Some properties of the Thom spectrum over loop suspension of complex projective space

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Abstract. This note provides a reference for some properties of the Thom spectrum $M_\xi$ over $\Omega \Sigma \mathbb{C}P^\infty$. Some of this material is used in recent work of Kitchloo and Morava. We determine the $M_\xi$-cohomology of $\mathbb{C}P^\infty$ and show that $M_\xi^*(\mathbb{C}P^\infty)$ injects into power series over the algebra of non-symmetric functions. We show that $M_\xi$ gives rise to a commutative formal group law over the non-commutative ring $\pi_*(M_\xi)$. We also discuss how $M_\xi$ and some real and quaternionic analogues behave with respect to spectra that are related to these Thom spectra by splittings and by maps.

Introduction

The map $\mathbb{C}P^\infty = BU(1) \to BU$ gives rise to a canonical loop map $\Omega \Sigma \mathbb{C}P^\infty \to BU$. Therefore the associated Thom spectrum has a strictly associative multiplication. But as is visible from its homology, which is a tensor algebra on the reduced homology of $\mathbb{C}P^\infty$, it is not even homotopy commutative. This homology ring coincides with the ring of non-symmetric functions, $\text{NSymm}$. We show that there is a map from the $M_\xi$-cohomology of $\mathbb{C}P^\infty$ to the power series over the ring of non-symmetric functions, $\text{NSymm}$. This result is used in [MK] in an application of $M_\xi$ to quasitoric manifolds.

Although $M_\xi$ maps to $MU$, there is no obvious map to it from $MU$, so a priori it is unclear whether there is a formal group law associated to $M_\xi$. However, analogues of the classical Atiyah-Hirzebruch spectral sequence calculations for $MU$ can be made for $M_\xi$, and these show that there is a ‘commutative formal group’ structure related to $M_\xi^*\mathbb{C}P^\infty$, even though the coefficient ring $M_\xi^*$ is not commutative and its elements do not commute with the variables coming from the choices of complex orientations. A formal group law in this context is an element $F(x, y) \in M_\xi^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ that satisfies the usual axioms for a commutative formal group law. We describe the precise algebraic structure arising here in Section 3.

For $MU$ the $p$-local splitting gives rise to a map of ring spectra $BP \to MU_{(p)}$. We show that despite the fact that $M_\xi_{(p)}$ splits into (suspended) copies of $BP$, there is no map of ring spectra $BP \to M_\xi_{(p)}$. For the canonical Thom spectrum

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over $\Omega \Sigma \Omega P^\infty$, $M\xi_R$, we show that the map of $E_2$-algebra spectra $HF_2 \to MO$ does not give rise to a ring map $HF_2 \to M\xi_R$.

For a map of ring spectra $MU \to E$ to some commutative $S$-algebra $E$ one can ask whether a map of commutative $S$-algebras $S \wedge_{\mathbb{P}(S)} \mathbb{P}(\Sigma^{-2}CP^\infty) \to E$ factors over $MU$. Here $\mathbb{P}(-)$ denotes the free commutative $S$-algebra functor. It is easy to see that $MU$ is not equivalent to $S \wedge_{\mathbb{P}(S)} \mathbb{P}(\Sigma^{-2}CP^\infty)$. We show that there are commutative $S$-algebras for which that is not the case. In the associative setting, the analogous universal gadget would be $S \Pi_{A(S)} A(\Sigma^{-2}CP^\infty)$, where $A(-)$ is the free associative $S$-algebra functor and $\Pi$ denotes the coproduct in the category of associative $S$-algebras. It is obvious that the homology of $S \Pi_{A(S)} A(\Sigma^{-2}CP^\infty)$ is much bigger than the one of $M\xi$. If we replace the coproduct by the smash product, there is still a canonical map $S \wedge_{\mathbb{P}(S)} \mathbb{P}(\Sigma^{-2}CP^\infty) \to M\xi$ due to the coequalizer property of the smash product. However, this smash product still has homology that is larger than that of $M\xi$. Therefore the freeness of $\Omega \Sigma \Sigma P^\infty$ is not reflected on the level of Thom spectra.

1. The Thom spectrum of $\xi$

Lewis showed \cite{LMSM} Theorem IX.7.1 that an $n$-fold loop map to $BF$ gives rise to an $E_n$-structure on the associated Thom spectrum. Here $E_n$ is the product of the little $n$-cubes operad with the linear isometries operad. For a more recent account in the setting of symmetric spectra see work of Christian Schlichtkrull \cite{Sch}.

The map $j: \Omega \Sigma \Sigma CP^\infty \to BU$ is a loop map and so the Thom spectrum $M\xi$ associated to that map is an $A_\infty$ ring spectrum and the natural map $M\xi \to MU$ is one of $A_\infty$ ring spectra, or equivalently of $S$-algebras in the sense of \cite{EKMM}.

Since the homology $H_*(M\xi)$ is isomorphic as a ring to $H_*(\Omega \Sigma \Sigma CP^\infty)$, we see that $M\xi$ is not even homotopy commutative. We investigated some of the properties of $M\xi$ in \cite{BR}.

For any commutative ring $R$, under the Thom isomorphism $H_*(\Omega \Sigma \Sigma CP^\infty; R) \cong H_*(M\xi; R)$, the generator $Z_i$ corresponds to an element $z_i \in H_{2i}(M\xi; R)$, where we set $z_0 = 1$. Thomifying the map $i: \Sigma P^\infty \to \Omega \Sigma \Sigma P^\infty$, we obtain a map $Mi: MU(1) \to \Sigma^2 M\xi$, and it is easy to see that

$$Mi \beta_{i+1} = z_i. \tag{1.1}$$

1.1. Classifying negatives of bundles. For every based space $X$, time-reversal of loops is a loop-map from $(\Omega X)^{\text{op}}$ to $\Omega X$, i.e.,

$$(\overset{\text{\textcircled{\textbullet}}}{}): (\Omega X)^{\text{op}} \to \Omega X; \quad w \mapsto \bar{w},$$

where $\bar{w}(t) = w(1 - t)$. Here $(\Omega X)^{\text{op}}$ is the space of loops on $X$ with the opposite multiplication of loops.

We consider $BU$ with the $H$-space structure coming from the Whitney sum of vector bundles and denote this space by $BU_{\oplus}$. A complex vector bundle of finite rank on a reasonable space $Y$ is represented by a map $f: Y \to BU$ and the composition

$$Y \xrightarrow{f} BU_{\oplus} \overset{(\overset{\text{\textcircled{\textbullet}}}{})}{\longrightarrow} BU_{\oplus}$$

classifies the negative of that bundle, switching the rôle of stable normal bundles and stable tangent bundles for smooth manifolds.
For line bundles \( g_i : Y \to \mathbb{C}P^\infty \to BU \) \((i = 1, \ldots, n)\) we obtain a map
\[
g = (g_n, \ldots, g_1) : Y \to Y^n \to (\mathbb{C}P^\infty)^n \to (\Omega \Sigma \mathbb{C}P^\infty)^{op} \to BU^{op}
\]
and the composition with loop reversal classifies the negative of the sum \( g_n \oplus \cdots \oplus g_1 \)
as indicated in the following diagram.

\[
\begin{array}{cccc}
Y & \xrightarrow{g} & Y^n & \xrightarrow{(\cdot)} \xrightarrow{g} (\Omega \Sigma \mathbb{C}P^\infty)^{op} \\
\downarrow & & \downarrow & \downarrow \\
\Omega \Sigma \mathbb{C}P^\infty & \xrightarrow{g} & BU
\end{array}
\]

In this way, the splitting of the stable tangent bundle of a toric manifold into a sum of line bundles can be classified by \( \Omega \Sigma \mathbb{C}P^\infty \). For work on an interpretation of \( \pi_* M\xi \) as the habitat for cobordism classes of quasitoric manifolds see [MK].

### 2. \( M\xi \)-(co)homology

We note that the composition of the natural map \( i : \mathbb{C}P^\infty \to \Omega \Sigma \mathbb{C}P^\infty \) with \( j : \Omega \Sigma \mathbb{C}P^\infty \to BU \) classifies the reduced line bundle \( \eta \) over \( \mathbb{C}P^\infty \). The associated map \( M\xi : \Sigma \infty \text{MU}(1) \to \Sigma^2 M\xi \) gives a distinguished choice of complex orientation \( x_\xi \in \tilde{M\xi}{^2}(\mathbb{C}P^\infty) \), since the zero-section \( \mathbb{C}P^\infty \to \text{MU}(1) \) is an equivalence.

We use the Atiyah-Hirzebruch spectral sequence
\[
E_2^{*,*} = H^*(\mathbb{C}P^\infty; M\xi^*) \Rightarrow M\xi^*(\mathbb{C}P^\infty).
\]
As \( M\xi \) is an associative ring spectrum, this spectral sequence is multiplicative and its \( E_2 \)-page is \( \mathbb{Z}[x] \otimes M\xi^* \). As the spectral sequence collapses, the associated graded is of the same form and we can deduce the following:

**Lemma 2.1.** As a left \( M\xi^* \) = \( M\xi_{-*} \)-module we have
\[
M\xi^*(\mathbb{C}P^\infty) = \{ \sum_{i \geq 0} a_i x_\xi^i : a_i \in M\xi^* \}.
\]

The filtration in the spectral sequence \([2.1]\) comes from the skeleton filtration of \( \mathbb{C}P^\infty \) and corresponds to powers of the augmentation ideal \( \tilde{M\xi}{^2}(\mathbb{C}P^\infty) \) in \( M\xi^*(\mathbb{C}P^\infty) \). Of course the product structure in the ring \( M\xi^*(\mathbb{C}P^\infty) \) is more complicated than in the case of \( \text{MU}^*(\mathbb{C}P^\infty) \) since \( x_\xi \) is not a central element.

In order to understand a difference of the form \( ux_\xi^k - x_\xi^ku \) with \( u \in M\xi_m \) and \( k \geq 1 \) we consider the cofibre sequence
\[
\Sigma^m \mathbb{C}P^{k-1} \subseteq \Sigma^m \mathbb{C}P^k \to \Sigma^m \Sigma^2 k.
\]
Both elements \( ux_\xi^k \) and \( x_\xi^ku \) restrict to the trivial map on \( \Sigma^m \mathbb{C}P^{k-1} \). The orientation \( x_\xi \) restricted to \( \Sigma^2 k \) is the 2-fold suspension of the unit of \( M\xi, \Sigma^2 i \in M\xi^2(\mathbb{S}^2) \).
Centrality of the unit ensures that the following square and outer diagram commute, so the difference $ux^k - x^k u$ is trivial. This yields

\begin{equation}
ux^k - x^k u \in (M\xi^* (\mathbb{C}P^\infty))^{[2k+2]}.
\end{equation}

Lemma 2.2. For every $u \in M\xi$, and $k \geq 1$, $u$ and $x^k u$ commute up to elements of filtration at least $2k + 2$, i.e.,

\begin{equation}
ux^k - x^k u \in (M\xi^* (\mathbb{C}P^\infty))^{[2k+2]}.
\end{equation}

Let $E$ be any associative $S$-algebra with an orientation class $u_E \in E^2(\mathbb{C}P^\infty)$. The Atiyah-Hirzebruch spectral sequence for $E^* \mathbb{C}P^\infty$ identifies $E^* (\mathbb{C}P^\infty)$ with the left $E_*$-module of power series in $u_E$ as in the case of $M\xi$:

$$E^*(\mathbb{C}P^\infty) = \{ \sum_{i \geq 0} \theta_i u^i : \theta_i \in E_* \}.$$ 

The orientation class $u_E \in E^2(\mathbb{C}P^\infty)$ restricts to the double suspension of the unit of $E$, $\Sigma^2 u_E \in E^2(\mathbb{C}P^1)$. Induction on the skeleta shows that for all $n$, $E_* (\mathbb{C}P^n)$ is free over $E^*$ and we obtain that

$$E_* (\mathbb{C}P^n) \cong E_* \{ \beta_0, \beta_1, \ldots \}$$

with $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ being dual to $u^i_E$. Let $\varphi: M\xi \to H \wedge M\xi$ be the map induced by the unit of $H$, and let $\Theta: M\xi^* (\mathbb{C}P^\infty) \to (H \wedge M\xi)^* (\mathbb{C}P^\infty)$ be the induced ring homomorphism (in fact $\Theta$ is a monomorphism as explained below). Then

\begin{equation}
\Theta(x_\xi) = \sum_{i \geq 0} z_i x_\xi^{i+1} = z(x_H),
\end{equation}

where $x_H \in (H \wedge M\xi)^2 (\mathbb{C}P^\infty)$ is the orientation coming from the canonical generator of $H^2(\mathbb{C}P^\infty)$. The proof is analogous to that for $MU$ in [Ad]. Note that $H \wedge M\xi$ is an algebra spectrum over the commutative $S$-algebra $H$ which acts centrally on $H \wedge M\xi$. Hence $x_H$ is a central element of $(H \wedge M\xi)^* (\mathbb{C}P^\infty)$. This contrasts with the image of $x_\xi$ in $(H \wedge M\xi)^* (\mathbb{C}P^\infty)$ which does not commute with all elements of $(H \wedge M\xi)_*$. We remark that the cohomology ring $M\xi^* (\mathbb{C}P^\infty)$ is highly non-commutative. Using [24], and noting that coefficient $z_i \in H_{2i}(M\xi)$ is an indecomposable in the algebra $H_*(M\xi)$, it follows that $x_\xi$ does not commute with any of the $z_i$. For example, the first non-trivial term in the commutator

$$z_1 z(x_H) - z(x_H) z_1$$

is $(z_1 z_2 - z_2 z_1)x_H^3 \neq 0$.

Let $\text{NSymm}$ denote the ring of non-symmetric functions. This ring can be identified with $H_*(\Omega \Sigma \mathbb{C}P^\infty)$. Using this and the above orientation we obtain
Proposition 2.3. The map $\Theta$ induces a monomorphism

$$\Theta : M\xi^*(\mathbb{C}P^\infty) \to \text{NSym}[[x_H]].$$

Proof. The right-hand side is isomorphic to the $H \wedge M\xi$-cohomology of $\mathbb{C}P^\infty$. As $M\xi$ is a wedge of suspensions of $BP$ at every prime and as the map is also rationally injective, we obtain the injectivity of $\Theta$. \qed

Note that for any $\lambda \in M\xi_*$ we can express $\Theta(x\xi\lambda)$ in the form

$$\sum_{i \geq 0} z_i x^{i+1} \lambda = \sum_{i \geq 0} z_i \lambda x^{i+1},$$

but as the coefficients are non-commutative, we cannot pass $\lambda$ to the left-hand side, so care has to be taken when calculating in $\text{NSym}[[x_H]]$.

3. A formal group law over $M\xi_*$

The two evident line bundles $\eta_1, \eta_2$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ can be tensored together to give a line bundle $\eta_1 \otimes \eta_2$ classified by a map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ and by naturality we obtain an element $\mu^* x_\xi \in M\xi^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. We also have

$$(3.1) \quad M\xi^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \left\{ \sum_{i,j \geq 0} a_{i,j}(x'_\xi)^i(x'''_\xi)^j : a_{i,j} \in M\xi_* \right\}$$

as a left $M\xi^*$ module, where $x'_\xi, x'''_\xi \in M\xi^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ are obtained by pulling back $x_\xi$ along the two projections. We have

$$\mu^* x_\xi = F_\xi(x'_\xi, x'''_\xi) = x'_\xi + x'''_\xi + \sum_{i,j \geq 1} a_{i,j}(x'_\xi)^i(x'''_\xi)^j,$$

where $a_{i,j} \in M\xi_{2(i+j)-2}$. The notation $F_\xi(x'_\xi, x'''_\xi)$ is meant to suggest a power series, but care needs to be taken over the use of such notation. For example, since the tensor product of line bundles is associative up to isomorphism, the formula

$$(3.2a) \quad F_\xi(F_\xi(x'_\xi, x'''_\xi), x'''_\xi) = F_\xi(x'_\xi, F_\xi(x'''_\xi, x'''_\xi))$$

holds in $M\xi^* (\mathbb{C}P^\infty \times \mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, where $x'_\xi, x'''_\xi, x''''_\xi$ denote the pullbacks of $x_\xi$ along the three projections. When considering this formula, we have to bear in mind that the inserted expressions do not commute with each other or coefficients. We also have the identities

$$(3.2b) \quad F_\xi(0, x_\xi) = x_\xi = F_\xi(0, x_\xi),$$

$$(3.2c) \quad F_\xi(x'_\xi, x'''_\xi) = F_\xi(x''''_\xi, x''''_\xi).$$

Let $\bar{x}_\xi = \gamma^* x_\xi$ denotes the pullback of $x_\xi$ along the map $\gamma : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the inverse $\eta^{-1} = \bar{\eta}$ of the canonical line bundle $\eta$. Then $\bar{x}_\xi \in M\xi^2 (\mathbb{C}P^\infty)$ and there is a unique expansion

$$\bar{x}_\xi = -x_\xi + \sum_{k \geq 1} c_k x_\xi^{k+1}$$

with $c_k \in M\xi_{2k}$. Since $\eta \otimes \bar{\eta}$ is trivial, this gives the identities

$$F_\xi(x_\xi, \bar{x}_\xi) = 0 = F_\xi(\bar{x}_\xi, x_\xi)$$

and so

$$(3.2d) \quad F_\xi(x_\xi, -x_\xi + \sum_{k \geq 1} c_k x_\xi^{k+1}) = 0 = F_\xi(-x_\xi + \sum_{k \geq 1} c_k x_\xi^{k+1}, x_\xi).$$
To summarize, we obtain the following result.

**Proposition 3.1.** The identities (3.2) together show that $F_{\xi}(x',x'')$ defines a commutative formal group law over the non-commutative ring $M_{\xi}$.

**Remark 3.2.** Note however, that most of the classical structure theory for formal group laws over (graded) commutative rings does not carry over to the general non-commutative setting. For power series rings over associative rings where the variable commutes with the coefficients most of the theory works as usual. If the variable commutes with the coefficients up to a controlled deviation, then the ring of skew power series still behaves reasonably (see for example [D]), but our case is more general.

### 4. The splitting of $M_{\xi}$ into wedges of suspensions of $BP$

In [BR] we showed that there is a splitting of $M_{\xi}$ into a wedge of copies of suspensions of $BP$ locally at each prime $p$. In the case of $MU$ the inclusion of the bottom summand is given by a map of ring spectra $BP \to MU(p)$. However, for $M_{\xi}$ this is not the case.

**Proposition 4.1.** For each prime $p$, there is no map of ring spectra $BP \to M_{\xi}(p)$

**Proof.** We give the proof for an odd prime $p$, the case $p = 2$ is similar. We set $H^*(BP) = \mathbb{F}_p[t_1, t_2, \ldots]$ where $t_r \in H_{2p^r-2}(BP)$ and the $A_*$-coaction on these generators is given by

$$
\psi(t_n) = \sum_{k=0}^{n} \zeta_k \otimes t_{n-k},
$$

where $\zeta_r \in A_2p^r-2$ is the conjugate of the usual Milnor generator $\xi_r$ defined in [Ad]. The right action of the Steenrod algebra satisfies

$$
P^1_1 t_1 = -1, \quad P^1_2 t_2 = -t_1^p, \quad P^p_2 t_2 = 0.
$$

Assume that a map of ring spectra $u: BP \to M_{\xi(p)}$ exists. Then $P^1_* u_*(t_1) = u_*(-1) = -1$, hence $w := u_*(t_1) \neq 0$. Notice that

$$
P^1_* (w^{p+1}) = -w^p, \quad P^p_* (w^{p+1}) = -w.
$$

Also, $P^p_* u_*(t_2) = 0$. This shows that $u_*(t_2)$ cannot be equal to a non-zero multiple of $w^{p+1}$. Therefore it is not contained in the polynomial subalgebra of $H_*(M_{\xi(p)})$ generated by $w^{p+1}$ and thus it cannot commute with $w$. This shows that the image of $u_*$ is not a commutative subalgebra of $H_*(M_{\xi(p)})$ which contradicts the commutativity of $H_*(BP)$.

**Remark 4.2.** Note that Proposition 4.1 implies that there is no map of ring spectra from $MU$ to $M_{\xi}$, because if such a map existed, we could precompose it $p$-locally with the ring map $BP \to MU(p)$ to get a map of ring spectra $BP \to M_{\xi(p)}$.
5. The real and the quaternionic cases

Analogous to the complex case, the map $\mathbb{R}P^\infty = BO(1) \to BO$ gives rise to a loop map $\xi_2 : \Omega \mathbb{R}P^\infty \to BO$, and hence there is an associated map of associative $S$-algebras $M\xi_2 \to MO$ on the level of Thom spectra. There is a splitting of $MO$ into copies of suspensions of $HF_2$. In fact a stronger result holds.

**Proposition 5.1.** There is a map of $E_2$-spectra $HF_2 \to MO$.

**Proof.** The map $\alpha : S^1 \to BO$ that detects the generator of the fundamental group of $BO$ gives rise to a double-loop map

$$\Omega^2 \Sigma^2 S^1 = \Omega^2 S^3 \to BO.$$ 

As the Thom spectrum associated to $\Omega \mathbb{R}P^\infty$ is a model of $HF_2$ by [Mah] with an $E_2$-structure [LMSM], the claim follows.

Generalizing an argument by Hu-Kriz-May [HKM], Gilmour [G] showed that there is no map of commutative $S$-algebras $HF_2 \to MO$.

The $E_2$-structure on the map from Proposition 5.1 cannot be extended to $\xi_2$. On the space level,

$$H_n(\Omega \Sigma \mathbb{R}P^\infty; F_2) = H_n(\mathbb{R}P^\infty; F_2),$$

where $H_n(\mathbb{R}P^\infty; F_2)$ is generated by an element $x_n$.

**Proposition 5.2.** There is no map of ring spectra $HF_2 \to M\xi_2$.

**Proof.** Assume $\gamma : HF_2 \to M\xi_2$ were a map of ring spectra. We consider $\gamma_* : (HF_2)_* \to (HF_2)_*$, $M\xi_2$. Note that $(HF_2)_*, M\xi_2$ is the free associative $F_2$-algebra generated by $z_1, z_2, \ldots$ with $z_i$ in degree $i$ being the image of $x_i$ under the Thom-isomorphism.

Under the action of the Steenrod-algebra on $HF_2$-homology $Sq^1(z_1) = 1$ and hence $Sq^1(z_1^2) = z_2^2$, by the derivation property of $Sq^1$.

In the dual Steenrod algebra we have $Sq^1_*(\xi_1) = 1$ and $Sq^2_*(\xi_2) = \xi_1$ and $Sq^1_*(\xi_2) = 0$.

Combining these facts we obtain

\begin{equation}
Sq^1_*(\gamma_*(\xi_1)) = \gamma_*(Sq^1(\xi_1)) = \gamma_*(1) = 1,
\end{equation}

in particular $\gamma_*(\xi_1) \neq 0$ and thus $\gamma_*(\xi_1) = z_1$.

Similarly,

$$Sq^2_*(\gamma_*(\xi_2)) = \gamma_*(Sq^2(\xi_2)) = \gamma_*(\xi_1) = z_1 \neq 0.$$ 

The image of $\gamma_*$ generates a commutative sub-$F_2$-algebra of $(HF_2)_*, M\xi_2$. The only elements in $(HF_2)_*, M\xi_2$ that commute with $z_1$ are polynomials in $z_1$. Assume that $\gamma_*(\xi_2) = z_1^3$. Then

$$0 = \gamma_*(Sq^1_*(\xi_2) = Sq^1_*(z_1^3) = z_1^2 \neq 0,$$

which is impossible. Therefore, $\gamma_*(\xi_2)$ does not commute with $z_1$, so we get a contradiction.

Note that Proposition 5.2 implies that there is no loop map $\Omega S^3 \to \Omega \Sigma \mathbb{R}P^\infty$ that is compatible with the maps to $BO$ since such a map would induce a map of associative $S$-algebras $HF_2 \to M\xi_2$.

A quaternionic model of quasisymmetric functions is given by $H^*(\Omega \Sigma \mathbb{R}P^\infty)$. Here, the algebraic generators are concentrated in degrees that are divisible by 4.
The canonical map \( \mathbb{H}P^{\infty} = BSp(1) \to BSp \) induces a loop-map \( \xi_{\mathbb{H}} : \Omega \mathbb{H}P^{\infty} \to BSp \) and thus gives rise to a map of associative \( S \)-algebras on the level of Thom spectra \( M\xi_{\mathbb{H}} \to MSp \).

Of course, the spectrum \( MSp \) is not as well understood as \( MO \) and \( MU \). There is a commutative \( S \)-algebra structure on \( MSp \) [May, pp. 22, 76], but for instance the homotopy groups of \( MSp \) are not known in an explicit form.

### 6. Associative versus commutative orientations

We work with the second desuspension of the suspension spectrum of \( \mathbb{C}P^\infty \). Such spectra are inclusion prespectra [EKMM, X.4.1] and thus a map of \( S \)-modules from \( S = \Sigma^\infty S^0 \) to \( \Sigma^\infty -2 \mathbb{C}P^\infty := \Sigma^{-2}\Sigma^\infty \mathbb{C}P^\infty \) is given by a map from the zeroth space of the sphere spectrum to the zeroth space of \( \Sigma^\infty -2 \mathbb{C}P^\infty \) which in turn is a colimit, namely \( \text{colim}_{2 \in \mathbb{W}} \Omega^W \mathbb{C}P^W - \mathbb{R}^2 \mathbb{C}P^\infty \). As a map \( \varrho : S \to \Sigma^\infty -2 \mathbb{C}P^\infty \) we take the one that is induced by the inclusion \( S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^\infty \).

The commutative \( S \)-algebra \( S \wedge (N_0)_+ = S[N_0] \) has a canonical map \( S[N_0] \to S \) which is given by the fold map. We can model this via the map of monoids that sends the additive monoid \( (N_0, 0, +) \) to the monoid \( (0, 0, +) \); thus \( S[N_0] \to S \) is a map of commutative \( S \)-algebras.

We get a map \( S[N_0] \to A(\Sigma^\infty -2 \mathbb{C}P^\infty) \) by taking the following map on the \( n \)th copy of \( S \) in \( S[N_0] \). We can view \( S \) as \( S \wedge \{ \ast \}^+_n \) where \( \{ \ast \} \) is a one-point space. The \( n \)-fold space diagonal gives a map

\[
\delta_n : S = S \wedge \{ \ast \}^+_n \to S \wedge \{ \{ \ast, \ldots, \ast \}^+_n \text{n-fold product} \}
\]

which fixes an equivalence of \( S \) with \( S^\wedge n \). We compose this map with the \( n \)-fold smash product of the map \( \varrho : S \to \Sigma^\infty -2 \mathbb{C}P^\infty \). The maps

\[
\varrho^{\wedge n} \circ \delta_n : S \to (\Sigma^\infty -2 \mathbb{C}P^\infty)^{\wedge n} \to A(\Sigma^\infty -2 \mathbb{C}P^\infty)
\]

together give a map of \( S \)-algebras

\[
\tau : S[N_0] \to A(\Sigma^\infty -2 \mathbb{C}P^\infty).
\]

Note, however, that \( S[N_0] \) is not central in \( A(\Sigma^\infty -2 \mathbb{C}P^\infty) \). Thus the coequalizer

\[
S \wedge S[N_0] \to A(\Sigma^\infty -2 \mathbb{C}P^\infty)
\]

does not possess any obvious \( S \)-algebra structure. Furthermore, there is a natural map

\[
S \wedge S[N_0] \to A(\Sigma^\infty -2 \mathbb{C}P^\infty) \to M\xi,
\]

but this is not a weak equivalence since the \( \mathbb{H} \mathbb{Z} \)-homology of the left-hand side is the quotient by the left ideal generated by \( z_0 - 1 \) and thus it is bigger than \( \mathbb{H} \mathbb{Z}_* M\xi \) which is the quotient by the two-sided ideal generated by \( z_0 - 1 \).

In the commutative context the pushout of commutative \( S \)-algebras is given by the smash product. Hence there is a natural morphism of commutative \( S \)-algebras

\[
\overline{P}(\Sigma^\infty -2 \mathbb{C}P^\infty) = S \wedge \mathbb{P}(S) \mathbb{P}(\Sigma^\infty -2 \mathbb{C}P^\infty) \to MU,
\]
where \( \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty) \) is the pushout in the following diagram of commutative \( S \)-algebras

\[
\begin{array}{c}
P(S) \longrightarrow \ P(\Sigma^\infty \mathbb{C}P^\infty) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
S \longrightarrow \ \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty).
\end{array}
\]

Here, we use the identity map on \( S \) to induce the left-hand vertical map of commutative \( S \)-algebras and the inclusion of the bottom cell of \( \Sigma^\infty \mathbb{C}P^\infty \) to induce the top map which is a cofibration and therefore \( \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty) \) is cofibrant. However, the map \( \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty) \to MU \) is not a weak equivalence as the next result shows.

**Lemma 6.1.** The canonical map of commutative \( S \)-algebras

\[
\widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty) \to MU
\]

is an equivalence rationally, but not globally. Furthermore, there is a morphism of ring spectra

\[
MU \to \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty)
\]

which turns \( MU \) into a retract of \( \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty) \).

**Proof.** Let \( k \) be a field. The Künneth spectral sequence for the homotopy groups of

\[
Hk \wedge (\widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty)) \simeq Hk \wedge_{P_{Hk}(Hk)} P_{Hk}(\Sigma^2 Hk \wedge \mathbb{C}P^\infty)
\]

has \( E^2 \)-term

\[
E^{2}_{*,*} = \text{Tor}^{\pi_*}_{*}(P_{Hk}(Hk))(k, \pi_*(P_{Hk}(\Sigma^2 Hk \wedge \mathbb{C}P^\infty))).
\]

When \( k = \mathbb{Q} \), \( \pi_*(P_{H\mathbb{Q}}(H\mathbb{Q})) \) is a polynomial algebra on a zero-dimensional class \( x_0 \) and

\[
(6.1) \quad \pi_*(P_{H\mathbb{Q}}(\Sigma^2 H\mathbb{Q} \wedge \mathbb{C}P^\infty)) \cong \mathbb{Q}[x_0, x_1, \ldots],
\]

where \( |x_i| = 2i \). Thus

\[
\pi_*(H\mathbb{Q} \wedge (\widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty)) \cong \mathbb{Q}[x_1, x_2, \ldots] \cong H\mathbb{Q}_*(MU).
\]

However, when \( k = \mathbb{F}_p \) for a prime \( p \), the freeness of the commutative \( S \)-algebras \( P(S) \) and \( P(\Sigma^\infty \mathbb{C}P^\infty) \) implies that \( (H\mathbb{F}_p)_*(P(\Sigma^\infty \mathbb{C}P^\infty)) \) is a free \( (H\mathbb{F}_p)_*(P(S)) \)-module and thus the \( E^2 \)-term reduces to the tensor product in homological degree zero. Note that this tensor product contains elements of odd degree, but \( (H\mathbb{F}_p)_*(MU) \) doesn’t.

Using the orientation for line bundles given by the canonical inclusion

\[
\Sigma^\infty \mathbb{C}P^\infty \to \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty),
\]

we have a map of ring spectra

\[
\varphi: MU \to \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty).
\]

The inclusion map \( \mathbb{C}P^\infty = BU(1) \to BU \) gives rise to the canonical map \( \sigma: \Sigma^\infty \mathbb{C}P^\infty \to MU \) and with this orientation we get a morphism of commutative \( S \)-algebras

\[
\theta: \widetilde{P}(\Sigma^\infty \mathbb{C}P^\infty) \to MU,
\]

such that the composite \( \theta \circ \varphi \circ \sigma \) agrees with \( \sigma \), hence \( \theta \circ \varphi \) is homotopic to the identity on \( MU \). \( \Box \)
Using topological André-Quillen homology, \( TAQ_*(-) \), we can show that the map of ring spectra \( \phi \) cannot be rigidified to a map \( \tilde{\phi} \) of commutative \( S \)-algebras in such a way that the composite \( \theta \circ \tilde{\phi} \) is a weak-equivalence. By Basterra-Mandell [BM],

\[
TAQ_*(MU|\Sigma; HF_p) \cong (HF_p)_*(\Sigma^2 ku),
\]

while [BGR] proposition 1.6] together with subsequent work of the first named author [Ba] gives

\[
TAQ_*(\Sigma(\Sigma^\infty-2\mathbb{C}P^\infty)|\Sigma; HF_p) \cong (HF_p)_*(\Sigma^\infty-2\mathbb{C}P^\infty),
\]

where \( \mathbb{C}P^\infty = \mathbb{C}P^\infty / \mathbb{C}P^1 \) is the cofiber of the inclusion of the bottom cell.

**Proposition 6.2.** For a prime \( p \), there can be no morphism of commutative \( S(p) \)-algebras

\[
\theta(p) : MU(p) \rightarrow (\Sigma^\infty-2\mathbb{C}P^\infty)(p)
\]

for which \( \sigma(p) \circ \theta(p) \) is a weak equivalence. Hence there can be no morphism of commutative \( S \)-algebras

\[\theta: MU \rightarrow \Sigma^\infty-2\mathbb{C}P^\infty\]

for which \( \sigma \circ \theta \) is a weak equivalence.

**Proof.** It suffices to prove the first result, and we will assume that all spectra are localised at \( p \). Assume such a morphism \( \theta \) existed. Then by naturality of the functor of Kähler differentials, \( \Omega_S \), there are (derived) morphisms of \( MU \)-modules and a commutative diagram

\[
\Omega_S(MU) \xrightarrow{\theta_*} \Omega_S(\Sigma^\infty-2\mathbb{C}P^\infty) \xrightarrow{\sigma_*} \Omega_S(MU)
\]

which by [BM] induce a commutative diagram in \( TAQ_*(-; HF_p) \) of the following form:

\[
H_*(\Sigma^2 ku; F_p) \xrightarrow{\theta_*} H_*(\Sigma^\infty-2\mathbb{C}P^\infty; F_p) \xrightarrow{\sigma_*} H_*(\Sigma^2 ku; F_p).
\]

It is standard that

\[
H_*(\Sigma^\infty-2\mathbb{C}P^\infty; F_p) = \begin{cases} F_p & \text{if } n \geq 2 \text{ and is even}, \\ 0 & \text{otherwise}. \end{cases}
\]

On the other hand, when \( p = 2 \),

\[
H_*(ku; F_2) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \subseteq \Lambda(2),
\]

with \( |\zeta_s| = 2^s - 1 \), while when \( p \) is odd

\[
\Sigma^2 ku(p) \sim \bigvee_{1 \leq r \leq p-1} \Sigma^{2r} \ell,
\]

where \( \ell \) is the Adams summand with

\[
H_*(\ell; F_2) = \mathbb{F}_p[\zeta_1, \zeta_2, \zeta_3, \ldots] \otimes \Lambda(\bar{r}_r : r \geq 2),
\]

for \( |\zeta_r| = 2p^r - 2 \) and \( |\bar{r}_r| = 2p^r - 1 \). Hence no such \( \theta \) can exist. \( \square \)
Proposition 6.3. There are commutative $S$-algebras $E$ which possess a map of commutative $S$-algebras
\[ \tilde{P}(\Sigma^{-2} CP^\infty) \to E \]
that cannot be extended to a map of commutative $S$-algebras $MU \to E$.

Proof. Matthew Ando [An] constructed complex orientations for the Lubin-Tate spectra $E_n$ which are $H_\infty$-maps $MU \to E_n$. However, in [JN], Niles Johnson and Justin Noël showed that none of these are $p$-typical for all primes up to at least 13 (and subsequently verified for primes up to 61). For any $p$-typical orientation there is a map of ring spectra $MU \to E_n$, but this map cannot be an $H_\infty$-map and therefore is not a map of commutative $S$-algebras. □

References

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