A lower bound for coherences on the Brown-Peterson spectrum

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Theorem

The Brown-Peterson spectrum BP at a prime p has at least a $(2p^2+2p-2)$-stage structure.

What are n-stages?

– n-stages approximate E_∞-structures.

Alan Robinson:

Consider the E_∞-operad $(\mathcal{B}_n)_n = (E\Sigma_n \times T_n)_n$. Here

- $(E\Sigma_n)_n$ is the topological version of the Barratt-Eccles operad and
- T_n is Boardman's tree operad.

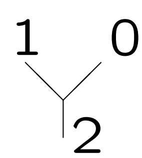
Trees

The space of n-trees, T_n, consists of abstract trees on $n + 1$ leaves. These leaves are labelled with the numbers $0, \ldots, n$ where each label appears exactly once. Internal edges get an assigned length $0 < \lambda \leqslant 1$.

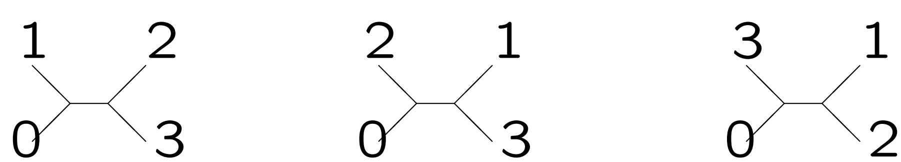
Examples

The tree space is set to consist of a point for $n \leqslant 2$

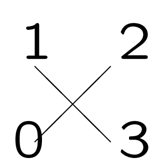
The only 2-tree is the tree



There are three different types of 3-trees, namely



with a corolla-shaped tree if the length of the only internal edge is zero.



Filtration

Robinson defines a filtration of this E_∞-operad as follows: set $\mathcal{B}_n^{(i)} := (E\Sigma_n)^{(i)} \times T_n$ where $(E\Sigma_n)^{(i)}$ is the i-th skeleton of the standard model for $E\Sigma_n$. Then define

$$\nabla^n \mathcal{B}_m := \mathcal{B}_m^{(n-m)}.$$

n-stages

An n-stage structure for an E_∞-structure on a spectrum E is a sequence of maps

$$\mu_m : \nabla^n \mathcal{B}_m \ltimes_{\Sigma_m} E^{\wedge m} \longrightarrow E$$

which on their restricted domain of definition satisfy the requirements for an operad action on E.

2-**stage structures**

A 2-stage structure on a spectrum E consist of action maps starting from $\nabla^2\mathcal{B}_m$ which is

$$\mathcal{B}_m^{(2-m)} = (E\Sigma_m)^{(2-m)} \times T_m.$$

Therefore the only requirement here is that we have a map

$$((E\Sigma_1)^{(1)} \times T_1) \ltimes_{\Sigma_1} E \cong E \xrightarrow{\varphi} E$$

and that E possesses a map

$$((E\Sigma_2)^{(0)} \times T_2) \ltimes_{\Sigma_2} E^{\wedge 2} \cong (\Sigma_2) \ltimes_{\Sigma_2} E^{\wedge 2} \longrightarrow E.$$

So we obtain a multiplication μ on E together with its twisted version $\mu \circ \tau$ if τ denotes the generator of Σ_2.

Iterates of μ and $\mu \circ \tau$ act on higher smash powers of E, but they do not have to satisfy any relations.

3-**stage structures**

A 3-stage structure on E comes with three kinds of maps, because non-trivial values for m are $1, 2, 3$.

$m = 2$:

$$((E\Sigma_2)^{(1)} \times T_2) \ltimes_{\Sigma_2} E^{\wedge 2} \longrightarrow E.$$

The 1-skeleton of $E\Sigma_2$ is the 1-circle, giving the homotopy between μ and $\mu \circ \tau$.

$$\begin{tikzcd}
\mu \arrow[r, bend left] & \mu \circ \tau \arrow[l, bend left]
\end{tikzcd}$$

In addition to that, the value $m = 3$ brings in the homotopies for associativity via the trees we described above.

A 3-stage structure on E is a homotopy commutative and associative multiplication.

Theorem [Robinson]

Assume that E is a homotopy commutative and associative ring spectrum which satisfies

$$E^*(E^{\wedge m}) \cong \mathrm{Hom}_{E_*}(E_*E^{\otimes m}, E_*)$$

for all $m \geqslant 1$.

If E has an $(n-1)$-stage structure which can be extended to an n-stage structure then possible obstructions to extending this further to an $(n+1)$-stage structure live in

$$\mathrm{H}\Gamma^{n,2-n}(E_*E|E_*; E_*).$$

If in addition $\mathrm{H}\Gamma^{n,1-n}(E_*E|E_*; E_*)$ vanishes, then this extension is unique.

What is H$\Gamma^{*,*}$?

Gamma cohomology – a cohomology theory for differential graded E_∞-algebras.

We will need it for graded commutative algebras.

How is it defined?

Let k be a (graded) commutative ring with unit, let A be a (graded) commutative k-algebra and let M be a (graded) A-module.

Robinson defines Gamma homology of A over k with coefficients in M as the homology of the total complex of a bicomplex $\Xi_{*,*}$ which we will now describe.

Let Lie(n) be the n-th term of the operad which codifies Lie-algebras over k, *i.e.*, Lie(n) is the free k-module generated by all Lie monomials in variables $x_1, \ldots, x_n$ such that each variable appears exactly once.

There is a canonical action of the symmetric group on n letters, Σ_n, on Lie(n) by permuting the variables x_i.

Let $\mathrm{Lie}(n)^*$ be the k-linear dual of $\mathrm{Lie}(n)$. Then the bicomplex for Gamma homology in bidegree (r, s) is defined as

$$\Xi_{r,s}(A|k; M) =$$

$$\mathrm{Lie}(s+1)^* \otimes k[\Sigma_{s+1}]^{\otimes r} \otimes A^{\otimes(s+1)} \otimes M.$$

Differentials

The horizontal differential is the differential of the bar construction, *i.e.*, the elements of the symmetric group are multiplied together or an action of Σ_{s+1} on the dual of the Lie monomials is induced or the elements in the $(s+1)$-st tensor power of the algebra A are permuted.

The vertical differential is more complicated...

Important is that both differential are homogeneous if k, A and M carry an internal grading.

From now on let k etc be graded (*e.g.* $k = E_*$, $A = E_*E$, $M = E_*$ or E_*E).

Following Robinson we denote by $\mathrm{H\Gamma}^{q,i}(A|k; M)$ the q-th cohomology of the homomorphism complex

$$\mathrm{Hom}^i_A(\mathrm{Tot}(\Xi_{*,*}(A|k; A)), M)$$

whose morphisms lower internal degree by i.

Properties of Gamma (co)homology For sake of simplicity assume that A is k-projective.

- In good cases (like BP) there is a universal coefficient spectral sequence with

  $$\mathsf{E}_2^{*,*} = \mathsf{Ext}_A^{*,*}(\mathsf{H\Gamma}_*(A|k; A), M)$$

  and converging to $\mathsf{H\Gamma}^*(A|k; M)$.

- Robinson and Whitehouse proved that Gamma cohomology vanishes if A is étale over k, and that Gamma cohomology satisfies Flat Base Change and has a Transitivity Sequence.

- **Theorem[Basterra-R]**

  Gamma cohomology is isomorphic to the obstruction groups which arise in the work of Goerss and Hopkins.

The case of BP

BP satisfies the necessary properties to apply Robinson's obstruction theory: BP is a homotopy commutative MU-ring spectrum at all primes, so we start with a 3-stage structure.

If we want to establish an $(2p^2 + 2p - 2)$-stage structure on BP, then we have to show that Gamma cohomology vanishes in bidegrees $(n, 2-n)$ for all $2p^2 + 2p - 3 \geqslant n \geqslant 3$.

Ingredients of the proof

- **Additivity**

  From $BP_*BP = BP_*[t_1, t_2, \ldots]$ we can deduce

  $\mathsf{H\Gamma}_{s,*}(BP_*BP|BP_*; BP_*BP) \cong$

  $$\bigoplus_{i \geqslant 1} \mathsf{H\Gamma}_{s,*}(BP_*[t_i]|BP_*; BP_*BP).$$

- **Flat base change**

  $\mathsf{H\Gamma}_{s,*}(BP_*[t_i]|BP_*; BP_*BP) \cong$

  $BP_* \otimes_{\mathbb{Z}_{(p)}} \mathsf{H\Gamma}_{s,*}(\mathbb{Z}_{(p)}[t_i]|\mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} BP_*BP.$

- **Calculation of $\mathsf{H}\Gamma_s$ for polynomial algebras [R-Robinson]**

  Summing over all internal degrees, we can identify Gamma homology of $\mathbb{Z}_{(p)}[t_i]$ as

$$\bigoplus_{t \geqslant 0} \mathsf{H}\Gamma_{s,t}(\mathbb{Z}_{(p)}[t_i] | \mathbb{Z}_{(p)}; \mathbb{Z}_{(p)}) \cong (H\mathbb{Z}_{(p)})_s H\mathbb{Z}.$$

Kochman's result

Kochman provides an explicit basis of the p-torsion in $H\mathbb{Z}_*H\mathbb{Z}$. The result is:

- There is only simple p-torsion.

- An explicit basis of $(H\mathbb{Z}_{(p)})_*H\mathbb{Z}$ over $\mathbb{Z}/p\mathbb{Z}$ consists of all expressions

$$P(n_1, \ldots, n_t)\bar{\zeta}_1^{e_1} \cdot \ldots \cdot \bar{\zeta}_s^{e_s}$$

  where $t \geqslant 0$, $t \neq 1$, $0 < n_1 < \ldots < n_t$, $e_i \geqslant 0$, $t + e_1 + \ldots + e_s > 0$ and $e_i = 0$ for $i < n_1$. Here, the degree of the $P(n_1, \ldots, n_t)$ is $2(p^{n_1} + \ldots + p^{n_t}) - t - 1$ and the degree of $\bar{\zeta}_i$ is $2(p^i - 1)$.

Cases $t = 0$ and $t = 2$.

For $t = 0$ the condition $e_i = 0$ for $i < n_1$ is void, therefore elements like $\bar{\zeta}_1^{e_1} \ldots \bar{\zeta}_s^{e_s}$ arise with at least one e_i being positive. These elements have total degree

$$\text{degree}(\bar{\zeta}_1^{e_1} \cdot \ldots \cdot \bar{\zeta}_s^{e_s}) = \sum_{i=1}^{s} e_i(2p^i - 2).$$

For $t = 2$ the element of lowest possible degree in this case is $P(1, 2)$ with

$$\text{degree}(P(1, 2)) = 2p + 2p^2 - 2 - 1 = 2p^2 + 2p - 3.$$

Obstructions for extending an n-stage to an $(n+1)$-stage structure live in $\mathsf{H}\Gamma^{n,2-n}$.

As we know that $(H\mathbb{Z}_{(p)})_* H\mathbb{Z}$ consists only of simple p-torsion it suffices to consider $\mathsf{Ext}^{0,*}$- and $\mathsf{Ext}^{1,*}$-terms in the Universal Coefficient spectral sequence.

We know as well, that the internal degree can only be of the form $\sum_{i=1}^{N} \lambda_i(2p^i - 2)$; consequently possible values for n have to be of the form $\sum_{i=1}^{N} \lambda_i(2p^i - 2) + 2$ with the λ_i being non-negative integers.

A degree count gives that the case $t = 0$ does not give any obstruction classes, but $t = 2$ gives a possible class:

Here the corresponding equation of degrees that has to be satisfied is

$$\sum_{i=1}^{N} \lambda_i(2p^i - 2) + 1 =$$

$$2p^n + 2p^m - 3 + \sum_{j=1}^{M} e_j(2p^j - 2).$$

The generator $P(1, 2)$ is of lowest possible degree and turns this requirement into

$$n - 1 = \sum_{i=1}^{N} \lambda_i(2p^i - 2) + 1 =$$

$$2p + 2p^2 - 3 = 2p - 2 + 2p^2 - 2 + 1.$$

Therefore such a homology class could occur for $n = 2p^2 + 2p - 2$. $\square$

Dyer-Lashof operations

An n-stage structure on a spectrum E gives rise to some Dyer-Lashof operations coming from the skeleton filtration of the Barratt-Eccles operad.

Proposition

If E has an n-stage structure with $n > p$ then there are Dyer-Lashof operations Q_i on the $\mathbb{F}_p$-homology of E for $i \leqslant n - p$.

The indecomposable element a_{p-1} in the group $(H\mathbb{F}_p)_{2p-2}(MU)$ is known to be in the image of $(H\mathbb{F}_p)_{2p-2}(BP)$. Consider an element $x = x_{2p-2}$ in $(H\mathbb{F}_p)_{2p-2}(BP)$ with image a_{p-1}.

For such an x the highest Dyer-Lashof operation Q^i which we get out of the $(2p^2+2p-2)$-stage structure is Q^{2p}.

Hu, Kriz and May proved that the inclusion from BP to MU cannot be a map of commutative S-algebras, and they used this particular Dyer-Lashof operation to show that.

The image of a_{p-1} under Q^{2p} is $a_{(2p+1)(p-1)}$ up to decomposable elements, but there is no indecomposable element in $(H\mathbb{F}_p)_{(2p+1)(p-1)}BP$. For $p = 2$ a similar argument works using a_1.

Theorem

The Brown-Peterson spectrum BP cannot be the Thom spectrum associated to an 4-fold loop map to BSF at $p = 2$ resp. a $(2p+4)$-fold loop map to BSF at any odd prime p.

Here, BSF is the classifying space of spherical fibrations.

Again, The proof uses a Dyer-Lashof operation argument.

Proof

Assume there were such a map from an n-fold loop space X to BSF

$$\gamma : X \longrightarrow BSF$$

which would allow to write BP as the Thom spectrum associated to γ, $BP = X^\gamma$.

Lewis: Then BP is an E_n-spectrum.

Thom isomorphism: the homology of BP is isomorphic to the homology of X, and the latter maps to the homology of BSF.

[Cohen-Lada-May]

If $p = 2$, the homology of BSF is

$$H_*(BSF) \cong H_*(BSO) \otimes C_*,$$

whereas at odd primes, the homology of BSF is isomorphic to

$$H_*(BSF) \cong H_*W \otimes C'_*.$$

The map from BSO to BSF is an infinite loop map and it is this map which includes the tensor factor $H_*(BSO)$ into $H_*(BSF)$. Therefore the tensor factor $H_*(BSO)$ is closed under the Dyer-Lashof operations.

A similar remark applies to W which is a summand of BO at odd primes, because there is a splitting of infinite loop spaces

$$BO_{(p)} \simeq W \times W^{\perp}.$$

Consider $x = x_{2p-2}$ in $H_*(BP)$ with $\mathcal{P}^1_*(x) = 1$. This gives a non-trivial class of degree $2p - 2$ in $H_*(BSF)$. There is no such class in the C_*- resp. C'_*-part. Therefore x has to have an image in $H_*(BSO)$ resp. $H_*(W)$.

In both cases, x has to hit an indecomposable element, whose Q^{2p}-image gives a generator up to decomposable elements. The lack of indecomposables in $H_*(BP)$ yields a contradiction.

□