covariant in G and contravariant in A: for homomorphisms $f\colon A \to B$ and $g\colon G \to H$ we get
$$f^*\colon \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G), g_*\colon \mathrm{Ext}(A,G) \to \mathrm{Ext}(A,H).$$
- For a family of abelian groups $(G_i, i \in I)$
$$\mathrm{Ext}(A, \prod_{i\in I} G_i) \cong \prod_{i\in I} \mathrm{Ext}(A, G_i)$$
and
$$\mathrm{Ext}(\bigoplus_{i\in I} G_i, B) \cong \prod_{i\in I} \mathrm{Ext}(G_i, B).$$
- The group $\mathrm{Ext}(A,G)$ is trivial if A is free abelian.
- Correspondingly, $\mathrm{Ext}(A,G)$ is trivial if G is divisible, *i.e.*, for all $g \in G$ and $n \in \mathbb{Z}\backslash\{0\}$ there is a $t \in G$ with $g = nt$. For example this holds if G is isomorphic to $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$, or $\mathbb{C}$.
- For natural numbers n and m
$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}.$$
- More generally,
$$\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG.$$

THEOREM 2.2. *(Universal coefficient theorem for cochain complexes) For every free chain complex C_* and $C^* = \mathrm{Hom}(C_*, G)$ the following sequence is exact and splits*
$$0 \to \mathrm{Ext}(H_{n-1}(C_*), G) \longrightarrow H^n(C^*) \xrightarrow{\kappa} \mathrm{Hom}(H_n(C_*), G) \to 0.$$

THEOREM 2.3. *(Universal coefficient theorem for singular cohomology) Let X be an arbitrary space. Then the sequence*
$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \xrightarrow{\kappa} \mathrm{Hom}(H_n(X), G) \to 0$$
is split exact.

Example We know that the homology of $\mathbb{C}P^n$ is free with
$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & 0 \leqslant k \leqslant 2n, k \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$
Therefore $H^k(\mathbb{C}P^n) \cong \mathrm{Hom}(H_k(\mathbb{C}P^n), \mathbb{Z})$, thus the cohomology is given by the $\mathbb{Z}$-dual of the homology.

PROOF OF THEOREM 2.2. Let C_* be a free chain complex and $C^* = \mathrm{Hom}(C_*, G)$. Then the sequence $0 \to Z_n \longrightarrow C_n \longrightarrow B_{n-1} \to 0$ is split exact. Therefore the G-dual sequence
$$0 \to B^{n-1} \longrightarrow C^n \longrightarrow Z^n \to 0$$
is short exact and it gives a short exact sequence of cochain complexes, where we view B^* and Z^* as cochain complexes with trivial differential. This yields a long exact sequence on the level of cohomology groups
$$\ldots Z^{n-1} \xrightarrow{\partial} B^{n-1} \longrightarrow H^n(C^*) \longrightarrow Z^n \xrightarrow{\partial} B^n \longrightarrow \ldots$$
Here, ∂ denotes the connecting homomorphism in the cohomological case. By the very definition of the connecting homomorphism we get that ∂ is the dual of the inclusion $i_n\colon B_n \subset Z_n$, $\partial = i_n^*$. We cut the long exact sequence above into the short one
$$0 \to \mathrm{coker}(i_{n-1}^*) \longrightarrow H^n(C^*) \longrightarrow \ker(i_n^*) \to 0$$
and hence we have to identify the kernel and the cokernel above.

The exact sequence
$$0 \to \mathrm{Hom}(H_n(C_*), G) \xrightarrow{\pi^*} \mathrm{Hom}(Z_n, G) \xrightarrow{i_n^*} \mathrm{Hom}(B_n, G)$$
tells us that the kernel of i_n^* is the image of π^* and due to the injectivity of π^* this is isomorphic to $\mathrm{Hom}(H_n(C_*), G)$.

The sequence

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_*) \to 0$$

is a free resolution of $H_{n-1}(C_*)$ and therefore the cokernel of i_{n-1}^* is $\mathrm{Ext}(H_{n-1}(C_*), G)$. $\square$

3. Axiomatic description of a cohomology theory

Before we give an axiomatic description of singular homology, we establish some consequences of some of the results we proved for singular homology.

- For a chain map $f\colon C_* \to C'_*$ (such as the barycentric subdivision) the G-dual map
$$f^* = \mathrm{Hom}(f, G)\colon \mathrm{Hom}(C'_*, G) \longrightarrow \mathrm{Hom}(C_*, G)$$
is a map of cochain complexes.
- If $(H_n\colon C_n \to C'_{n+1})_n$ is a chain homotopy, then the G-dual
$$(H^n := \mathrm{Hom}(H_n, G)\colon \mathrm{Hom}(C'_{n+1}, G) \to \mathrm{Hom}(C_n, G))_n$$
is a cochain homotopy. Thus if $\partial H_n + H_{n-1}\partial = f_n - g_n$, then $H^n\delta + \delta H^{n-1} = f^n - g^n$.
- For a split exact sequence $0 \to B_1 \longrightarrow B_2 \longrightarrow B_3 \to 0$ the dual sequence $0 \to \mathrm{Hom}(B_3, G) \longrightarrow \mathrm{Hom}(B_2, G) \longrightarrow \mathrm{Hom}(B_1, G) \to 0$ is exact. For instance, if A is a subspace of X, then the short exact sequence
$$0 \to S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) \to 0$$
is split. We define $r_n\colon S_n(X) \to S_n(A)$ on $\alpha\colon \Delta^n \to X$ via
$$r_n(\alpha) \begin{cases} \alpha, & \text{if } \alpha(\Delta^n) \subset A, \\ 0, & \text{otherwise.} \end{cases}$$
Therefore $0 \to S^*(X, A) \longrightarrow S^*(X) \longrightarrow S_*(A) \to 0$ is a short exact sequence.

With the help of these facts we can show that singular cohomology satisfies the *axioms of a cohomology theory*:

(a) The assignment $(X, A) \mapsto H^n(X, A)$ is a contravariant functor from the category of pairs of topological spaces to the category of abelian groups.

(b) For any subspace $A \subset X$ there is a natural homomorphism $\partial\colon H^n(A) \to H^{n+1}(X, A)$

(c) If $f, g\colon (X, A) \to (Y, B)$ are two homotopic maps of pairs of topological spaces, then $H^n(f) = H^n(g)\colon H^n(Y, B) \to H^n(X, A)$.

(d) For any subspace $A \subset X$ we get a long exact sequence
$$\ldots \xrightarrow{\partial} H^n(X, A) \longrightarrow H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial} \ldots$$

(e) Excision holds, *i.e.*, for $W \subset \bar{W} \subset \mathring{A} \subset A \subset X$
$$H^n(i)\colon H^n(X, A) \cong H^n(X \backslash W, A \backslash W), \text{ for all } n \geqslant 0.$$

(f) Let pt be the one-point space, then
$$H^n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$
This is called the axiom about the coefficients or the dimension axiom.

(g) Singular cohomology is additive:
$$H^n(\bigsqcup_{i \in I} X_i) \cong \prod_{i \in I} H^n(X_i).$$

For singular cohomology with coefficients in G we have an analoguous set of axioms.

There are generalized cohomology theories like topological K-theory or cobordism theories that satisfy all axioms but the dimension axiom.

Note that

$$H^m(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & m = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geqslant 1$. For later use we choose $\nu_n \in H^n(\mathbb{S}^n)$ with $\langle \nu_n, \mu_n \rangle = 1$.

4. Cap product

The rough idea of the cap product is to digest a piece of a chain with a cochain of smaller or equal degree.

Definition 4.1. Let $a\colon \Delta^n \to X$ and let $0 \leqslant q \leqslant n$.

- The $(n-q)$-dimensional front face of a is

$$V(a) = V^{n-q}(a) = a \circ v\colon \Delta^{n-q} \hookrightarrow \Delta^n \xrightarrow{a} X$$

where v is the inclusion $v\colon \Delta^{n-q} \hookrightarrow \Delta^n$ with $v(e_i) = e_i$ for $0 \leqslant i \leqslant n-q$.

- The q-dimensional back face of a is

$$H(a) = H^q(a) = a \circ h\colon \Delta^q \hookrightarrow \Delta^n \xrightarrow{a} X$$

where $h\colon \Delta^q \hookrightarrow \Delta^n$ is the inclusion with $h(e_0) = e_{n-q}, \ldots, h(e_q) = e_n$, *i.e.*, $h(e_i) = e_{n-(q-i)}$.

Definition 4.2. Let R be an associative ring with unit. We define

$$\cap\colon S^q(X,A;R) \otimes S_n(X,A;R) = \mathrm{Hom}(S_q(X,A),R) \otimes S_n(X,A) \otimes R \longrightarrow S_{n-q}(X;R) = S_{n-q}(X) \otimes R$$

as

$$\alpha \cap (a \otimes r) := V(a) \otimes \langle \alpha, H^q(a) \rangle r.$$

(a) The map $\cap$ is well-defined: for $a = a' \in S_n(X,A)$, *i.e.*, $a = a' + b$ with $\mathrm{im}(b) \subset A$ we get

$$\alpha \cap (a \otimes r) = \alpha \cap ((a'+b) \otimes r) = \alpha \cap (a' \otimes r) + V(b) \otimes \langle \alpha, H(b) \rangle r.$$

The image of $H(b)$ is contained in A, but $\alpha \in \mathrm{Hom}(S_q(X,A),R)$, thus $\alpha\colon S_q(X) \to R$ with $\alpha|_{S_q(A)} = 0$ and $\langle \alpha, H(b) \rangle = 0$.

(b) We can express the $(n-q)$-dimensional front face of a can be expressed as

$$V^{n-q}(a) = \partial_{n-q+1} \circ \ldots \circ \partial_n(a).$$

Similarly,

$$H^q(a) = \partial_0 \circ \ldots \circ \partial_0(a).$$

(c) There is a more general version of the cap product. If there is a pairing of abelian groups

$$G \otimes G' \to G''$$

then we can define

$$\cap\colon S^q(X,A;G) \otimes S_n(X,A;G') \to S_{n-q}(X;G'').$$

Proposition 4.3. *The Leibniz formular holds for the cap product,* i.e.,

$$\partial(\alpha \cap (a \otimes r)) = (-1)^q (\delta\alpha) \cap (a \otimes r) + \alpha \cap (\partial a \otimes r).$$

For a map of pairs of spaces $f\colon (X,A) \to (X,B)$

$$f_*(f^*(\beta) \cap (a \otimes r)) = \beta \cap (f_*(a) \otimes r).$$

Here, $f_*\colon S_*(X,A) \to S_*(Y,B)$ *and* $f^*\colon S^*(Y,B) \to S^*(X,A)$.

For the proof we suppress the tensor product with R. It just adds to notational complexity.

Proof. For the first claim we calculate

$$\begin{aligned}\partial(\alpha \cap a) &= \partial(V(a) \otimes \langle \alpha, H(a)\rangle)\\ &= \partial(V(a)) \otimes \langle \alpha, H(a)\rangle\\ &= \sum_{i=0}^{n-q} (-1)^i \partial_i(\partial_{n-q-1} \circ \ldots \circ \partial_n(a)) \otimes \langle \alpha, H(a)\rangle\end{aligned}$$

and

$$\begin{aligned}(\delta\alpha) \cap a &= V(a) \otimes \langle \delta\alpha, H(a)\rangle\\ &= V(a) \otimes \langle \alpha, \partial H(a)\rangle\\ &= \sum_{i=0}^{q} (-1)^i V(a) \otimes \langle \alpha, \partial_i \partial_0^{n-q}(a)\rangle.\end{aligned}$$

Finally,

$$\begin{aligned}\alpha \cap \partial a &= \sum_{j=0}^{n} (-1)^j \alpha \cap \partial_j a\\ &= \sum_{j=0}^{n} (-1)^j V(\partial_j a) \otimes \langle \alpha, H(\partial_j a)\rangle\\ &= \sum_{j=0}^{n} (-1)^j V(\partial_j a) \otimes \langle \alpha, H(\partial_j(a))\rangle\\ &= \sum_{j=0}^{n} (-1)^j \partial_{n-q-1} \circ \ldots \circ \partial_{n-1} \circ \partial_j a \otimes \langle \alpha, \partial_0^{n-q-1} \partial_i a\rangle.\end{aligned}$$

In order to get the result, use that $\partial_j \partial_i = \partial_{i-1}\partial_j$ for $0 \leqslant j < i \leqslant n$.

For the claim about naturality we plug in the definitions and obtain

$$\begin{aligned}f_*(f^*(\beta) \cap a) &= f_*(V(a) \otimes \langle f^*\beta, H(a)\rangle)\\ &= f_*(V(a) \otimes \langle \beta, f_* H(a)\rangle)\\ &= V(f_*(a)) \otimes \langle \beta, H(f_*(a))\rangle)\\ &= \beta \cap f_*(a).\end{aligned}$$

□

From the Leibniz formula we get that the cap product satisfies that

- a cocycle cap a cycle is a cycle,
- a cocycle cap a boundary is a boundary,
- a coboundary cap a cycle is a boundary.

Therefore we obtain the following result

Proposition 4.4. *The cap product induces a map*

$$\cap \colon H^q(X, A; R) \otimes H_n(X, A; R) \longrightarrow H_{n-q}(X; R)$$

via

$$[\alpha] \cap [a] := [V(a) \otimes \langle \alpha, H(a)\rangle]$$

Examples Let R be a ring and consider $1 \in S^0(X; R)$, *i.e.*, $1(a) = 1$ for all $a \in S_0(X)$. We claim that $1 \cap a = a$. We have $V(a) = a$ because $q = 0$ and $H(a)(e_0) = a(e_n)$. Therefore, $1 \cap a = a \otimes \langle a, a(e_n)\rangle = a \otimes 1$ and we identify the latter with a.

For a space X and $\alpha \in S^n(X; G)$, $a \in S_n(X; G)$ we get

$$\alpha \cap a = a(e_0) \otimes \langle \alpha, a\rangle$$

because $V(a)(e_0) = a(e_0)$ and $H(a) = a$. In this sense, the cap product generalizes the Kronecker pairing: if X is path-connected, then $[a(e_0)] \in H_0(X)$ is a generator, and thus we can identify it with $1 \in Z$.

There is also a version of the cap product of the form

$$\cap\colon H^q(X;R) \otimes H_n(X,A;R) \longrightarrow H_{n-q}(X,A;R).$$

Where does the notation $\cap$ come from? For instance if we take a torus T and the meridian $b \subset T$, then we consider the class $\beta \in H^1(T)$ dual to $[b] \in H_1(T)$. We know that $H_2(T) \cong \mathbb{Z}$ and we denote the generator by σ. Then $\beta \cap \sigma$ can be represented by the 1-dimensional submanifold of T given by the longitude $a \subset T$. This submanifold is transversal to b.

5. Cup product

In the following, let R be a commutative ring with unit and we will consider homology and cohomology with coefficients in R, but we will suppress the R in our notation, so $H_n(X,A)$ will stand for $H_n(X,A;R)$ and similarly $S_n(X,A)$ is $S_n(X,A;R)$. We'll use analogous abbreviations for cochains and cohomology. Sometimes, if we have to be explicit, we denote the multiplication in R by μ.

Definition 5.1. For $\alpha \in S^p(X,A)$ and $\beta \in S^q(Y,B)$ we define the *cohomology cross product*, $\times$, as

$$\alpha \times \beta := \mu \circ (\alpha \otimes \beta) \circ \mathrm{EZ}$$

where EZ is any Eilenberg-Zilber map

$$S_*(X \times Y; X \times B \cup A \times Y) \longrightarrow S_*(X,A) \otimes S_*(Y,B)$$

Thus

$$\begin{tikzcd}
S_n(X \times Y; X \times B \cup A \times Y) \arrow[d, "\mathrm{EZ}"'] \arrow[ddrr, "\alpha\times\beta"] & & \\
\bigoplus_{p+q=n} S_p(X,A) \otimes S_q(Y,B) \arrow[d] & & \\
S_p(X,A) \otimes S_q(Y,B) \arrow[r, "\alpha\otimes\beta"'] & R \otimes R \arrow[r, "\mu"'] & R
\end{tikzcd}$$

- The cohomology cross product is natural, *i.e.*, for maps of pairs of spaces $f\colon (X,A) \to (X',A')$, $g\colon (Y,B) \to (Y',B')$
$$(f,g)^*(\alpha \times \beta) = (f^*\alpha) \times (g^*\beta).$$
- The Leibniz formula holds
$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^{|\alpha|}\alpha \times (\delta\beta).$$
Here, $|\alpha|$ denotes the degree of α.
- For the Kronecker pairing we have for cohomology classes α, β and homology classes a, b of a corresponding degree
$$\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \langle \beta, b \rangle.$$
- For $1 \in R$ and thus $1 \in S^0(X,A)$
$$1 \times \beta = p_2^*(\beta), \alpha \times 1 = p_1^*(\alpha)$$
where p_i $(i = 1,2)$ denotes the projection onto the i-th factor in $X \times Y$.
- The cohomology cross product is associative
$$\alpha \times (\beta \times \gamma) = (\alpha \times \beta) \times \gamma$$
on the level of cohomology groups.
- It satisfies a graded version of commutativity. The twist map $\tau\colon X \times Y \to Y \times X$ yields on cohomology
$$\alpha \times \beta = (-1)^{|\alpha||\beta|}\tau^*(\beta \times \alpha).$$

We will use the cohomology cross product in order to obtain a multiplication on H^*. Let $\Delta\colon X \to X \times X$.

Definition 5.2. For $\alpha \in H^p(X,A)$ and $\beta \in H^q(X,B)$ we define the *cup-product of α and β* as

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

$$\begin{tikzcd} H^p(X,A) \otimes H^q(X,B) \arrow[r, "\times"] \arrow[dr, "\cup"'] & H^{p+q}(X \times X, X \times B \cup A \times X) \arrow[d, "\Delta^*"] \\ & H^{p+q}(X, A \cup B) \end{tikzcd}$$

Proposition 5.3. *Let α, β, γ be cohomology classes. The cup product satisfies*

(a)
$$\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma.$$

(b)
$$\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha.$$

(c) *For $\partial \colon H^*(A) \to H^{+*1}(X,A)$, $\alpha \in H^*(A)$, $\beta \in H^*(X)$*
$$\partial(\alpha \cup i^*\beta) = (\partial\alpha) \cup \beta \in H^*(X,A).$$

(d) *For $f \colon X \to Y$*
$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

(e) *We can express the cohomology cross product via the cup product*
$$\alpha \times \beta = p_1^*(\alpha) \cup p_2^*(\beta).$$

Proof. The properties can be deduced from the properties of the cross product, thus we only prove the last claim. So let $\alpha \in H^p(X)$, $\beta \in H^q(X)$.

$$p_1^*(\alpha) \cup p_2^*(\beta) = (\alpha \times 1) \cup (1 \times \beta).$$

Here, $\alpha \times 1$ and $1 \times \beta$ live in the cohomology of $X \times X$. By definition, the cup product is the pull-back of the cross product by the diagonal. Here, $\Delta_{X \times X} \colon X \times X \to X^4$. Therefore, the above is equal to

$$\Delta^*_{X \times X}((\alpha \times 1) \times (1 \times \beta)) = \alpha \times \beta.$$

□

We will get an explicit formula of the cup product by choosing a nice version of the Eilenberg-Zilber map.

Definition 5.4. A *diagonal approximation* is a natural chain map $D \colon S_*(X) \longrightarrow S_*(X) \otimes S_*(X)$ with $D(x) = x \otimes x$ for $x \in S_0(X)$.

With the method of acyclic methods one can prove

Proposition 5.5. *Any two diagonal approximations are chain homotopic.*

Definition 5.6. The *Alexander-Whitney map* is the diagonal approximation

$$\mathrm{AW}(a) = \sum_{p+q=n} V^p(a) \otimes H^q(a)$$

for $a \in S_n(X)$.

It is obvious that AW is a chain map and this map yields

$$(\alpha \cup \beta)(a) = \mu \circ (\alpha \otimes \beta)\mathrm{AW}(a) = \mu \circ (\alpha \otimes \beta) \sum_{p+q=n} (V^p(a) \otimes H^q(a)) = (-1)^{pq} \alpha(V^p(a))\beta(H^q(a)).$$

This formula gives that $\cup$ is associative on cochain level and not just on the level of cohomology groups. But note, that is does not give a (graded) commutative product on singular cochains. (The cup product is homotopy commutative and in fact it is homotopy commutative up to coherent homotopies, it is an *E_∞-algebra*.)

Note, that with this model of the cup product, the properties in Proposition 5.3 can be checked directly.

Proposition 5.7. (a) *For all pairs of spaces (X,A) the cohomology groups $H^*(X,A;R)$ have a structure of a graded commutative ring with unit $1 \in H^0(X,A;R)$.*

(b) *The ring $H^*(X,A;R)$ acts on $H_*(X,A;R)$ via the cap product*

$$H^*(X,A;R) \otimes H_*(X,A;R) \ni \alpha \otimes a \mapsto \alpha \cap a,$$

i.e., $1 \cap a = a$, $(\alpha \cup \beta) \cap a = \alpha \cap (\beta \cap a)$. *Thus $H_*(X,A;R)$ is a graded module over the graded ring $H^*(X,A;R)$.*

Examples Many cup products are trivial for degree reasons.

(a) Let $\mathbb{S}^n$ be a sphere of dimension $n \geqslant 1$. We know that $H^0(\mathbb{S}^n) \cong \mathbb{Z} \cong H^n(\mathbb{S}^n)$ and the cohomology is trivial in all other degrees. We have $1 \in H^0(\mathbb{S}^n)$ and $\nu_n \in H^n(\mathbb{S}^n)$. We know that

$$1 \cup \nu_n = \nu_n = \nu_n \cup 1, 1 \cup 1 = 1$$

but $\nu_n \cup \nu_n = 0 \in H^{2n}(\mathbb{S}^n)$. Thus, $H^*(\mathbb{S}^n)$ has the structure of a so-called graded exterior algebra with the generator ν_n.

(b) More generally, if X is a CW complex of finite dimension, then $\alpha \cup \beta = 0$ for all α, β for $|\alpha| + |\beta|$ exceeding the dimension of X.

(c) In particular, $H^*(X)$ often has nilpotent elements: if

$$\alpha^r := \underbrace{\alpha \cup \ldots \cup \alpha}_{r} = 0,$$

then $(\alpha \cup \beta)^r = \pm \alpha^r \cup \beta^r = 0$.

(d) Assume that $\alpha \in H^p(X;R)$ with an odd p, then

$$\alpha^2 = (-1)^{p^2} \alpha^2 = -\alpha^2.$$

Therefore $2\alpha^2 = 0$ and if R is a field of characteristic not equal to 2 or if R is torsionfree, then $\alpha^2 = 0$.

(e) If $X = X_1 \vee X_2$ and X_1, X_2 are well-pointed, then $H^*(X) \cong H^*(X_1) \oplus H^*(X_2)$ as rings: for $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ with $\alpha_i, \beta_i \in H^*(X_i)$ the cup product is

$$\alpha \cup \beta = (\alpha_1 + \alpha_2) \cup (\beta_1 + \beta_2) = \alpha_1 \cup \beta_1 + \alpha_2 \cup \beta_2.$$

(f) If X can be covered like $X = X_1 \cup \ldots \cup X_r$ with $H^*(X_i) = 0$ for $* \geqslant 1$, then in $H^*(X)$ all r-fold cup products of elements of positive degree vanish. We prove the case where $r = 2$; the general claim then follows by induction. So assume $X = X_1 \cup X_2$ such that the X_i have vanishing cohomology groups in positive degrees and let $i_j\colon X_j \hookrightarrow X$ be the inclusion of X_j into X ($j = 1, 2$). Then for all $\alpha \in H^*(X)$, $i_j^*(\alpha) = 0$. Consider the exact sequence

$$H^*(X, X_j) \longrightarrow H^*(X) \longrightarrow H^*(X_j).$$

Therefore, for all α there is an $\alpha' \in H^*(X, X_1)$ that is mapped isomorphically to α. Similarly, for $\beta \in H^*(X)$ there is an $\beta' \in H^*(X, X_2)$ that corresponds to β. The cup product $\alpha \cup \beta$ then corresponds to $\alpha' \cup \beta'$ but this is an element of $H^*(X, X_1 \cup X_2) = H^*(X, X) = 0$.

(g) Consider a product of spheres, $X = \mathbb{S}^n \times \mathbb{S}^m$ with $n, m \geqslant 1$. The Künneth formula and the universal coefficient theorem tell us that

$$H^*(\mathbb{S}^n \times \mathbb{S}^m) \cong H^*(\mathbb{S}^n) \otimes H^*(\mathbb{S}^m).$$

We have three additive generators

$$\alpha_n = \nu_n \times 1, \beta_m = 1 \times \nu_m, \quad \text{and } \gamma_{n+m} = \nu_n \times \nu_m.$$

The square α_n^2 is trivial:

$$\alpha_n^2 = (\nu_n \times 1) \cup (\nu_n \times 1) = (\nu_n \cup \nu_n) \times (1 \cup 1) = 0.$$

Similarly, $\beta_m^2 = 0 = \gamma_{n+m}^2$. But the product

$$\alpha_n \cup \beta_m = \nu_n \times \nu_m = \gamma_{n+m}.$$

This determines the ring structure of $H^*(\mathbb{S}^n \times \mathbb{S}^m)$. In particular, the cohomology ring $H^*(\mathbb{S}^n \times \mathbb{S}^m)$ is *not* isomorphic to the cohomology ring $H^*(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$. Additively, both graded abelian groups are isomorphic, thus the graded cohomology ring is a finer invariant than the cohomology groups.

Note that the cohomology rings of $\Sigma(\mathbb{S}^n \times \mathbb{S}^m)$ and $\Sigma(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m})$ are isomorphic (namely here cup products of elements of positive degree are trivial due to example (f)). But here, we actually have

$$\Sigma(\mathbb{S}^n \times \mathbb{S}^m) \simeq \Sigma(\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m}).$$

6. Orientability of manifolds

Definition 6.1. A topological space X is called *locally euclidean*, if every point $x \in X$ has an open neighborhood U which is homeomorphic to an open subset $V \subset \mathbb{R}^m$.

- A homeomorphism $\varphi\colon U \to V$ is called a *chart*.
- A set of charts is called *atlas*, if the corresponding $U \subset X$ cover X.
- The number m is *the dimension of* X.

Example Consider the line with two origins, *i.e.*, let

$$X = \{(x, 1) | x \in \mathbb{R}\} \cup \{(x, -1) | x \in \mathbb{R}\}/\sim, \quad (x, 1) \sim (x, -1) \text{ for } x \neq 0.$$

Then X is locally euclidean, but X is not a particularly nice space. For instance, it is not hausdorff: you cannot separate the two origins.

Definition 6.2. A topological space X is an *m-dimensional (topological) manifold* (or m-manifold for short) if X is a locally euclidean space of dimension m that is hausdorff and has a countable basis for its topology.

Examples

(a) Let $U \subset \mathbb{R}^m$ an open subset, then U is a topological manifold of dimension m.
(b) The n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an n-manifold and $\mathbb{S}^n = \mathbb{S}^n \backslash N \cup \mathbb{S}^n \backslash S$ is an atlas of $\mathbb{S}^n$.
(c) The 2-dimensional torus $T \cong \mathbb{S}^1 \times \mathbb{S}^1$ is a 2-manifold and more generally, the surfaces F_g are 2-manifolds. Charts can be easily given via the $4g$-gon whose quotient is F_g.
(d) The open Möbius strip $[-1, 1] \times (-1, 1)/\sim$ with $(-1, t) \sim (1, -t)$ is a 2-manifold.

Let M be a connected manifold of dimension $m \geqslant 2$. We denote the open charts by $U_\alpha \subset M$. Without loss of generality we can assume that

$$\varphi\colon U_\alpha \cong \mathring{\mathbb{D}}^m \subset \mathbb{R}^m$$

and for an $x \in M$ we can choose charts with $\varphi(x) = 0$. Excision tells us that for all $x \in M$

$$H_m(M, M\backslash x) \cong H_m(\mathring{\mathbb{D}}^m, \mathring{\mathbb{D}}^m \backslash\{0\}) \cong H_{m-1}(\mathring{\mathbb{D}}^m \backslash\{0\}) \cong \mathbb{Z}$$

for $m \geqslant 2$.

For a triple $B \subset A \subset M$ there are maps of pairs

$$\varrho_{B,A}\colon (M, M\backslash A) \longrightarrow (M, M\backslash B).$$

Definition 6.3. An m-manifold M is *orientable* (with respect to $\mathbb{Z}$) if there is a coherent choice of generators $o_x \in H_m(M, M\backslash x)$, *i.e.*, for all $x \in M$ and for all neighborhoods U of x with $U \subset U_\alpha$ with $U \cong \mathring{\mathbb{D}}^m$ and for all $x_1, x_2 \in U$ we have

$$o_{x_2} = \varrho_{x_2,U} \circ (\varrho_{x_1,U})^{-1}(o_{x_1}).$$

$$\begin{tikzcd}
& H_m(M, M\backslash U) \arrow[dl, "\varrho_{x_1,U}"'] \arrow[dr, "\varrho_{x_2,U}"] & \\
o_{x_1} \in H_m(M, M\backslash x_1) & & H_m(M, M\backslash x_2) \ni o_{x_2}
\end{tikzcd}$$

Definition 6.4. If such a choice is possible, then $(o_x | x \in M)$ is an *orientation of* M.

Note that for an orientation $(o_x|x \in M)$ the family $(-o_x|x \in M)$ is an orientation of M as well.

Example If M is the open Möbius strip and you pick a generator $o_x \in H_2(M, M\backslash x)$ and you walk once around the Möbius strip, you end up at $-o_x$.

If we choose other coefficients, these problems can disappear. For instance for $G = \mathbb{Z}/2\mathbb{Z}$ there is no problem to choose coherent generators for $H_2(M, M\backslash x; \mathbb{Z}/2\mathbb{Z})$, so the Möbius strip is $\mathbb{Z}/2\mathbb{Z}$-*orientable*.

Now, we consider integral coefficients again. What we want to have is a global class $o_M \in H_m(M;\mathbb{Z}) = H_m(M)$ with

$$\varrho_{x,M} =: \varrho_x \colon H_m(M) \to H_m(M, M\backslash x), \quad \varrho_x(o_M) = o_x$$

if $(o_x|x \in M)$ is an orientation of M.

Example If $M = \mathbb{R}P^2$, then $H_2(\mathbb{R}P^2) = 0$, but $H_2(\mathbb{R}P^2, \mathbb{R}P^2\backslash x) \cong \mathbb{Z}$, so here we cannot have such a class.

Definition 6.5. Let $K \subset M$ be a compact subset of M. We call an $o_K \in H_m(M, M\backslash K)$ an *orientation of* M *along* K, if the classes $o_x := (\varrho_{x,K})_*(o_K)$ constitute a coherent choice of generators for all $x \in K$.

Of course, if we have a global class o_M which gives coherent generators o_x for all $x \in M$, then of course we get a class o_K as above for all compact $K \subset M$.

Lemma 6.6. *Let M be a connected topological manifold of dimension m and assume that M is orientable. Let $K \subset M$ be compact. Then*

- $H_q(M, M\backslash K) = 0$ *for all* $q > m$, *and*
- *if* $a \in H_m(M, M\backslash K)$, *then* a *is trivial if and only if* $(\varrho_{x,K})_*(a) = 0$ *for all* $x \in K$.

The following method of proof is a standard method in the theory of manifolds.

Proof.

(a) First, let $M = \mathbb{R}^m$ and let K be convex and compact in M. In this case we can assume without loss of generality that $K \subset \mathring{\mathbb{D}}^m$. We calculate

$$H_q(M, M\backslash K) = H_q(\mathbb{R}^m, \mathbb{R}^m\backslash K) H_q(\mathbb{R}^m, \mathbb{R}^m\backslash x) = 0, \text{ for } q > m.$$

All identifications are isomorphisms and this gives the second claim as well.

(b) Let M be again $\mathbb{R}^m$ and let $K = K_1 \cup K_2$ with K_1, K_2 as in (a). In this case the claims follow with the help of the Mayer-Vietoris sequence, because K_1, K_2 and $K_1 \cap K_2$ satisfy the assumptions as in (a).

(c) An induction shows the case of $M = \mathbb{R}^m$ and $K = K_1 \cup \ldots \cup K_r$ with K_i as in (a).

(d) Let $M = \mathbb{R}^m$ and let K be an arbitrary compact subset and let $a \in H_q(M, M\backslash K)$. Choose a $\psi \in S_q(\mathbb{R}^m)$ with $[j_*(\psi)] = a$ with $j \colon \mathbb{R}^m \to (\mathbb{R}^m, \mathbb{R}^m\backslash K)$. The boundary of ψ, $\partial(\psi)$, has to be of the form

$$\partial(\psi) = \sum_{j=1}^{\ell} \lambda_j \tau_j$$

with $\tau_j \colon \Delta^{q-1} \to \mathbb{R}^m\backslash K$. As Δ^{q-1} is compact, the union

$$\bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1}) \subset \mathbb{R}^m\backslash K$$

is compact.

There exists an open neighborhood U of K in $\mathbb{R}^m$ with

$$\bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1}) \cap U = \varnothing.$$

Therefore ψ gives a cycle in $S_*(\mathbb{R}^m, \mathbb{R}^m\backslash U)$ and we let $a' \in H_q(\mathbb{R}^m, \mathbb{R}^m\backslash U)$ be the corresponding class. Thus

$$(\varrho_{K,U})_*(a') = a.$$

Choose closed balls $B_1, \dots, B_r \subset \mathbb{R}^m$ with $B_i \subset U$ for all i and $K \cap B_i \neq \varnothing$ such that $K \subset \bigcup_{i=1}^r B_i$. Consider the restriction maps

$$(\mathbb{R}^m, \mathbb{R}^m \backslash U) \xrightarrow{\varrho_{\bigcup B_i, U}} (\mathbb{R}^m, \mathbb{R}^m \backslash \bigcup_{i=1}^r B_i) \xrightarrow{\varrho_{K, \bigcup B_i}} (\mathbb{R}^m, \mathbb{R}^m \backslash K).$$

Define a'' as $a'' := (\varrho_{\bigcup B_i, U})_*(a')$. Note that $(\varrho_{K,\bigcup B_i})_*(a'') = a$.

The B_i are convex and compact and therefore

$$(\varrho_{\bigcup B_i, U})_*(a') = 0 = a'', \text{ for all } q > m$$

and hence $a = 0$.

Let $q = m$ and assume that $(\varrho_{x,K})_*(a) = 0$ for all $x \in K$. We have to show that a is trivial. We express $(\varrho_{x,K})_*(a)$ as above as

$$(\varrho_{x,K})_*(a) = (\varrho_{x,K})_* \circ (\varrho_{K,\bigcup B_i})_*(a'') = (\varrho_{x,\bigcup B_i})_*(a'') = 0$$

for all $x \in K$. For every $x \in B_i \cap K$ the above composition is equal to $(\varrho_{x,B_i})_* \circ (\varrho_{B_i,\bigcup B_i})_*(a'')$, but $(\varrho_{x,B_i})_*$ is an isomorphism and hence $(\varrho_{B_i,\bigcup B_i})_*(a'') = 0$. This implies $(\varrho_{y,B_i})_* \circ (\varrho_{B_i,\bigcup B_i})_*(a'') = 0$ for all $y \in B_i$ and in addition $(\varrho_{y,\bigcup B_i})_*(a'') = 0$ for all $y \in \bigcup B_i$. According to case (c) this implies that $a'' = 0$ and therefore $a = (\varrho_{K,\bigcup B_i})_*(a'')$ is trivial as well.

(e) Now let M be arbitrary and $K \subset U_\alpha \cong \mathbb{R}^m$. Therefore

$$H_q(M, M \backslash K) \cong H_q(U_\alpha, U_\alpha \backslash K) \cong H_q(\mathbb{R}^m, \mathbb{R}^m \backslash \mathrm{im}(K)).$$

As the image of K is compact in $\mathbb{R}^m$, the claim follows from (d).

(f) If M and K are arbitrary, then $K = K_{\alpha_1} \cup \ldots \cup K_{\alpha_r}$ with $K_{\alpha_i} = K \cap U_{\alpha_i} \neq \varnothing$. An induction as in (c) then proves the claim.

□

Proposition 6.7. *Let $K \subset M$ be compact and assume that M is oriented with $(o_x \in H_m(M, M \backslash x) | x \in M)$. Then there is a unique orientation of M along K, which is compatible with the orientation of M,* i.e., *there is a class $o_K \in H_m(M, M \backslash K)$ such that $(\varrho_{xK})_*(o_K) = o_x$ for all $x \in K$.*

Proof. First we show uniqueness. Let o_K and $\tilde{o}_K$ be two orientations of M along K. By assumption we have that

$$(\varrho_{xK})_*(o_K) - (\varrho_{xK})_*(\tilde{o}_K) = (\varrho_{xK})_*(o_K - \tilde{o}_K) = 0.$$

According to Lemma 6.6 this is only the case if $o_K - \tilde{o}_K = 0$.

In order to prove existence we first consider the case where $K \subset U_\alpha \cong \mathring{\mathbb{D}}^m$ and hence $M \backslash U_\alpha \subset M \backslash K$. Let $x \in K$. We denote the isomorphism $H_m(M, M \backslash U_\alpha) \cong H_m(M, M \backslash x)$ by ϕ.

We define o_K as

$$o_K := (\varrho_{K,U_\alpha})_*((\phi^{-1})(o_x)).$$

For $K = K_1 \cup K_2$ with K_i contained in the source of a chart we get that o_{K_1} and o_{K_2} exist. Let $K_0 = K_1 \cap K_2$ and consider the Mayer-Vietoris sequence

$$0 \to H_m(M, M \backslash K) \xrightarrow{i} H_m(M, M \backslash K_1) \oplus H_m(M, M \backslash K_2) \xrightarrow{\kappa} H_m(M, M \backslash K_0) \to \ldots$$

The uniqueness of the orientation along K_0 implies that

$$\kappa(o_{K_1}, o_{K_2}) = (\varrho_{K_0,K_1})_*(o_{K_1}) - (\varrho_{K_0,K_2})_*(o_{K_2}) = 0.$$

Therefore there is a unique class $o_K \in H_m(M, M \backslash K)$ with $i(o_K) = (o_{K_1}, o_{K_2})$.

For the last case we consider a compact subset K and we know that $K = K_1 \cup \ldots \cup K_r$ with $K_i \subset U_{\alpha_i}$. An induction then finishes the proof. □

Theorem 6.8. *Let M be a connected and compact manifold of dimension m. The following are equivalent*

(a) *M is orientable,*
(b) *there is an orientation class $o_M \in H_m(M; \mathbb{Z})$,*
(c) *$H_m(M; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Proposition 6.7 yields that (a) implies (b). Now assume that (b) holds, thus there is a class $o_M \in H_m(M)$ restricting to the local orientation classes o_x. Then the class o_M satisfies, that o_M is not trivial, because its restriction $(\varrho_{x,M})_* o_M = o_x$ is a generator and hence non-trivial. Furthermore, o_m cannot be of finite order: if $ko_M = 0$, then this would imply $ko_x = 0$ for all $x \in M$ contradicting the generating property of the o_x. Let $a \in H_m(M)$ be an arbitrary element. Thus $(\varrho_{x,M})_*(a) = ko_x$ for some integer k. As the o_x are coherent in x, this k has to be constant and if we set $b := ko_M - a$ then $(\varrho_{x,M})_* b = 0$ for all x and this implies that $b = 0$. Therefore $a = ko_M$, thus every element in $H_m(M)$ is a multiple of o_M and $H_m(M) \cong \mathbb{Z}$.

Assuming (c) there are two possible generators in $H_m(M)$. Choose one of them and call it o_M. Then $((\varrho_{x,M})_* o_M | x \in M)$ is an orientation of M. □

The o_M as in Theorem 6.8 is also called *fundamental class of M* and is often denoted by $[M] = o_M$.

Example For the m-sphere, $M = \mathbb{S}^m$ we can choose $\mu_m \in H_m(\mathbb{S}^m)$ as a generator, thus

$$[\mathbb{S}^m] = o_{\mathbb{S}^m} = \mu_m.$$

All results about orientations can be transferred to a setting with coefficients in a commutative ring R with unit 1_R.

- Then M is called R-orientable if and only if there is a coherent choice of generators $H_m(M, M\backslash x; R)$ for all $x \in M$.
- The results we had have formulations relative R: Lemma 6.6 goes through, and if M has an R-orientation $(o_x^R | x \in M)$, then for all compact $K \subset M$ there is an R-orientation of M along K, *i.e.*, a class $o_K^R \in H_m(M, M\backslash K; R)$ that restricts to the local classes. The R-version of Theorem 6.8 yields a class $o_M^R \in H_m(M; R)$ restricting to the o_x^R. The class o_M^R is then called the *fundamental class of M with respect to R* and is denoted by $[M; R]$.

Returning to integral coefficients, we know that for compact orientable manifolds of the same dimension we get a copy of the integers in the homology of the highest degree.

Definition 6.9. Let M and N be two oriented compact connected manifolds of the same dimension $m \geqslant 2$ and let $f\colon M \to N$ be continuous. Then the *degree of f* is the integer $\mathrm{grad}(f)$ that is given by

$$H_m(f)[M] = \mathrm{grad}(f)[N].$$

Of course, this definition extends the notion of the degree of a map that we had for self-maps of spheres.

Proposition 6.10. *Let M, N_1, N_2 be as above and let $f\colon M \to N_1$, $g\colon N_1 \to N_2$.*

(a) *The degree is multiplicative,* i.e.,

$$\mathrm{grad}(g \circ f) = \mathrm{grad}(g)\mathrm{grad}(f).$$

(b) *If $\bar{M}$ is the same manifold as M but with opposite orientation, then*

$$\mathrm{grad}(f) = \mathrm{grad}(f\colon \bar{M} \to \bar{N}) = -\mathrm{grad}(f\colon \bar{M} \to N) = -\mathrm{grad}(f\colon M \to \bar{N}).$$

(c) *If the degree of f is not trivial, then f is surjective.*

Proof. The first claim follows directly from the definition of the degree. For (b) note that $[\bar{M}] = -[M]$, because we have to have

$$(\varrho_{x,M})_*[\bar{M}] = -o_x$$

if $(o_x | x \in M)$ is the orientation of M.

For (c) assume that f is not surjective, thus there is a $y \in N$, that is not contained in the image of M under f. Consider the composition

$$H_m(M) \xrightarrow{H_m(f)} H_m(N) \xrightarrow{(\varrho_{y,N})_*} H_m(N, N\backslash y).$$

This composition is trivial by assumption. On the other hand $(\varrho_{y,N})_*$ is an isomorphism. Hence $H_m(f) = 0$. □

7. Cohomology with compact support

So far, orientation theory works fine if we restrict our attention to compact manifolds. We are aiming at *Poincaré duality*: if M is a compact connected oriented manifold of dimension m, then taking the cap product with $[M] = o_M$ gives a map

$$(-) \cap o_M \colon H^q(M) \to H_{m-q}(M).$$

Our aim is to show that this gives an isomorphism, but we also want to extend the result to non-compact M. To this end we define the following.

Definition 7.1. Let X be an arbitrary topological space and let R be a commutative ring with unit 1_R. Then the *singular n-cochains with compact support* are

$$S^n_c(X;R) = \{\varphi\colon S_n(X) \to R | \exists K_\varphi \subset X \text{ compact }, \varphi(\sigma) = 0 \text{ for all } \sigma\colon \Delta^n \to X \text{ with } \sigma(\Delta^n) \cap K_\varphi = \varnothing.\}$$

The n-cohomology with compact support of X with coefficients in R is

$$H^n_c(X;R) := H^n(S^*_c(X;R)).$$

Note that $S^*_c(X;R) \subset S^*(X;R)$ is a sub-complex. This inclusion of complexes induces a map on cohomology

$$H^n_c(X;R) \longrightarrow H^n(X;R).$$

If X is compact, then $H^n_c(X;R) \cong H^n(X;R)$ for all n.

Do we get a map from singular cohomology to singular cohomology with compact support? Well, yes, but only in a relative setting: Let $K \subset X$ be compact. The restriction map

$$\varrho_{K,X}\colon (X, X\backslash X) = (X, \varnothing) \longrightarrow (X, X\backslash K)$$

induces a map

$$\varrho^n_{K,X}\colon S^n(X, X\backslash K; R) \longrightarrow S^n(X;R)$$

whose image is contained in $S^n_c(X;R)$: for a φ in the image there is a $\psi \in S^n(X, X\backslash K; R)$ with $\varrho^n_{K,X}(\psi) = \varphi$. The functional ψ is trivial on all simplices $\sigma\colon \Delta^n \to X$ with $\sigma(\Delta^n) \cap K = \varnothing$. Therefore,

$$\varphi(\sigma) = \varrho^n_{K,X}(\psi)(\sigma) = 0$$

for such σ.

Lemma 7.2. (a) *Each $\varrho^*_{K,X}$ gives a cochain map $S^*(X, X\backslash K; R) \longrightarrow S^*_c(X;R)$ and in particular we get an induced map*

$$H^*(\varrho_{K,X})\colon H^*(X, X\backslash K; R) \longrightarrow H^*_c(X;R).$$

(b) *For compact subsets $K \subset L \subset X$ we have*

$$\varrho_{K,L} \circ \varrho_{L,X} = \varrho_{K,X}$$

and therefore

$$\begin{tikzcd}
S^*(X, X\backslash K; R) \arrow[d, "\varrho^*_{K,L}"'] \arrow[dr, "\varrho^*_{K,X}"] & \\
S^*(X, X\backslash L; R) \arrow[r, "\varrho^*_{L,X}"'] & S^*_c(X;R)
\end{tikzcd}$$

commutes.

Lemma 7.2 says that the system $K \mapsto S^*(X, X\backslash K; R)$ is a *direct system of cochain complexes*: For $K \subset L \subset L'$ we have

$$\varrho^*_{K,L'} = \varrho^*_{L,L'} \circ \varrho^*_{K,L}$$

and we even have that for compact K and L we can consider the inclusions $K \subset K \cup L$ and $L \subset K \cup L$, thus these maps meet again.

We recall some facts about direct limits of R-modules and (co)chain complexes of R-modules.

Let I be a partially ordered set which we consider as a diagram, *i.e.*, for all $i < j$ there is a unique map $f_{ji}\colon i \to j$ and for $i = j$ we have $f_{ii} = \mathrm{id}_i$. The poset I is called *directed*, if for all $i, j \in I$ there is a $k \in I$ with $i, j \leqslant k$.

Let M_i for $i \in I$ be a family of R-modules together with maps $f_{ji}\colon M_i \to M_j$ with $f_{kj} \circ f_{ji} = f_{ki}$ for $i \leqslant j \leqslant k$. Then we call $(M_i)_{i\in I}$ a *direct system.* If I is directed, then we call the system $(M_i)_{i\in I}$ a *directed system.*

The *direct limit of* (M_i), $\varinjlim M_i$ is the R-module that is determined (up to canonical isomorphism) by the following universal property: there are R-linear maps $h_i\colon M_i \to \varinjlim M_i$ such that for every family of R-module maps $g_i\colon M_i \to M$ that satisfy $g_j \circ f_{ji} = g_i$ for all $i \leqslant j$, there is a unique morphism of R-modules $g\colon \varinjlim M_i \to M$ such that $g \circ h_i = g_i$ for all $i \in I$.

For a direct system $(M_i, i \in I)$ of R-modules we can construct $\varinjlim M_i$ as

$$\varinjlim M_i = \left(\bigoplus_{i\in I} M_i\right)/U$$

where U is the submodule of $\bigoplus_{i\in I} M_i$ generated by all $m_i - f_{ji}(m_i), i \leqslant j$.

For (co)chain complexes the construction is similar. For a direct system of chain complexes $((C_i)_*)_{i\in I}$ we set

$$(\varinjlim(C_i))_n := \varinjlim((C_i)_n).$$

The boundary operators $d_i\colon (C_i)_n \to (C_i)_{n-1}$ induce a boundary map

$$d\colon (\varinjlim(C_i))_n \longrightarrow (\varinjlim(C_i))_{n-1}.$$

Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ be three direct systems of R-modules. If

$$0 \to A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i \to 0$$

is a short exact sequence for all $i \in I$ and if $f_{ji} \circ \varphi_i = \varphi_j \circ f_{ji}$, $f_{ji} \circ \psi_i = \psi_j \circ f_{ji}$ for all $i \leqslant j$, then we call

$$0 \to (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \to 0$$

a short exact sequence of direct systems.

Lemma 7.3. (a) *If*

$$0 \to (A_i) \xrightarrow{(\phi_i)} (B_i) \xrightarrow{(\psi_i)} (C_i) \to 0$$

is a short exact sequence of directed systems of R-modules, then the sequence of R-modules

$$0 \to \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \to 0$$

is short exact.

(b) *If $(A_i)_{i\in I}$ is a directed system of chain complexes, then*

$$\varinjlim H_m(A_i) \cong H_m(\varinjlim A_i).$$

Proof. The maps $\varphi_i\colon A_i \to B_i$ give via composition with $h_i\colon B_i \to \varinjlim B_i$ maps $A_i \to \varinjlim B_i$ and by the universal property this yields a unique map

$$\varphi\colon \varinjlim A_i \longrightarrow \varinjlim B_i.$$

One has to show that i) φ is injective, ii) the kernel of ψ is the image of φ and iii) ψ is surjective.

We show i) and leave ii) and iii) as an exercise.

Let $a \in \varinjlim A_i$ with $\varphi(a) = 0 \in \varinjlim B_i$. Write $a = [\sum_{j=1}^n \lambda_j a_j]$ with $a_j \in A_{j_i}$. Choose $k \geqslant i_1, \ldots, i_n$, then $a = [a_k]$ for some $a_k \in A_k$. By assumption $\varphi(a) = [\varphi_k(a_k)] = 0$. Thus there is an $N \geqslant k$ with $f_{Nk}\varphi_k(a_k) = 0$ and by the coherence of the maps φ_k we have $f_{Nk}\varphi_k(a_k) = \varphi_N \circ f_{Nk}(a_k)$. But φ_N is a monomorphism and therefore $f_{Nk}(a_k) = 0 \in \varinjlim A_i$, but $a = [f_{Nk}(a_k)] = 0$.

For (b) we observe that (a) holds as well for short exact sequences of directed systems of chain complexes and thus (a) implies (b). $\square$

We can use this algebraic result to approximate singular cohomology with compact support via relative singular cohomology groups.

Proposition 7.4. *For all spaces X we have*

$$\varinjlim S^*(X, X\backslash K; R) \xrightarrow{\cong} S_c^*(X; R)$$

and hence

$$\varinjlim H^*(X, X\backslash K; R) \xrightarrow{\cong} H_c^*(X; R).$$

Here the directed system runs over the poset of compact subsets of X.

Proof. A cochain $\varphi \in S^n(X; R)$ is an element of $S_c^n(X; R)$ if and only if there is a compact $K = K_\varphi$ such that $\varphi(\sigma) = 0$ for all σ with $\sigma(\Delta^n) \cap K = \varnothing$ and this is the case if and only if $\varphi \in S^n(X, X\backslash K; R)$. The remaining part of the claim follows from Lemma 7.2. □

What is the compact cohomology of $\mathbb{R}^m$?

Proposition 7.5.

$$H_c^*(\mathbb{R}^m; R) \cong H^*(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}; R) \cong \begin{cases} R, & * = m, \\ 0, & * \neq m. \end{cases}$$

Proof. If $K \subset \mathbb{R}^m$ is compact, then there is a closed ball of radius r_K around the origin, $B_{r_K}(0)$, with $K \subset B_{r_K}(0)$. Without loss of generality we can assume that r_K is a natural number. Thus

$$\varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m\backslash K; R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m\backslash B_r(0))$$

where the direct system on the right runs over all natural numbers r. But

$$H^*(\mathbb{R}^m, \mathbb{R}^m\backslash B_r(0)) \cong H^*(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\})$$

for all r and the diagrams

$$\begin{tikzcd} H^*(\mathbb{R}^m, \mathbb{R}^m\backslash B_r(0)) \arrow[r] \arrow[d] & H^*(\mathbb{R}^m, \mathbb{R}^m\backslash B_{r+1}(0)) \arrow[d] \\ H^*(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \arrow[r] & H^*(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}) \end{tikzcd}$$

commute. Therefore

$$\varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m\backslash B_r(0)) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\})$$

is an isomorphism, but the system on the right is constant and therefore

$$H_c^*(\mathbb{R}^m; R) \cong \varinjlim H^*(\mathbb{R}^m, \mathbb{R}^m\backslash B_r(0)) \cong H^*(\mathbb{R}^m, \mathbb{R}^m\backslash\{0\}).$$

□

8. Poincaré duality

Let M be a connected m-dimensional manifold with an R-orientation $(o_x | x \in M)$. For a compact $K \subset M$ let o_K be the orientation of M along K. For $K \subset L$ compact we have that

$$(\varrho_{K,L})_*(o_L) = o_K$$

because $(\varrho_{x,K})_*(o_K) = o_x = (\varrho_{x,L})_*(o_L) = (\varrho_{x,K})_* \circ (\varrho_{K,L})_*(o_L)$ and o_K is unique with this property. Consider

$$(-) \cap o_K \colon H^{m-p}(M, M\backslash K; R) \longrightarrow H_p(M; R), \quad \alpha \mapsto \alpha \cap o_K = V(o_K) \otimes \langle \alpha, H(o_K)\rangle.$$

For $K \subset L$ we have $(\varrho_{K,L})^*(\alpha) \in H^{m-p}(M, M\backslash L; R)$ and

$$(\varrho_{K,L})^*(\alpha) \cap o_L = \alpha \cap (\varrho_{K,L})_*(o_L) = \alpha \cap o_K.$$

because the cap product is natural.

Therefore the cap product yields a map

$$\varinjlim(- \cap o_k) \colon \varinjlim H^{m-p}(M, M \backslash K; R) = H_c^{m-p}(M; R) \longrightarrow H_p(M; R).$$

DEFINITION 8.1. The map

$$\varinjlim(- \cap o_K) \colon H_c^{m-p}(M; R) \to H_p(M; R)$$

is called *Poincaré duality map* and is denoted by PD or PD_M.

THEOREM 8.2. *(Poincaré Duality) Let M be a connected m-manifold with R-orientation $(o_x | x \in M)$. Then* PD *is an isomorphism* $\mathrm{PD} \colon H_c^{m-p}(M; R) \longrightarrow H_p(M; R)$ *for all $p \in \mathbb{Z}$.*

COROLLARY 8.3. *(Poincaré duality for compact manifolds) Let M be a connected compact manifold of dimension m with an R-orientation $(o_x | x \in M)$ and let $[M] = o_M$ be the fundamental class of M, then*

$$\mathrm{PD} = (-) \cap [M] \colon H^{m-p}(M; R) \longrightarrow H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Example Any connected compact manifold of dimension m possesses a $\mathbb{Z}/2\mathbb{Z}$-orientation and thus a fundamental class $o_M^{\mathbb{Z}/2\mathbb{Z}} \in H_m(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and thus for all p

$$(-) \cap o_M^{\mathbb{Z}/2\mathbb{Z}} \colon H^{m-p}(M; \mathbb{Z}/2\mathbb{Z}) \cong H_p(M; \mathbb{Z}/2\mathbb{Z}).$$

For instance the cohomology of $\mathbb{R}P^n$ and its homology satisfy Poincaré duality with $\mathbb{Z}/2\mathbb{Z}$-coefficients regardless of the parity of n.

PROOF OF THEOREM 8.2.

(a) First we consider the case of $M = \mathbb{R}^m$ and we know that

$$H_c^{m-p}(\mathbb{R}^m) \cong \begin{cases} R, & p = 0, \\ 0, & p \neq 0 \end{cases} \cong H_p(\mathbb{R}^m; R)$$

therefore, abstractly, both R-modules are isomorphic. Let B_r be the closed r-ball centered at the origin. We have to understand

$$(-) \cap o_{B_r} \colon H_c^m(\mathbb{R}^m) \to H_0(\mathbb{R}^m; R).$$

We know that $\langle 1, \alpha \cap o_{B_r} \rangle = \langle \alpha, o_{B_r} \rangle$ for all $\alpha \in H^m(\mathbb{R}^m, \mathbb{R}^m \backslash B_r; R)$. But

$$\langle -, o_{B_r} \rangle \colon H^m(\mathbb{R}^m, \mathbb{R}^m \backslash B_r; R) \longrightarrow R, \quad u \mapsto \langle u, o_{B_r} \rangle$$

is bijective because

$$H^m(\mathbb{R}^m, \mathbb{R}^m \backslash B_r; R) \cong \mathrm{Hom}(H_m(\mathbb{R}^m, \mathbb{R}^m \backslash B_r), R) \oplus \mathrm{Ext}(H_{m-1}(\mathbb{R}^m, \mathbb{R}^m \backslash B_r), R)$$

but the last summand is trivial because $H_{m-1}(\mathbb{R}^m, \mathbb{R}^m \backslash B_r) = 0$. Thus we obtain that for all r the map $(-) \cap o_{B_r}$ is bijective and therefore its direct limit

$$\varinjlim (-) \cap o_{B_r} \colon \varinjlim H^m(\mathbb{R}^m, \mathbb{R}^m \backslash B_r; R) \longrightarrow H_0(\mathbb{R}^m; R)$$

is an isomorphism as well.

(b) Now assume that $M = U \cup V$ such that the claim holds for the open subsets U, V and $U \cap V$, *i.e.*, the maps $\mathrm{PD}_U, \mathrm{PD}_V$ and $\mathrm{PD}_{U \cap V}$ are isomorphisms and each of them uses the orientation that is induced from the orientation of M. Assume that $K \subset U$ and $L \subset V$ are compact and consider the relative version of the Mayer-Vietoris sequences in cohomology

$$\cdots \longrightarrow H^p(M, M \backslash (K \cap L); R) \longrightarrow H^p(M, M \backslash K; R) \oplus H^p(M, M \backslash L; R) \longrightarrow H^p(M, M \backslash (K \cup L); R)$$
$$\longrightarrow H^{p+1}(M, M \backslash (K \cap L); R) \longrightarrow \cdots$$

Excision tells us that

$$\begin{aligned} H^p(M, M\backslash(K\cap L); R) &\cong H^p((U\cap V), (U\cap V)\backslash(K\cap L); R) \\ H^p(M, M\backslash K; R) &\cong H^p(U, U\backslash K; R) \\ H^p(M, M\backslash L; R) &\cong H^p(V, V\backslash L; R). \end{aligned}$$

Here our W for excision is $M\backslash(U\cap V)$, $M\backslash U$ and $M\backslash V$ respectively, and the corresponding A is $M\backslash(K\cap L)$, $M\backslash K$ respectively $M\backslash L$. We obtain a map of exact sequences

$$\begin{tikzcd} H_c^{m-p}(U\cap V; R) \arrow[r, "\cap o_{U\cap V}"] \arrow[d] & H_p(U\cap V; R) \arrow[d] \\ H_c^{m-p}(U; R)\oplus H_c^{m-p}(V; R) \arrow[r, "\cap o_U \oplus \cap o_V"] \arrow[d] & H_p(U; R)\oplus H_p(V; R) \arrow[d] \\ H_c^{m-p}(M; R) \arrow[r, "\cap o_M"] \arrow[d] & H_p(M; R) \arrow[d] \\ H_c^{m-p+1}(U\cap V; R) \arrow[r, "\cap o_{U\cap V}"] \arrow[d] & H_{p-1}(U\cap V; R) \arrow[d] \\ H_c^{m-p+1}(U; R)\oplus H_c^{m-p+1}(V; R) \arrow[r, "\cap o_U \oplus \cap o_V"] & H_{p-1}(U; R)\oplus H_{p-1}(V; R) \end{tikzcd}$$

The five lemma thus proves the case $M = U\cup V$.

(c) Now assume $M = \bigcup_{i=1}^{\infty} U_i$ with open U_i such that $U_1 \subset U_2 \subset \ldots$. We will show that if the claim holds for all U_i with the orientation induced by the one of M, then the claim holds for M.

To that end, let $U \subset M$ be an arbitrary open subset and let $K \subset U$ be compact. Excision gives us

$$H^p(M, M\backslash K; R) \cong H^p(U, U\backslash K; R)$$

and we denote by φ_K the inverse of this map. The direct limit of these φ_K induces a map

$$\varphi_U^M := \varinjlim \varphi_K \colon H_c^p(U; R) \longrightarrow H_c^p(M; R).$$

In general, this map is *not* an iso (U is 'too small'), but now we let U vary. For $U \subset V \subset W$ we get

$$\varphi_U^W = \varphi_V^W \circ \varphi_U^V, \quad \varphi_U^U = \mathrm{id}.$$

As the excision isomorphism is induced by the inclusion $(U, U\backslash K) \hookrightarrow (M, M\backslash K)$, we get that the following diagram commutes:

$$\begin{tikzcd} H_c^{m-p}(U; R) \arrow[r, "\varphi_U^M"] \arrow[d, "\mathrm{PD}_U"] & H_c^{m-p}(M; R) \arrow[d, "\mathrm{PD}_M"] \\ H_p(U; R) \arrow[r, "(i_U^M)_*"] & H_p(M; R) \end{tikzcd}$$

and hence the corresponding diagram

$$\begin{tikzcd} \lim_{\longrightarrow} H_c^{m-p}(U_i; R) \arrow[r, "\lim_{\longrightarrow} \varphi_{U_i}^M"] \arrow[d, "\lim_{\longrightarrow} \mathrm{PD}_{U_i}"] & H_c^{m-p}(M; R) \arrow[d, "\mathrm{PD}_M"] \\ \lim_{\longrightarrow} H_p(U_i; R) \arrow[r, "\lim_{\longrightarrow} (i_{U_i}^M)_*"] & H_p(M; R) \end{tikzcd}$$

commutes as well. The map $\lim_{\longrightarrow} \varphi^M_{U_i}$ is an isomorphism because every K ends up in some U_i eventually. By assumption, each PD_{U_i} is an isomorphism and so is their limit. Similarly the limit of the $(i^M_{U_i})_*$ is an iso and therefore PD_M is.

(d) We show that the claim is valid for arbitrary open subsets $M \subset \mathbb{R}^m$. We express M as a union $M = \bigcup_{r=1}^{\infty} \mathring{B}_r$ where the B_r are m-balls. This is possible because $\mathbb{R}^m$ has a countable basis of its topology. Set $U_i := \bigcup_{r=1}^{i} \mathring{B}_r$, then of course

$$U_1 \subset U_2 \subset \dots$$

The claim holds for the U_i and because of (c) it then holds for M.

(e) Finally we assume that M is as in the theorem with some fixed R-orientation. Every point in M has a neighborhood which is homeomorphic to some open subset of $\mathbb{R}^m$ and we can choose the homeomorphism in such a way that it preserves the orientation. We know that M has a countable basis for its topology and thus there are open subsets $V_1, V_2, \ldots \subset M$ such that $V_i \cong W_i \subset \mathbb{R}^m$ and the V_i cover M. Define $U_i := \bigcup_{j=1}^{i} V_j$, thus $M = \bigcup_i U_i$. The claim holds for the V_j and therefore it holds for the U_i and thus for M.

$\square$

9. Alexander-Lefschetz duality

We will derive a relative version of Poincaré duality and some geometric applications. First, we will consider Čech cohomology.

Let X be an arbitrary topological space and let $A \subset B \subset X$. We consider open neighborhoods (V, U) of (B, A), *i.e.*, open subsets $U \subset V \subset X$ with $A \subset U$ and $B \subset V$. The rough idea of Čech cohomology is to approximate $H^q(B, A)$ by $H^q(V, U)$ where the open neighborhoods come closer and closer to (B, A).

Note that for $(V, U) \subset (V', U')$ we get induced maps

$$H^q(V', U') \longrightarrow H^q(V, U).$$

We use this property to construct a directed system, so we set $(V', U') \leqslant (V, U)$ if and only if $V \subset V'$ and $U \subset U'$.

Definition 9.1. We define the *Čech cohomology of the pair* (B, A) with $A \subset B \subset X$ as

$$\check{H}^p(B, A) = \varinjlim H^p(V, U).$$

In this generality, Čech cohomology has very bad properties.

For subsets $A \subset B \subset X$ where X is a so-called euclidean neighborhood retract and if A and B are locally compact, then $\check{H}^p(B, A)$ only depends on B and A and *not* on X. (A space Y is a euclidean neighborhood retract, if there is a space $X \subset \mathbb{R}^n$ for some n such that X is a retract of a neighborhood $X \subset U \subset \mathbb{R}^n$ and Y is homeomorphic to X.)

If in addition A and B are euclidean neighborhood retracts, then $\check{H}^p(B, A)$ is actually isomorphic to $H^p(B, A)$.

Now let M be a connected m-dimensional manifold and let $K \subset L \subset M$ be compact subsets in M. We assume that there is an orientation class $o_L \in H_m(M, M \backslash L)$ of M along L (possibly with coefficients in R but we will suppress this from the notation). We aim at a cap-pairing of $\check{H}^*(L, K)$ with $H_*(M, M \backslash L)$.

For $(L, K) \subset (V, U)$ we get a map on the level of chains and cochains

$$S^p(V, U) \otimes \left(\frac{S_k(U) + S_k(V \backslash K)}{S_k(V \backslash L)} \right) \longrightarrow S_{k-p}(V \backslash K, V \backslash L).$$

For this note that $V \backslash L \subset (U \cup (V \backslash K)) = V$. Thus for $\alpha \in S^p(V, U)$ and $a + b \in \frac{S_k(U)+S_k(V\backslash K)}{S_k(V\backslash L)}$ we have

$$\alpha \cap (a + b) = \alpha \cap a + \alpha \cap b = 0 + \alpha \cap b$$

and this ends up in the correct chain group.

The homology of $\frac{S_*(U)+S_*(V\backslash K)}{S_*(V\backslash L)}$ is isomorphic to $H_*(V, V \backslash L)$ and this is isomorphic to $H_*(M, M \backslash L)$ via excision.

Excision tells us as well that

$$H_*(V\backslash K, V\backslash L; R) \cong H_*(M\backslash K, M\backslash L; R).$$

As Čech cohomology is the direct limit $\lim_{\longrightarrow} H^*(V,U)$ and as everything is compatible (which we did not really show), the above gives a well-defined map

$$\mathrm{PD}\colon \check{H}^*(L,K) \otimes H_k(M, M\backslash L) \longrightarrow H_{k-p}(M\backslash K, M\backslash L), \quad \alpha \otimes o_L \mapsto \alpha \cap o_L.$$

Proposition 9.2. *(Alexander-Lefschetz duality) Let M be a connected m-dimensional manifold and let $K \subset L \subset M$ with K, L compact. Let M be oriented along L with respect to R. Then the map*

$$\mathrm{PD} = (-) \cap o_L \colon \check{H}^q(L,K;R) \longrightarrow H_{m-q}(M\backslash K, M\backslash L; R)$$

is an isomorphism for all integers q.

Before we prove this result, we collect some properties of this form of the Poincaré duality map.

(a) This PD map still satisfies that $\mathrm{PD}(1) = o_L$ for $K = \varnothing$ and $1 \in H^0(L;R)$.

(b) The PD-map is natural in the following sense: for $(L,K) \hookrightarrow (L',K')$ the diagram

$$\begin{tikzcd} \check{H}^q(L',K') \arrow[r, "(-)\cap o_{L'}"] \arrow[d, "\check{H}^q(i)"] & H_{m-q}(M\backslash K', M\backslash L') \arrow[d, "H_{m-q}(i)"] \\ \check{H}^q(L,K) \arrow[r, "(-)\cap o_L"] & H_{m-q}(M\backslash K, M\backslash L) \end{tikzcd}$$

commutes.

(c) We won't prove the following fact. The diagram

$$\begin{tikzcd} \cdots \arrow[r] & \check{H}^q(L,K) \arrow[r] \arrow[d, "\cap o_L"] & \check{H}^q(L) \arrow[r] \arrow[d, "\cap o_L"] & \check{H}^q(K) \arrow[r] \arrow[d, "\cap o_K"] & \check{H}^{q+1}(L,K) \arrow[r] \arrow[d, "\cap o_L"] & \cdots \\ \cdots \arrow[r] & H_{m-q}(M\backslash K, M\backslash L) \arrow[r] & H_{m-q}(M, M\backslash L) \arrow[r] & H_{m-q}(M, M\backslash K) \arrow[r] & H_{m-q-1}(M\backslash K, M\backslash L) \arrow[r] & \cdots \end{tikzcd}$$

commutes, and therefore (using the five lemma) it suffices to show the absolute version of Alexander-Lefschetz duality.

Lemma 9.3. *If K and L are compact subsets of M with an orientation class $o_{K\cup L}$ along $K \cup L$ and induced orientation classes o_K and o_L. Then the diagram*

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & \check{H}^q(K\cup L) \arrow[r] \arrow[d, "\cap o_{K\cup L}"] & \check{H}^q(K)\oplus\check{H}^q(L) \arrow[r] \arrow[d, "\cap o_K \oplus \cap o_L"] & \check{H}^q(K\cap L) \arrow[r, "\partial"] \arrow[d, "o_{K\cap L}"] & \cdots \\ \cdots \arrow[r, "\delta"] & H_{m-q}(M, M\backslash(K\cup L)) \arrow[r] & H_{m-q}(M, M\backslash K)\oplus H_{m-q}(M, M\backslash L) \arrow[r] & H_{m-q}(M, M\backslash(K\cap L)) \arrow[r, "\delta"] & \cdots \end{tikzcd}$$

commutes and has exact rows.

Proof. The only critical squares are the ones that are slightly out of the focus of the above diagram, the ones with the connecting homomorphisms. The $\check{H}^*$-sequence comes from direct limits of

$$0 \to \mathrm{Hom}(S_*(U) + S_*(V), R) \longrightarrow \mathrm{Hom}(S_*(U), R) \oplus \mathrm{Hom}(S_*(V), R) \longrightarrow \mathrm{Hom}(S_*(U \cap V), R) \to 0$$

for open U, V with $K \subset U$ and $L \subset V$.

Let $\alpha \in \check{H}^q(K \cap L; R)$. Choose a representing cocycle f with $\alpha = [f]$, *i.e.*, $\delta f = 0$ on $U \cap V$ and let ∂ be the conneting homomorphism for ordinary singular cohomology. What is $\partial(\alpha)$? A preimage for f in the direct sum is $(f, 0)$ and its coboundary is $(\delta f, 0)$, so if we choose an $h \in \mathrm{Hom}(S_*U + S_*V, R)$ with the property $h(u + v) = \delta f(u)$ for $u \in S_*(U)$, $v \in S_*(V)$, then

$$\partial(\alpha) = [h].$$

We can extend h to a cochain on M (for instance by letting it be trivial on the complement).

We want to compare $\partial(\alpha) \cap o_{K\cup L}$ and $\delta(\alpha \cap o_{K\cap L})$. For the first term we express $o_{K\cup L} = [a]$ as a sum

$$a = b + c + d + e \in S_*(U \cap V) \oplus S_*(U\backslash L) + S_*(V\backslash K) + S_*(M\backslash(K \cup L)).$$

The subsets $U \cap V, U\backslash L, V\backslash K$ and $M\backslash(K \cup L)$ are open and therefore we can work with these small chains. With the notation as above we get

$$\partial(\alpha) \cap o_{K\cup L} = [h \cap (b + c + d + e)] = [h \cap c]$$

because h is only non-trivial on chains in U and as $\delta(f)$ is trivial on $U \cap V$ h is only non-trivial on the complement of V in U.

For $\alpha \cap o_{K\cap L}$ we write $[f \cap a]$ and as the lower exact row comes from the short exact sequence

$$0 \to \frac{S_*(M)}{S_*(M\backslash K \cap L)} \longrightarrow \frac{S_*(M)}{S_*(M\backslash K)} \oplus \frac{S_*(M)}{S_*(M\backslash L)} \longrightarrow \frac{S_*(M)}{S_*(M\backslash K) + S_*(M\backslash L)} \to 0$$

we view the latter as an element modulo $S_*(M\backslash K) + S_*(M\backslash L)$. The connecting homomorphism picks $(f \cap a, 0)$ as a pre-image of $f \cap a$ takes its boundary $(\partial(f \cap a), 0)$ but the latter is up to sign

$$(\partial(f \cap a), 0) = (\delta(f) \cap a), 0) \pm (f \cap \partial a, 0).$$

Writing a as $a = b + c + d + e$ and using that f ignores b and e we obtain that the above is $(\delta f \cap c + \delta f \cap d \pm f \cap \partial a, 0)$. But $\delta f \cap d$ and $f \cap \partial a$ are elements in $S_*(M\backslash K)$ and hence all that remains when we pick a preimage is $(\delta f \cap c, 0)$, thus

$$\delta(\alpha \cap o_{K\cap L}) = [\delta f \cap c] = [h \cap c].$$

□

Now we can prove Alexander-Lefschetz duality.

Proof of Proposition 9.2. The lemma above tells us that it suffices to prove the absolute case, *i.e.*, to show that

$$(-) \cap o_K : \check{H}^q(K) \longrightarrow H_{m-q}(M, M\backslash K)$$

is an isomorphism for all q.

If K is empty, then we get the true statement that $0 = 0$. For K a point we only get something non-trivial for $q = 0$ and here $1 \in R = \check{H}^0(K)$ is sent to $o_K = o_x$ via Poincaré duality.

Similarly, if $M = \mathbb{R}^m$ and K is convex and compact we can proceed as in the case of a point.

If $K = K_1 \cup \ldots \cup K_r$ and M is still $\mathbb{R}^m$ an induction over r proves the claim.

For $M = \mathbb{R}^m$ and K arbitrary we can find a neighborhood U of K of the form $U = \bigcup_{i=1}^N U_i$ with the U_i being convex. Such U suffice to calculate the direct limit $\lim_{\longrightarrow} H^q(U)$ for the Čech cohomology of K. For such U we have

$$H_{m-q}(\mathbb{R}^m, \mathbb{R}^m\backslash K) \cong \varinjlim H_{m-q}(\mathbb{R}^m, \mathbb{R}^m\backslash U)$$

because $\mathbb{R}^m\backslash K = \bigcup_U \mathbb{R}^m\backslash U$. The U satisfy Alexander-Lefschetz duality and hence K does.

Finally let M and K be arbitrary, but satisfying the conditions of Proposition 9.2. Express $K = K_1 \cup \ldots \cup K_r$ such that the K_i are contained in a chart that is homeomorphic to $\mathbb{R}^m$ and proceed as in the case before. □

10. Application of duality

Proposition 10.1. *(Classical Alexander duality) Let $K \subset \mathbb{R}^m$ be compact. Then*

$$\check{H}^q(K) \cong H_{m-q}(\mathbb{R}^m, \mathbb{R}^m\backslash K) \cong H_{m-q-1}(\mathbb{R}^m\backslash K).$$

Here the first isomorphism is Alexander-Lefschetz duality and the second one is a result of the long exact sequence of pairs in homology.

This is bad news for knot complements. A knot K is the homeomorphic image of $\mathbb{S}^1$ in $\mathbb{R}^3$ and the above tells us that

$$H_1(\mathbb{R}^3\backslash K) \cong \check{H}^1(K)$$

but the circle is a euclidean neighborhood retract and therefore Čech cohomology concides with ordinary singular cohomology, but $H^1(K) \cong \mathbb{Z}$, thus the first homology group of any knot complement is isomorphic to the integers, thus it does not help to distinguish knots. The fundamental group of the knot complement does a better job. Here the un-knot gives the integers, but for instance the complement of the trefoil knot has a fundamental group that is *not* isomorphic to the integers, but is isomorphic to the group $\langle a, b | a^2 = b^3 \rangle$. This group is actually isomorphic to the braid group on three strands.

PROPOSITION 10.2. *Let M be a compact oriented connected m-manifold and let $\varnothing \neq K \subset M$ be compact. If $H_1(M)$ is trivial, then $\check{H}^{m-1}(K)$ is free abelian and $M \backslash K$ has* $\mathrm{rank}\check{H}^{m-1}(K) + 1$ *components.*

PROOF. Let $k = |\pi_0(M\backslash K)|$ be the number of components of the complement of K in M. Therefore

$$k = \mathrm{rank} H_0(M\backslash K) = 1 + \mathrm{rank}\tilde{H}_0(M\backslash K).$$

By assumption $H_1(M) = 0 = \tilde{H}_0(M)$ and therefore we know from the long exact sequence and duality that

$$\tilde{H}_0(M\backslash K) \cong H_1(M, M\backslash K) \cong \check{H}^{m-1}(K).$$

□

PROPOSITION 10.3. *If M is a compact connected orientable m-manifold and if the first homology group of M with integral coefficients vanishes, then all compact submanifolds without boundary of dimension $m-1$ are orientable.*

Compact manifolds without boundary are often called *closed*.

PROOF. A submanifold $N \subset M$ is a euclidean neighborhood retract and therefore

$$H^{m-1}(N) \cong \check{H}^{m-1}(N) \cong H_1(M, M\backslash N) \cong \tilde{H}_0(M\backslash N)$$

thus $H^{m-1}(N)$ is free abelian. This implies that the components of N are orientable. □

COROLLARY 10.4. *It is not possible to embed $\mathbb{R}P^2$ into $\mathbb{R}^3$.*

If one could, then one could embed $\mathbb{R}P^2$ into $\mathbb{S}^3$ as the one-point compactification of $\mathbb{R}^3$. Due to $H_1(\mathbb{S}^3) = 0$, the 2-manifold $\mathbb{R}P^2$ would be orientable, but we know that it's not. At the math institute in Oberwolfach there is a model of the Boy surface. That's a model of an immersion of $\mathbb{R}P^2$ into three-space. `http://www.mfo.de/general/boy/`

PROPOSITION 10.5. *Let M be a compact connected and orientable m-manifold and let β_i be the i-th* Betti number *of M, $\beta_i = \dim_{\mathbb{Q}} H_i(M; \mathbb{Q})$. Then*

$$\beta_i = \beta_{m-i}.$$

PROOF. Note that by the very definition of Čech cohomology, $\check{H}^*(M)$ is isomorphic to $H^*(M)$ because $L = M$ and $K = \varnothing$. Duality then tells us that

$$\dim_{\mathbb{Q}} H_{m-i}(M; \mathbb{Q}) = \dim_{\mathbb{Q}} H^i(M; \mathbb{Q})$$

As $\mathbb{Q}$ is divisible, there is no Ext-term arising in the universal coefficient theorem and thus

$$\dim_{\mathbb{Q}} H^i(M; \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathrm{Hom}(H_i(M), \mathbb{Q}))$$

but this is equal to the dimension of the vector space of the homomorphisms from the free part of $H_i(M)$ to $\mathbb{Q}$ which is equal to the rank of $H_i(M)$ and this in turn is equal to β_i. □

PROPOSITION 10.6. *If M is as above of odd dimension, then the Euler characteristic of M vanishes.*

Just recall that

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \beta_i.$$

PROPOSITION 10.7. *If M is a compact connected orientable m-manifold with boundary, then*

$$\check{H}^q(M, \partial M) \cong H_{m-q}(M).$$

Proof. Glue a collar to M, *i.e.*, consider

$$W := M \cup (\partial M \times [0,1)) =: M \cup W'.$$

Then W is an m-manifold without boundary such that duality applies and as $W \simeq M$ we obtain

$$\check{H}^q(M, \partial M) \cong H_{m-q}(W \backslash \partial M, W \backslash M) \cong H_{m-q}(M \backslash \partial M) \cong H_{m-q}(M).$$

For this note that $W \backslash M \simeq \partial M$, $W \backslash \partial M \simeq M \backslash \partial M \sqcup W' \backslash \partial M$, $W \backslash M = W' \backslash \partial M$ and that taking the complement of ∂M in M gives something that is homotopy equivalent to M. □

Corollary 10.8. *If M is as above then the Euler characteristic of ∂M is always even.*

Proof. Note that $\chi(M) = \chi(W)$ and the long exact sequence of the pair $W \backslash M \subset W$ gives

$$\chi(W) = \chi(W \backslash M) + \chi(W, W \backslash M).$$

Homotopy invariance yields $\chi(W \backslash M) = \chi(\partial M)$ and duality guarantees that $\chi(W, W \backslash M) = (-1)^m \chi(M)$. Therefore

$$\chi(\partial M) = (1 + (-1)^{m-1}) \chi(M)$$

and this is always an even number. □

An important consequence is that $\mathbb{R}P^{2m}$ can never be a boundary of a manifold, because its Euler characteristic is 1. Similarly, as

$$\chi(\mathbb{C}P^{2m}) = \sum_{i=0}^{2m} (-1)^{2i} = 2m + 1$$

and

$$\chi(\mathbb{H}P^{2m}) = \sum_{i=0}^{2m} (-1)^{4i} = 2m + 1$$

all these projective spaces do not occur as boundaries of other manifolds. For the calculations of χ note that for complex projective space of dimension $2m$ we have cells in dimension up to $4m$, but only in even dimensions. Similarly, for quaterion projective space of dimension $2m$ cells occur up to dimension $8m$, but only in degrees divisible by 4.

These facts are important in *bordism theory*: one can introduce an equivalence relation on manifolds by saying that two m-manifolds M and N are bordant, if there is an $(m+1)$-manifold W whose boundary is the disjoint union of M and N, $\partial W = M \sqcup N$. Thus the above projective spaces don't give trivial equivalence classes under the bordism relation.

11. Duality and cup products

Let M be a connected closed M-manifold with an R-orientation for some commutative ring R. We consider the composition

$$\begin{tikzcd} H^k(M;R) \otimes_R H^{m-k}(M;R) \arrow[r, "\cup"] & H^m(M;R) \arrow[d, "(-)\cap o_M^R"] \\ & H_0(M;R) \cong R \end{tikzcd}$$

Definition 11.1. For $\alpha \in H^k(M;R)$, $\beta \in H^{m-k}(M;R)$ the map

$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, o_M^R \rangle$$

is called *cup product pairing of M*.

Proposition 11.2. *The cup product pairing is non-singular if R is a field or if $R = \mathbb{Z}$ and all homology groups of M are torsion-free.*

Here, non-singular means that the induced maps

$$H^k(M;R) \to \mathrm{Hom}_R(H^{m-k}(M;R),R) \text{ and } H^{m-k}(M;R) \to \mathrm{Hom}_R(H^k(M;R),R)$$

are both isomorphisms.

Proposition 11.2 holds as long as one restricts attention to the free part of the cohomology groups: let $FH^k(M;R)$ denote the free part of $H^k(M;R)$ then there is a non-singular pairing

$$FH^k(M;R) \otimes_R FH^{m-k}(M;R) \to R.$$

In geometric applications the ground ring is often $R = \mathbb{R}$, so then you are dealing with a pairing over the real numbers and methods of linear algebra apply.

Proof. The Kronecker pairing yields a map

$$\kappa\colon H^k(M;R) \to \mathrm{Hom}_R(H_k(M;R),R)$$

and Poincaré duality tells us that capping with o_M^R is an isomorphism between $H_k(M;R)$ and $H^{m-k}(M;R)$. The composite is

$$H^k(M;R) \to \mathrm{Hom}_R(H_k(M;R),R) \cong \mathrm{Hom}_R(H^{m-k}(M;R),R),\ \alpha \mapsto \langle \alpha, (-) \cap o_M^R \rangle.$$

Over a field, κ and hence the composite is an isomorphism. In the torsion-free setting we obtain an isomorphism as well. □

Dual to the cup product pairing there is the *intersection form*:

$$H_p(M) \otimes H_{m-p}(M) \to \mathbb{Z}$$

with $a \otimes b \mapsto \langle \mathrm{PD}^{-1}(a) \cup \mathrm{PD}^{-1}(b), o_M \rangle$. In particular for even-dimensional manifolds, the signature of this form is an important invariant in differential topology. For instance one can show that for a compact oriented manifold W such that $\partial W = M$ with a $4n$-dimensional manifold M the signature of the intersection form on M is trivial.

Lemma 11.3. *Let M be as in 11.2 with torsion-free homology groups. If $H^p(M) \cong \mathbb{Z} \cong H^{m-p}(M)$ and if $\alpha \in H^p(M)$, $\beta \in H^{m-p}(M)$ are generators, then $\alpha \cup \beta$ is a generator of $H^m(M) = \mathbb{Z}$.*

Proof. For α there exists a $\beta' \in H^{m-p}(M)$ with

$$\langle \alpha \cup \beta', o_M \rangle = 1.$$

As β is a generator we know that $\beta' = k\beta$ for some integer k and hence

$$1 = \langle \alpha \cup \beta', o_M \rangle = \langle \alpha \cup k\beta, o_M \rangle = k\langle \alpha \cup \beta, o_M \rangle.$$

But $\langle \alpha \cup \beta', o_M \rangle$ is an integer, so k has to be ± 1 and therefore $\alpha \cup \beta$ generates $H^m(M)$. □

We will use this result to calculate the cohomology rings of projective spaces.

Lemma 11.4. *If $\alpha \in H^2(\mathbb{C}P^m)$ is a generator, then $\alpha^q \in H^{2q}(\mathbb{C}P^m)$ is a generator as well.*

Proof. We have to show by induction that α^{q-1} is an additive generator of $H^{2q-2}(\mathbb{C}P^m)$ and we do that by induction over m because we will use the argument in this proof later again.

For $m = 1$ there is nothing to prove because $\mathbb{C}P^1 \cong \mathbb{S}^2$ and there $\alpha^2 = 0$.

Consider the inclusion $i\colon \mathbb{C}P^{m-1} \hookrightarrow \mathbb{C}P^m$. The CW structure of $\mathbb{C}P^m$ is $\mathbb{C}P^{m-1} \cup_f \mathbb{D}^{2m}$. For $m > 1$ $i^*\colon H^{2i}(\mathbb{C}P^m) \to H^{2i}(\mathbb{C}P^{m-1})$ is an isomorphism for $1 \leqslant i \leqslant m-1$ and $i^*(\alpha)$ generates $H^2(\mathbb{C}P^{m-1})$. Induction over m then shows that $(i^*(\alpha))^q$ generates $H^{2q}(\mathbb{C}P^{m-1})$ for all $1 \leqslant q \leqslant m-1$. But $(i^*(\alpha))^q = i^*(\alpha^q)$ and therefore α^q generates $H^{2q}(\mathbb{C}P^m)$ for $1 \leqslant q \leqslant m-1$. Lemma 11.3 then shows that $\alpha \cup \alpha^{m-1} = \alpha^m$ generates $H^{2m}(\mathbb{C}P^m)$. □

Corollary 11.5. *As a graded ring*

$$H^*(\mathbb{C}P^m) \cong \mathbb{Z}[\alpha]/\alpha^{m+1} \text{ with } |\alpha| = 2.$$

Similarly,

$$H^*(\mathbb{R}P^m;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/\alpha^{m+1} \text{ with } |\alpha| = 1.$$

There are two geometric consequences that follow from this calculation.

Proposition 11.6. *For $0 < m < n$ the inclusion $j\colon \mathbb{C}P^m \hookrightarrow \mathbb{C}P^n$ is no weak retract.*

Proof. Let us assume that there is an $r\colon \mathbb{C}P^n \to \mathbb{C}P^m$ with $r \circ j \simeq \mathrm{id}$. On second cohomology groups j induces an isomorphism
$$j^*\colon H^2(\mathbb{C}P^n) \to H^2(\mathbb{C}P^m).$$
Let $\alpha \in H^2(\mathbb{C}P^m)$ be a generator, then $\beta := r^*(\alpha)$ is a generator as well. As $\alpha^{m+1} = 0$ we get
$$\beta^{m+1} = r^*(\alpha)^{m+1} = r^*(\alpha^{m+1}) = r^*(0) = 0.$$
But $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\beta]/\beta^{n+1}$ and hence $\beta^{m+1} \neq 0$. □

Proposition 11.7. *The attaching map of the $2n$-cell in $\mathbb{C}P^n$ is not null-homotopic.*

Proof. Let $\varphi\colon \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ be the attaching map, thus
$$\mathbb{C}P^n = C_\varphi = \mathbb{C}P^{n-1} \cup_\varphi \mathbb{D}^{2n}.$$
If φ were null-homotopic, then
$$\mathbb{C}P^{n-1} \cup_\varphi \mathbb{D}^{2n} \simeq \mathbb{C}P^{n-1} \vee \mathbb{S}^{2n}$$
and $\mathbb{C}P^{n-1}$ were a weak retract of $\mathbb{C}P^n$. □

A famous example of this phenomenon is the Hopf fibration $\varphi = \eta\colon \mathbb{S}^3 \to \mathbb{S}^2 = \mathbb{C} \cup \infty$. Consider $\mathbb{S}^3 \subset \mathbb{C}^2$ and send $\mathbb{S}^3 \ni (u, v)$ to
$$\eta(u, v) := \begin{cases} \frac{u}{v}, & v \neq 0, \\ \infty, & v = 0. \end{cases}$$
Then this map is not null-homotopic, $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$, and in fact it generates $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

12. The Milnor sequence

The aim is to calculate the cohomology rings of infinite dimensional projective spaces and more generally to understand cohomology groups for infinite dimensional CW complexes.

Let $(M_i)_{i \in \mathbb{N}_0}$ be a family of R-modules together with a sequence of maps
$$M_0 \xleftarrow{f_1} M_1 \xleftarrow{f_2} M_2 \xleftarrow{f_3} \ldots$$
We call such a family $(M_i, f_i)_{i \in \mathbb{N}_0}$ an *inverse system*.

Definition 12.1. The *inverse limit of the inverse system* $(M_i)_{i \in \mathbb{N}_0}$ is the R-module
$$\varprojlim M_i = \{(x_0, x_1, \ldots) \in \prod_{i \in \mathbb{N}_0} M_i | f_{i+1}(x_{i+1}) = x_i, i \geqslant 0\}.$$

If ξ denotes the map that sends $(x_0, x_1, \ldots) \in \prod_{i \in \mathbb{N}_0} M_i$ to $(x_0 - f_1(x_1), x_1 - f_2(x_2), \ldots)$ then we can express the inverse limit as the kernel of ξ:
$$0 \to \varprojlim M_i \longrightarrow \prod_{i \in \mathbb{N}_0} M_i \xrightarrow{\xi} \prod_{i \in \mathbb{N}_0} M_i.$$

Definition 12.2. Let $\varprojlim{}^1 M_i$ be the R-module $\mathrm{coker}(\xi)$.

Thus we have an exact sequence
$$0 \to \varprojlim M_i \longrightarrow \prod_{i \in \mathbb{N}_0} M_i \xrightarrow{\xi} \prod_{i \in \mathbb{N}_0} M_i \longrightarrow \varprojlim{}^1 M_i \to 0.$$

Lemma 12.3. *If*
$$0 \to (M_i, f_i) \longrightarrow (N_i, g_i) \longrightarrow (Q_i, h_i) \to 0$$
is a short exact sequence of inverse systems, then the sequence
$$0 \to \varprojlim M_i \longrightarrow \varprojlim N_i \longrightarrow \varprojlim Q_i \longrightarrow \varprojlim{}^1 M_i \longrightarrow \varprojlim{}^1 N_i \longrightarrow \varprojlim{}^1 Y_i \to 0$$
is exact.

Proof. Consider $\xi\colon \prod_i M_i \to \prod_i M_i$ as a chain complex C_*. Then the first homology group is the inverse limit and the zeroth homology group is the lim-one term

$$H_1 C_* \cong \varprojlim M_i, \quad H_0 C_* \cong \varprojlim{}^1 M_i.$$

We can translate the short exact sequence of inverse systems into a short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & \prod_i M_i \arrow[r] \arrow[d, "\xi"] & \prod_i N_i \arrow[r] \arrow[d, "\xi"] & \prod_i Q_i \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & \prod_i M_i \arrow[r] & \prod_i N_i \arrow[r] & \prod_i Q_i \arrow[r] & 0
\end{tikzcd}$$

and the associated long exact sequence is precisely what we want. □

Therefore the lim-one terms measure how non-exact inverse limits are. We are interested in the case where we have exactness.

Lemma 12.4. *(Mittag-Leffler condition) Assume that for every $n \geqslant 0$ there is an $N = N(n)$ such that we have for all $m \geqslant N$ that the image of $f_{n+1} \circ \ldots \circ f_m \colon M_m \to M_n$ is equal to the image of $f_{n+1} \circ \ldots \circ f_N \colon M_N \to M_n$. Then*

$$\varprojlim M_i = 0.$$

Proof. Without loss of generality we can assume that the sequence $N(n)$ is monoton increasing in n. We have to show that the cokernel of ξ is trivial. Let $(a_i)_i \in \prod_i M_i$. We have to show that this sequence is in the image of ξ.

As a first case we deal with sequences (a_i) such that a_i is in the image of $f_{i+1} \circ \ldots \circ f_N \colon M_N \to M_i$. We construct elements $b_0, \ldots, b_k$ with $a_i = b_i - f_{i+1} b_{i+1}$ for $i < k$ by induction on k such that

$$b_i \in \mathrm{im}(f_{i+1} \circ \ldots \circ f_N).$$

We start with $a_0 = b_0$. Assume the claim is shown for i up to k. Choose a $y \in M_{N(k+1)}$ with

$$a_k - b_k = f_{k+1} \circ \ldots \circ f_{N(k+1)}(y).$$

This is possible by the assumption that the image of $f_{k+1} \circ \ldots \circ f_{N(k+1)}$ is equal to the image of $f_{k+1} \circ \ldots \circ f_{N(k)}$. Define

$$b_{k+1} := f_{k+2} \circ \ldots \circ f_{N(k+1)}(y).$$

Then

$$f_{k+1} b_{k+1} - b_k = a_k - b_k + b_k = a_k.$$

If a_i is not in the image $f_{i+1} \circ \ldots \circ f_N \colon M_N \to M_i$, then we define

$$a_i' = a_i + f_{i+1} a_{i+1} + \ldots + f_{i+1} \circ \ldots \circ f_{N(i)}(a_{N(i)}).$$

We check that

$$\begin{aligned}
a_i - (a_i' - f_{i+1}(a_{i+1}')) &= a_i - a_i - f_{i+1}(a_{i+1}) - \ldots - f_{i+1} \circ \ldots \circ f_{N(i)}(a_{N(i)}) \\
&\quad + f_{i+1}(a_{i+1}) + f_{i+1} \circ f_{i+2}(a_{i+2}) + \ldots + f_{i+1} \circ \ldots \circ f_{N(i+1)}(a_{N(i+1)}) \\
&= f_{i+1} \circ \ldots \circ f_{N(i)+1}(a_{N(i)+1}) + \ldots + f_{i+1} \circ \ldots \circ f_{N(i+1)}(a_{N(i+1)})
\end{aligned}$$

and therefore $a_i - (a_i' - f_{i+1}(a_{i+1}'))$ is in the image of $f_{i+1} \circ \ldots \circ f_{N(i)}$. As in case one we write $a_i - (a_i' - f_{i+1}(a_{i+1}'))$ as $b_i - f_{i+1} b_{i+1}$. Thus

$$a_i = c_i - f_{i+1}(c_{i+1})$$

with $c_i = b_i - a_i'$. □

Examples If every map f_i is surjective, then the system (M_i, f_i) satisfies the Mittag-Leffler criterion. For instance the inverse system

$$\mathbb{Z}/p\mathbb{Z} \longleftarrow \mathbb{Z}/p^2\mathbb{Z} \longleftarrow \mathbb{Z}/p^3\mathbb{Z} \longleftarrow \ldots$$

satisfies this condition. The inverse limit of this system is called the *p-adic integers*. These are denoted by $\mathbb{Z}_p^{\hat{}}$ and they are the p-adic completion of the ring of integers.

We want to apply this result to inverse systems of cochain complexes.

Assume that X is a CW complex and that $(X_n)_n$ is a sequence of subcomplexes with $X_n \subset X_{n+1}$ and $X = \bigcup_n X_n$, for instance, we could have $X_n = X^n$, the n-skeleton of X. Consider

$$S_n^*(X) := X^*(X_n).$$

The inclusion maps $X_n \subset X_{n+1}$ induce maps

$$f_{n+1}\colon S_{n+1}^*(X) \longrightarrow S_n^*(X).$$

We therefore have

$$S_0^*(X) \xleftarrow{f_1} S_1^*(X) \xleftarrow{f_2} \dots$$

and these maps commute with the coboundary maps

$$\begin{tikzcd} S_{n+1}^i(X) \arrow[r, "f_{n+1}"] \arrow[d, "\delta"] & S_n^i(X) \arrow[d, "\delta"] \\ S_{n+1}^{i+1}(X) \arrow[r, "f_{n+1}"] & S_n^{i+1}(X). \end{tikzcd}$$

Lemma 12.5. *If (C_n^*, f_n) is an inverse system of cochain complexes, such that for every cochain degree m the system (C_n^m, f_n) satisfies the Mittag-Leffler condition, then the sequence*

$$0 \to \varprojlim{}^1 H^{m-1}(C_n^*) \longrightarrow H^m(\varprojlim C_n^*) \longrightarrow \varprojlim H^m(C_n^*) \to 0$$

is exact.

Proof. We consider the two exact sequences

$$0 \to B_n^m \longrightarrow Z_n^m \longrightarrow H^m(C_n^*) \to 0 \tag{12.1}$$

and

$$0 \to Z_n^m \longrightarrow C_n^m \longrightarrow B_n^{m+1} \to 0. \tag{12.2}$$

As the C_n^m satisfy the Mittag-Leffler condition we know that

$$\varprojlim{}^1 C_n^m = 0, \quad \text{for all } m. \tag{12.3}$$

Lemma 12.3 tells us that the sequence

$$\varprojlim{}^1 C_n^m \longrightarrow \varprojlim{}^1 B_n^{m+1} \to 0$$

is exact and thus $\varprojlim{}^1 B_n^{m+1} = 0$. Therefore the sequence 12.1 yields that

$$\varprojlim{}^1 Z_n^m \cong \varprojlim{}^1 H^m(C_n^*).$$

In addition we know that

$$0 \to \varprojlim Z_n^m \longrightarrow \varprojlim C_n^m \longrightarrow \varprojlim B_n^{m+1}$$

is exact and hence the inverse limit of the cocycles is equal to the module of cocycles in the inverse limit

$$\varprojlim Z_n^m \cong Z^m(\varprojlim C_n^*).$$

As the lim-one term on the inverse system of coboundaries is trivial we obtain that

$$0 \to \varprojlim B_n^m \longrightarrow \varprojlim Z_n^m \longrightarrow \varprojlim H^m(C_n^*) \to 0$$

is exact as well. The isomorphism 12.3 tells us that the kernel of the connecting homomorphism

$$\partial\colon \varprojlim B_n^m \longrightarrow \varprojlim{}^1 Z_n^{m-1} \to 0$$

is isomorphic to the coboundaries

$$B^m(\varprojlim C_n^*).$$

Therefore we get the following sequence of inclusions

$$B^m(\varprojlim C_n^*) \subset \varprojlim B_n^m \subset \varprojlim Z_n^m = Z^m(\varprojlim C_n^*)$$

and this gives that

$$0 \to \frac{\lim_{\longleftarrow} B_n^m}{B^m(\lim_{\longleftarrow} C_n^*)} \longrightarrow \frac{Z^m(\lim_{\longleftarrow} C_n^*)}{B^m(\lim_{\longleftarrow} C_n^*)} \longrightarrow \frac{\lim_{\longleftarrow} Z_n^m}{\lim_{\longleftarrow} B_n^m} \to 0$$

is exact. The middle term is $H^m(\lim_{\longleftarrow} C_n^*)$, the right term is isomorphic to $\lim_{\longleftarrow} H^m(C_n^*)$ and the left term is isomorphic to the lim-one term $\lim_{\longleftarrow}{}^1 H^{m-1}(C_n^*)$ because $\lim_{\longleftarrow}{}^1 Z_n^{m-1} \cong \lim_{\longleftarrow}{}^1 H^{m-1}(C_n^*)$.. □

Theorem 12.6. *(Milnor sequence) If X is a CW complex with a filtration $X_0 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$ of subcomplexes with $X = \bigcup_n X_n$, then the sequence*

$$0 \to \varprojlim{}^1 H^{m-1}(X_n; G) \longrightarrow H^m(X; G) \longrightarrow \varprojlim H^m(X_n; G) \to 0$$

is exact for all abelian groups G.

Proof. We define $C_n^* = \mathrm{Hom}(S_*(X_n), G)$. This system satisfies the Mittag-Leffler condition because the inclusions

$$S_m(X_n) \hookrightarrow S_m(X_{n+1})$$

dualize to epimorphisms

$$\mathrm{Hom}(S_m(X_{n+1}), G) \longrightarrow \mathrm{Hom}(S_m(X_n), G).$$

The only thing we have to show is that

$$H^m(X; G) \cong H^m(\varprojlim \mathrm{Hom}(S_*(X_n), G)).$$

The inverse limit has a universal property dual to the one of the direct limit and the maps

$$\mathrm{Hom}(S_*(X), G) \longrightarrow \mathrm{Hom}(S_*(X_n), G)$$

can be used to show that $\mathrm{Hom}(S_*(X), G)$ has the universal property of

$$\varprojlim \mathrm{Hom}(S_*(X_n), G).$$

□

Example We consider the infinite complex projective space $\mathbb{C}P^\infty$. The arguments are analogous for the infinite real and quaternionic projective spaces, $\mathbb{R}P^\infty$ and $\mathbb{H}P^\infty$.

For $\mathbb{C}P^\infty$ we consider the skeleton filtration, *i.e.*,

$$X_0 = \mathrm{pt} \subset X_1 = \mathbb{C}P^1 \subset X_2 = \mathbb{C}P^2 \subset \ldots$$

so X_n is the $2n$-skeleton of $\mathbb{C}P^\infty$. The Milnor sequence in this case is

$$0 \to \varprojlim{}^1 H^{m-1}(\mathbb{C}P^n) \longrightarrow H^m(\mathbb{C}P^\infty) \longrightarrow \varprojlim H^m(\mathbb{C}P^n) \to 0.$$

However, the maps $H^{m-1}(\mathbb{C}P^{n+1}) \to H^{m-1}(\mathbb{C}P^n)$ are surjective and therefore this inverse system satisfies the Mittag-Leffler condition as well and thus

$$\varprojlim{}^1 H^{m-1}(\mathbb{C}P^n) = 0$$

and therefore

$$H^m(\mathbb{C}P^\infty) \cong \varprojlim H^m(\mathbb{C}P^n).$$

The inverse limit of truncated polynomial rings $\mathbb{Z}[\alpha]/\alpha^{n+1}$ is isomorphic to the ring of formal power series.

Corollary 12.7.

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[[\alpha]],\ |\alpha| = 2$$

where $\mathbb{Z}[[\alpha]]$ denotes the ring of formal power series in α.

Corollary 12.8.

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[[\alpha]],\ |\alpha| = 1$$

and

$$H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[[\alpha]],\ |\alpha| = 4.$$

Often we consider the cohomology of a space as a direct sum

$$H^*(X;G) = \bigoplus_{n\geqslant 0} H^n(X;G).$$

From that point of view we only finite sums in $H^*(X;G)$ so that this interpretation yields the identification of $H^m(\mathbb{C}P^\infty)$ and $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2\mathbb{Z})$ as a polynomial ring and you will find

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[\alpha],\ |\alpha| = 2$$

and

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[[\alpha]],\ |\alpha| = 1.$$

in the literature as well. However, if you view $H_*(X)$ as $\bigoplus_n H_n(X)$ and for free $H_*(X)$ the cohomology as a dual, then the description of $H^*(X)$ as a product $\prod_n H^n(X)$ is more natural.

13. Lens spaces

Let $m \in \mathbb{N}$ and let $\ell_1, \ldots, \ell_n$ be natural numbers with $\gcd(m, \ell_i) = 1$ for all i and assume $n \geqslant 2$. We consider the action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{S}^{2n-1}$ given by

$$\varrho\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1},\ (\zeta; z_1, \ldots, z_n) \mapsto (\zeta^{\ell_1} z_1, \ldots, \zeta^{\ell_n} z_n).$$

Here, $\mathbb{Z}/m\mathbb{Z} = \langle\zeta\rangle$ with $\zeta = e^{\frac{2\pi i}{m}}$ and we view $\mathbb{S}^{2n-1}$ as a subspace of $\mathbb{C}^n$.

This action is free: if $\varrho(\zeta^r; z_1, \ldots, z_n) = (z_1, \ldots, z_n)$, then we have $\zeta^{r\ell_i} z_i = z_i$ for all i, but that implies $\zeta^{r\ell_i} = 1$ for all i and thus $r\ell_i = km$ for some k. As the ℓ_i have no non-trivial common divisor with m this implies that r is a multiple of m.

Example If $m = 2$, then the ℓ_i are odd and therefore the action

$$\varrho\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$$

is the antipodal action.

We consider the quotient spaces $\mathbb{S}^{2n-1}/(\mathbb{Z}/m\mathbb{Z})$.

Definition 13.1. The space $L = L(m; \ell_1, \ldots, \ell_n) = \mathbb{S}^{2n-1}/(\mathbb{Z}/m\mathbb{Z})$ with parameters $(m; \ell_1, \ldots, \ell_n)$ as above is called *lens space*.

Example For $m = 2$ we get the real projective spaces $L(2; \ell_1, \ldots, \ell_n) = \mathbb{R}P^{2n-1}$ as lens spaces.

Note that the projection map $\pi\colon \mathbb{S}^{2n-1} \longrightarrow L(2; \ell_1, \ldots, \ell_n)$ is a covering map because of the freeness of the $\mathbb{Z}/m\mathbb{Z}$-action.

We now want to consider CW structures on lens spaces that generalize the CW structures on projective spaces.

We start with a CW structure on $\mathbb{S}^1$ that has zero cells $\{e^{\frac{2\pi ij}{m}}, 1 \leqslant j \leqslant m\}$.

Let B_j^{2n-2} be the set

$$B_j^{2n-2} = \{\cos\theta(0, \ldots, 0, e^{\frac{2\pi ij}{m}}) + \sin\theta(z_1, \ldots, z_{n-1}, 0) | 0 \leqslant \theta \leqslant \pi/2, (z_1, \ldots, z_{n-1}) \in \mathbb{S}^{2n-3}\},$$

i.e., we connect the point $(0, \ldots, 0, e^{\frac{2\pi ij}{m}})$ with the point $(z_1, \ldots, z_{n-1}) \in \mathbb{S}^{2n-3}$ via a quarter of a circle.

A calculation shows that $B_j^{2n-2} \subset \mathbb{S}^{2n-1}$ and we have $B_j^{2n-2} \cong \mathbb{D}^{2n}$.

If we connect the circular arc between $e^{\frac{2\pi ij}{m}}$ and $e^{\frac{2\pi i(j+1)}{m}}$ with $\mathbb{S}^{2n-3}$ we get B_j^{2n-1} with boundary

$$\partial B_j^{2n-1} = B_j^{2n-2} \cup B_{j+1}^{2n-2}.$$

We have to understand the $\mathbb{Z}/m\mathbb{Z}$-action on these cells. If we restrict ϱ to $\mathbb{S}^{2n-3}$, then $\varrho(\mathbb{S}^{2n-3}) \subset \mathbb{S}^{2n-3}$. The arcs between the $e^{\frac{2\pi ij}{m}}$ and $e^{\frac{2\pi i(j+1)}{m}}$ are permuted by ϱ and therefore ϱ permutes the balls B_j^{2n-2} and the balls B_j^{2n-1}.

Assume that $r \in \mathbb{N}$ with $r\ell_n = 1 \bmod m$, then ϱ^r has order m as well and

$$\varrho|_{B_j^{2n-2}}\colon B_j^{2n-2} \longrightarrow B_{j+1}^{2n-2},$$

because

$$\zeta^{r\ell_n} e^{\frac{2\pi ij}{m}} = e^{\frac{2\pi ijr\ell_n}{m}} e^{\frac{2\pi ij}{m}} = e^{\frac{2\pi i(j+1)}{m}}.$$

The B_j^{2n-1} are a fundamental domain of the ϱ^r-action. Thus

$$L \cong B_j^{2n-1}/\varrho^r = B_j^{2n-1}/B_j^{2n-2} \sim B_{j+1}^{2n-2}$$

for any j.

There is a natural inclusion

$$L(m; \ell_1, \dots, \ell_{n-1}) \subset L(m; \ell_1, \dots, \ell_n)$$

which is given by mapping the class $[(z_1, \dots, z_{n-1})]$ to $[(z_1, \dots, z_{n-1}, 0)]$. Representing $L(m; \ell_1, \dots, \ell_{n-1})$ as $B_j^{2n-3}/\sim$ we see that we can build $L(m; \ell_1, \dots, \ell_n)$ out of $L(m; \ell_1, \dots, \ell_{n-1})$ by attaching the $(2n-1)$-cell B_j^{2n-1} and a $(2n-2)$-cell B_j^{2n-2}. Note that we really just have to take one of the latter because B_j^{2n-2} is glued to its neighbor B_{j-1}^{2n-2} in the quotient.

Inductively we get a cell structure of L with one cell in each dimension up to $2n-1$.

Example Let n be 2, so the lens spaces are quotients of $\mathbb{S}^3$. Let $m = 5$ and $\ell_1 = 1$ and $\ell_2 = 2$, so $\zeta = e^{\frac{2\pi i}{5}}$.

We have B_j^3 being a 3-ball with boundary B_j^2 and B_{j+1}^2. The B_j^2 have elements $\cos\theta(0, e^{\frac{2\pi i j}{5}}) + \sin\theta(z, 0)$ for $z \in \mathbb{S}^1$ so these are pairs

$$(\sin\theta z, \cos\theta e^{\frac{2\pi i j}{5}}) \in \mathbb{S}^3.$$

Let us consider the cellular chain complex of the lens spaces. We saw that

$$C_*(L) = \mathbb{Z}, \quad * = 0, \dots, 2n-1.$$

and let σ^k be the cell corresponding to the ball B_j^k.

The top cell has trivial boundary

$$d(\sigma^{2n-1}) = \sigma^{2n-2} - \sigma^{2n-2} = 0$$

because the topological boundary of B_j^{2n-1} is the union of two balls one dimension lower which are identified in the quotient.

For calculating the boundary of σ_j^{2n-2} the boundary of that cell is $\mathbb{S}^{2n-3}$ and the attaching map is the quotient map

$$\mathbb{S}^{2n-3} \longrightarrow L(m; \ell_1, \dots, \ell_{n-1}).$$

As the action ϱ permutes the cells cyclically, we get that

$$d(\sigma_j^{2n-2}) = m\sigma_j^{2n-3}.$$

By induction we see that the boundary maps are given by multiplication by zero respectively m. Thus

$$H_*(L(m; \ell_1, \dots, \ell_n)) = H_*(0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \dots \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0, 2n-1, \\ \mathbb{Z}/m\mathbb{Z}, & * \text{ odd and } < 2n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that we get $H_1(L) = \pi_1(L) = \mathbb{Z}/m\mathbb{Z}$ from covering theory because $\pi_1\mathbb{S}^{2n-1} = 0$.

As the top homology group is $\mathbb{Z}$ we see that the lens spaces are compact connected orientable manifolds of dimension $2n-1$.

Lemma 13.2. *The additive cohomology groups are*

$$H^*(L; \mathbb{Z}/m\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/m\mathbb{Z}, & 0 \leqslant * \leqslant 2n-1 \\ 0, & * > 2n-1. \end{cases}$$

The universal coefficient theorem gives the result immediately.

Note that the homology groups of L with coefficients in $\mathbb{Z}/m\mathbb{Z}$ are isomorphic to the cohomology groups just by using

$$H_k(L; \mathbb{Z}/m\mathbb{Z}) \cong H_k(L; \mathbb{Z}) \otimes \mathbb{Z}/m\mathbb{Z} \oplus \operatorname{Tor}(H_{k-1}(L), \mathbb{Z}/m\mathbb{Z}).$$

We now focus on the case where $m = p$ is a prime.

Proposition 13.3. *The cohomology group $H^j(L(p;\ell_1,\dots,\ell_{n+1});\mathbb{Z}/p\mathbb{Z})$ is generated by*

$$\begin{cases}\beta^i, & j=2i\\ \alpha\beta^i, & j=2j+1.\end{cases}$$

Here $\alpha \in H^1(L;\mathbb{Z}/p\mathbb{Z})$ and $\beta \in H^2(L;\mathbb{Z}/p\mathbb{Z})$ are generators.

Proof. We prove the claim by induction on n. For $n=1$ we have $L = L(p;\ell_1,\ell_2)$ and if $\alpha \in H^1(L;\mathbb{Z}/p\mathbb{Z})$ and $\beta \in H^2(L;\mathbb{Z}/p\mathbb{Z})$ are generators, then a cup product pairing argument shows that $\alpha \cup \beta$ is a generator in degree three. We have to understand what α^2 is: if p is odd, then $\alpha^2 = 0$. For $p=2$ we know that the lens space is $\mathbb{R}P^3$ and hence in that case α^2 is a generator so it is equal to β. In all other degrees, the cohomology groups are trivial.

Assume now that the claim is true up to degree n. We consider the inclusion

$$L(p;\ell_1,\dots,\ell_n) \hookrightarrow L(p;\ell_1,\dots,\ell_{n+1}) =: L^{2n+1}.$$

Up to degree $2n-1$ this inclusion gives rise to an isomorphism on cohomology groups. We know that β^i generates the cohomology groups up in degrees $j=2i<2n-1$ and $\alpha\beta^i$ generates the cohomology groups in degrees $j = 2i+1 \leqslant 2n-1$. An argument as for projective spaces then shows that $\beta \cup \beta^{n-1}$ generates $H^{2n}(L^{2n+1};\mathbb{Z}/p\mathbb{Z})$ and $\beta \cup \alpha\beta^{n-1} = \alpha\beta^n$ generates $H^{2n+1}(L^{2n+1};\mathbb{Z}/p\mathbb{Z})$. $\square$

Corollary 13.4. *As graded rings*

$$H^*(L(p;\ell_1,\dots,\ell_{n+1});\mathbb{Z}/p\mathbb{Z}) \cong \begin{cases}\Lambda(\alpha)\otimes \mathbb{Z}/p\mathbb{Z}[\beta]/\beta^{n+1}, & p>2,\\ \mathbb{Z}/p\mathbb{Z}[\alpha]/\alpha^{2n+2}, & p=2.\end{cases}$$

Let L denote the direct limit of any system of the form

$$L(p;\ell_1,\dots,\ell_{n+1}) \subset L(p;\ell_1,\dots,\ell_{n+2}) \subset \dots$$

then

$$H^*(L;\mathbb{Z}/p\mathbb{Z}) \cong \begin{cases}\Lambda(\alpha)\otimes \mathbb{Z}/p\mathbb{Z}[[\beta]], & p>2,\\ \mathbb{Z}/p\mathbb{Z}[[\alpha]], & p=2.\end{cases}$$

The second claim follows with the help of the Milnor sequence.

Lens spaces of dimension three give rise to important examples of orientable connected and compact 3-manifolds that have the same fundamental group and homology groups but that are not homotopy equivalent. For instance the lens spaces $L(5;1,1)$ and $L(5;1,2)$ are of that type. You can find a proof of that fact for instance in Bredon's book.

We can interpret the generator β in terms of the so-called *Bockstein-homomorphism*. The short exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0, \quad 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

give rise to short exact sequences of cochain complexes

$$0 \to S^*(X;\mathbb{Z}) \to S^*(X;\mathbb{Z}) \to S^*(X;\mathbb{Z}/p\mathbb{Z}) \to 0, \quad 0 \to S^*(X;\mathbb{Z}/p\mathbb{Z}) \to S^*(X;\mathbb{Z}/p^2\mathbb{Z}) \to S^*(X;\mathbb{Z}/p\mathbb{Z}) \to 0$$

and we get corresponding long exact sequences of cohomology groups. Let $\tilde{\beta}\colon H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z})$ be the connecting homomorphism for the first sequence, let $\beta\colon H^n(X;\mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$ be the one for the second sequence and let $\rho\colon H^{n+1}(X;\mathbb{Z}) \to H^{n+1}(X;\mathbb{Z}/p\mathbb{Z})$ be induced by the reduction mod p. Then β is called the Bockstein homomorphism.

Lemma 13.5. *The diagram*

$$\begin{tikzcd} H^n(X;\mathbb{Z}/p\mathbb{Z}) \arrow[r, "\tilde{\beta}"] \arrow[dr, "\beta"'] & H^{n+1}(X;\mathbb{Z}) \arrow[d, "\rho"] \\ & H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}) \end{tikzcd}$$

commutes.

For the proof just note that the diagram of the corresponding short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "\rho"] & \mathbb{Z} \arrow[r, "\rho"] \arrow[d, "\rho_2"] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r, "p"] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

commutes and therefore we obtain the commutativity of

$$\begin{tikzcd}
H^n(X;\mathbb{Z}/p\mathbb{Z}) \arrow[r, "\tilde{\beta}"] \arrow[d, "\mathrm{id}"] & H^{n+1}(X;\mathbb{Z}) \arrow[d, "\rho"'] \\
H^n(X;\mathbb{Z}/p\mathbb{Z}) \arrow[r, "\beta"] & H^{n+1}(X;\mathbb{Z}/p\mathbb{Z}).
\end{tikzcd}$$

With the help of this auxiliary result it is easy to see that $\beta \in H^2(L(p;\ell_1,\dots,\ell_{n+1});\mathbb{Z}/p\mathbb{Z})$ deserves its name: this class is the image of the Bockstein homomorphism applies to α, *i.e.*, $\beta = \beta(\alpha)$.

The Bockstein homomorphism is just one example of a cohomology operation.