A lower bound for coherences on the Brown-Peterson spectrum

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Rosendal, 20th of August 2005
Theorem

The Brown-Peterson spectrum $BP$ at a prime $p$ has at least a $(2p^2 + 2p - 2)$-stage structure.
What are $n$-stages?

- $n$-stages approximate $E_\infty$-structures.

Alan Robinson:

Consider the $E_\infty$-operad $(B_n)_n = (E\Sigma_n \times T_n)_n$. Here

- $(E\Sigma_n)_n$ is the topological version of the Barratt-Eccles operad and
- $T_n$ is Boardman’s tree operad.
Trees

The space of $n$-trees, $T_n$, consists of abstract trees on $n + 1$ leaves. These leaves are labelled with the numbers $0, \ldots, n$ where each label appears exactly once. Internal edges get an assigned length $0 < \lambda \leq 1$. 
Examples

The tree space is set to consist of a point for $n \leq 2$

The only 2-tree is the tree

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0.5,0) node[midway] {1};
\draw (0,0) -- (0,-0.5) node[midway] {0};
\draw (0.5,0) -- (1,-0.5) node[midway] {2};
\end{tikzpicture}
\end{center}

There are three different types of 3-trees, namely

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0.5,0) node[midway] {1};
\draw (0,0) -- (0,-0.5) node[midway] {2};
\draw (0.5,0) -- (1,-0.5) node[midway] {3};
\end{tikzpicture}
\quad
\begin{tikzpicture}
\draw (0,0) -- (0.5,0) node[midway] {2};
\draw (0,0) -- (0,-0.5) node[midway] {3};
\draw (0.5,0) -- (1,-0.5) node[midway] {1};
\end{tikzpicture}
\quad
\begin{tikzpicture}
\draw (0,0) -- (0.5,0) node[midway] {3};
\draw (0,0) -- (0,-0.5) node[midway] {2};
\draw (0.5,0) -- (1,-0.5) node[midway] {1};
\end{tikzpicture}
\end{center}

with a corolla-shaped tree if the length of the only internal edge is zero.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0.5,0) node[midway] {1};
\draw (0,0) -- (0,-0.5) node[midway] {2};
\draw (0.5,0) -- (1,-0.5) node[midway] {3};
\end{tikzpicture}
\end{center}
Filtration

Robinson defines a filtration of this $E_\infty$-operad as follows: set $B_n^{(i)} := (E\Sigma_n)^{(i)} \times T_n$ where $(E\Sigma_n)^{(i)}$ is the $i$-th skeleton of the standard model for $E\Sigma_n$. Then define

$$\nabla^n B_m := B_m^{(n-m)}.$$
An $n$-stage structure for an $E_\infty$-structure on a spectrum $E$ is a sequence of maps

$$\mu_m : \nabla^n B_m \times \Sigma_m E^\wedge m \to E$$

which on their restricted domain of definition satisfy the requirements for an operad action on $E$. 
2-stage structures

A 2-stage structure on a spectrum $E$ consist of action maps starting from $\nabla^2 B_m$ which is

$$B_m^{(2-m)} = (E\Sigma_m)^{(2-m)} \times T_m.$$ 

Therefore the only requirement here is that we have a map

$$((E\Sigma_1)^{(1)} \times T_1) \times \Sigma_1 E \cong E \xrightarrow{\varphi} E$$

and that $E$ possesses a map

$$((E\Sigma_2)^{(0)} \times T_2) \times \Sigma_2 E^{\wedge 2} \cong (\Sigma_2) \times \Sigma_2 E^{\wedge 2} \rightarrow E.$$

So we obtain a multiplication $\mu$ on $E$ together with its twisted version $\mu \circ \tau$ if $\tau$ denotes the generator of $\Sigma_2$.

Iterates of $\mu$ and $\mu \circ \tau$ act on higher smash powers of $E$, but they do not have to satisfy any relations.
3-stage structures

A 3-stage structure on $E$ comes with three kinds of maps, because non-trivial values for $m$ are $1, 2, 3$.

$m = 2$:

$$((E\Sigma_2)^{(1)} \times T_2) \ltimes \Sigma_2 E^{\wedge 2} \rightarrow E.$$  

The 1-skeleton of $E\Sigma_2$ is the 1-circle, giving the homotopy between $\mu$ and $\mu \circ \tau$.

In addition to that, the value $m = 3$ brings in the homotopies for associativity via the trees we described above.

A 3-stage structure on $E$ is a homotopy commutative and associative multiplication.
Theorem [Robinson]

Assume that $E$ is a homotopy commutative and associative ring spectrum which satisfies

$$E^*(E^\wedge m) \cong \text{Hom}_{E^*}(E^*E^\otimes m, E^*)$$

for all $m \geq 1$.

If $E$ has an $(n-1)$-stage structure which can be extended to an $n$-stage structure then possible obstructions to extending this further to an $(n+1)$-stage structure live in

$$\text{H}_n^{\Gamma, 2-n}(E^*E|E^*; E^*).$$

If in addition $\text{H}_n^{n, 1-n}(E^*E|E^*; E^*)$ vanishes, then this extension is unique.
What is $H\Gamma^*,*$?

Gamma cohomology – a cohomology theory for differential graded $E_\infty$-algebras.

We will need it for graded commutative algebras.

How is it defined?

Let $k$ be a (graded) commutative ring with unit, let $A$ be a (graded) commutative $k$-algebra and let $M$ be a (graded) $A$-module.

Robinson defines Gamma homology of $A$ over $k$ with coefficients in $M$ as the homology of the total complex of a bicomplex $\Xi_{*,*}$ which we will now describe.
Let $\text{Lie}(n)$ be the $n$-th term of the operad which codifies Lie-algebras over $k$, i.e., $\text{Lie}(n)$ is the free $k$-module generated by all Lie monomials in variables $x_1, \ldots, x_n$ such that each variable appears exactly once.

There is a canonical action of the symmetric group on $n$ letters, $\Sigma_n$, on $\text{Lie}(n)$ by permuting the variables $x_i$. 
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Let $\text{Lie}(n)^*$ be the $k$-linear dual of $\text{Lie}(n)$. Then the bicomplex for Gamma homology in bidegree $(r, s)$ is defined as

$$\Xi_{r,s}(A|k; M) = \text{Lie}(s+1)^* \otimes k[\Sigma_{s+1}]^r \otimes A^{(s+1)} \otimes M.$$
Differentials

The horizontal differential is the differential of the bar construction, *i.e.*, the elements of the symmetric group are multiplied together or an action of $\Sigma_{s+1}$ on the dual of the Lie monomials is induced or the elements in the $(s+1)$-st tensor power of the algebra $A$ are permuted.

The vertical differential is more complicated...

Important is that both differential are homogeneous if $k$, $A$ and $M$ carry an internal grading.
From now on let $k$ etc be graded (e.g. $k = E_*$, $A = E_*E$, $M = E_*$ or $E_*E$).

Following Robinson we denote by $HΓ^{q,i}(A|k; M)$ the $q$-th cohomology of the homomorphism complex

$$\text{Hom}^i_A(\text{Tot}(Ξ_*,|A|k; A)), M)$$

whose morphisms lower internal degree by $i$. 
Properties of Gamma (co)homology For sake of simplicity assume that $A$ is $k$-projective.

- In good cases (like $BP$) there is a universal coefficient spectral sequence with
  \[ E_2^{*,*} = \text{Ext}_{A}^{*,*} (\mathcal{H}\Gamma_\ast (A|k; A), M) \]
  and converging to $\mathcal{H}\Gamma_\ast (A|k; M)$.

- Robinson and Whitehouse proved that Gamma cohomology vanishes if $A$ is étale over $k$, and that Gamma cohomology satisfies Flat Base Change and has a Transitivity Sequence.

- **Theorem[Basterra-R]**
  Gamma cohomology is isomorphic to the obstruction groups which arise in the work of Goerss and Hopkins.
The case of $BP$

$BP$ satisfies the necessary properties to apply Robinson’s obstruction theory: $BP$ is a homotopy commutative $MU$-ring spectrum at all primes, so we start with a 3-stage structure.

If we want to establish an $(2p^2 + 2p - 2)$-stage structure on $BP$, then we have to show that Gamma cohomology vanishes in bidegrees $(n, 2 - n)$ for all $2p^2 + 2p - 3 \geq n \geq 3$. 
Ingredients of the proof

• Additivity

From $BP_*BP = BP*[t_1,t_2,\ldots]$ we can deduce

$$H_{\Gamma s,*}(BP_*BP|BP_*; BP_*BP) \cong \bigoplus_{i \geq 1} H_{\Gamma s,*}(BP_*[t_i]|BP_*; BP_*BP).$$

• Flat base change

$$H_{\Gamma s,*}(BP_*[t_i]|BP_*; BP_*BP) \cong BP_* \otimes_{\mathbb{Z}(p)} H_{\Gamma s,*}(\mathbb{Z}(p)[t_i]|\mathbb{Z}(p); \mathbb{Z}(p)) \otimes_{\mathbb{Z}(p)} BP_*BP.$$
• Calculation of $H\Gamma_s$ for polynomial algebras [R-Robinson]

Summing over all internal degrees, we can identify Gamma homology of $\mathbb{Z}_p[t_i]$ as

$$\bigoplus_{t \geq 0} H\Gamma_{s,t}(\mathbb{Z}_p[t_i]|\mathbb{Z}_p;\mathbb{Z}_p) \cong (H\mathbb{Z}_p)_s H\mathbb{Z}.$$
Kochman’s result

Kochman provides an explicit basis of the $p$-torsion in $H\mathbb{Z} \ast H\mathbb{Z}$. The result is:

- There is only simple $p$-torsion.

- An explicit basis of $(H\mathbb{Z}_{(p)}) \ast H\mathbb{Z}$ over $\mathbb{Z}/p\mathbb{Z}$ consists of all expressions

$$P(n_1, \ldots, n_t)\bar{\zeta}_1^{e_1} \cdots \bar{\zeta}_s^{e_s}$$

where $t \geq 0$, $t \neq 1$, $0 < n_1 < \ldots < n_t$, $e_i \geq 0$, $t + e_1 + \ldots + e_s > 0$ and $e_i = 0$ for $i < n_1$. Here, the degree of the $P(n_1, \ldots, n_t)$ is $2(p^{n_1} + \ldots + p^{n_t}) - t - 1$ and the degree of $\bar{\zeta}_i$ is $2(p^i - 1)$. 
Cases $t = 0$ and $t = 2$.

For $t = 0$ the condition $e_i = 0$ for $i < n_1$ is void, therefore elements like $\bar{\zeta}^{e_1}_1 \ldots \bar{\zeta}^{e_s}_s$ arise with at least one $e_i$ being positive. These elements have total degree

$$\text{degree}(\bar{\zeta}^{e_1}_1 \ldots \bar{\zeta}^{e_s}_s) = \sum_{i=1}^{s} e_i (2p^i - 2).$$

For $t = 2$ the element of lowest possible degree in this case is $P(1, 2)$ with

$$\text{degree}(P(1, 2)) = 2p + 2p^2 - 2 - 1 = 2p^2 + 2p - 3.$$
Obstructions for extending an $n$-stage to an $(n + 1)$-stage structure live in $\mathrm{H}^\Gamma_{n, 2-n}$.

As we know that $\left(\mathrm{H} \mathbb{Z}(p)\right)_* \mathbb{H}$ consists only of simple $p$-torsion it suffices to consider $\mathrm{Ext}^0, *$- and $\mathrm{Ext}^1, *$-terms in the Universal Coefficient spectral sequence.
We know as well, that the internal degree can only be of the form \( \sum_{i=1}^{N} \lambda_i (2p^i - 2) \); consequently possible values for \( n \) have to be of the form \( \sum_{i=1}^{N} \lambda_i (2p^i - 2) + 2 \) with the \( \lambda_i \) being non-negative integers.
A degree count gives that the case $t = 0$ does not give any obstruction classes, but $t = 2$ gives a possible class:

Here the corresponding equation of degrees that has to be satisfied is

$$\sum_{i=1}^{N} \lambda_i (2p^i - 2) + 1 = 2p^n + 2p^m - 3 + \sum_{j=1}^{M} e_j (2p^j - 2).$$

The generator $P(1, 2)$ is of lowest possible degree and turns this requirement into

$$n - 1 = \sum_{i=1}^{N} \lambda_i (2p^i - 2) + 1 = 2p + 2p^2 - 3 \equiv 2p - 2 + 2p^2 - 2 + 1.$$

Therefore such a homology class could occur for $n = 2p^2 + 2p - 2$. □
Dyer-Lashof operations

An $n$-stage structure on a spectrum $E$ gives rise to some Dyer-Lashof operations coming from the skeleton filtration of the Barratt-Eccles operad.

Proposition

If $E$ has an $n$-stage structure with $n > p$ then there are Dyer-Lashof operations $Q_i$ on the $\mathbb{F}_p$-homology of $E$ for $i \leq n - p$. 
The indecomposable element $a_{p-1}$ in the group $(H\mathbb{F}_p)_{2p-2}(MU)$ is known to be in the image of $(H\mathbb{F}_p)_{2p-2}(BP)$. Consider an element $x = x_{2p-2}$ in $(H\mathbb{F}_p)_{2p-2}(BP)$ with image $a_{p-1}$.

For such an $x$ the highest Dyer-Lashof operation $Q^i$ which we get out of the $(2p^2 + 2p - 2)$-stage structure is $Q^{2p}$. 
Hu, Kriz and May proved that the inclusion from $BP$ to $MU$ cannot be a map of commutative $S$-algebras, and they used this particular Dyer-Lashof operation to show that.

The image of $a_{p-1}$ under $Q^{2p}$ is $a_{(2p+1)(p-1)}$ up to decomposable elements, but there is no indecomposable element in $(H\mathbb{F}_p)(2p+1)(p-1)BP$. For $p = 2$ a similar argument works using $a_1$. 
Theorem

The Brown-Peterson spectrum $BP$ cannot be the Thom spectrum associated to an 4-fold loop map to $BSF$ at $p = 2$ resp. a $(2p+4)$-fold loop map to $BSF$ at any odd prime $p$.

Here, $BSF$ is the classifying space of spherical fibrations.

Again, The proof uses a Dyer-Lashof operation argument.
Proof

Assume there were such a map from an \(n\)-fold loop space \(X\) to \(BSF\)

\[\gamma : X \longrightarrow BSF\]

which would allow to write \(BP\) as the Thom spectrum associated to \(\gamma\), \(BP = X^\gamma\).

Lewis: Then \(BP\) is an \(E_n\)-spectrum.

Thom isomorphism: the homology of \(BP\) is isomorphic to the homology of \(X\), and the latter maps to the homology of \(BSF\).

[Cohen-Lada-May]

If \(p = 2\), the homology of \(BSF\) is

\[H_*(BSF) \cong H_*(BSO) \otimes C_*,\]

whereas at odd primes, the homology of \(BSF\) is isomorphic to

\[H_*(BSF) \cong H_*W \otimes C'_*.\]
The map from $BSO$ to $BSF$ is an infinite loop map and it is this map which includes the tensor factor $H_*(BSO)$ into $H_*(BSF)$. Therefore the tensor factor $H_*(BSO)$ is closed under the Dyer-Lashof operations.

A similar remark applies to $W$ which is a summand of $BO$ at odd primes, because there is a splitting of infinite loop spaces

$$BO(p) \cong W \times W^\perp.$$
Consider \( x = x_{2p-2} \) in \( H_\ast(BP) \) with \( P_\ast^1(x) = 1 \). This gives a non-trivial class of degree \( 2p-2 \) in \( H_\ast(BSF) \). There is no such class in the \( C_\ast \)- resp. \( C'_\ast \)-part. Therefore \( x \) has to have an image in \( H_\ast(BSO) \) resp. \( H_\ast(W) \).

In both cases, \( x \) has to hit an indecomposable element, whose \( Q^{2p} \)-image gives a generator up to decomposable elements. The lack of indecomposables in \( H_\ast(BP) \) yields a contradiction.