A strictly commutative model for the cochain algebra of a space

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(joint work with Steffen Sagave)

For a simplicial set X and an arbitrary commutative ring k, the cochain algebra on X is given by the cochain complex $C^*(K;k)$ whose cochain module in degree n is $C^n(X;k) = \mathrm{Sets}(X_n,k)$. As the coboundary map one takes the alternating sum of the coface maps. The cup-product of two cochains is induced by restricting along front and back inclusions. In particular, the cup-product is not graded-commutative on cochain level, but it induces a graded commutative k-algebra structure on homology. The cup-i-products witness the non-commutativity on cochain level, for instance

$$\delta(f \cup_1 g) = f \cup g - (-1)^{|f||g|} g \cup f,$$

so we get a homotopy that compares $f \cup g$ and $(-1)^{|f||g|} g \cup f$. The full structure is rather involved: $C^*(X;k)$ is an E_∞-algebra, i. e., its multiplication is commutative up to all higher homotopies. Mike Mandell showed that for nilpotent spaces of finite type the E_∞-algebra of the integral cochains determines the homotopy type [2].

Over the rationals the situation is drastically different. The Sullivan algebra of polynomial differential forms, $A^*_{PL}(X)$, is a differential graded commutative model of $C^*(X;\mathbb{Q})$ and rational spaces can be classified using rational commutative differential graded algebras.

Our project replaces the E_∞-algebra of cochains over an arbitrary commutative ring k by a suitable commutative monoid. We know that there exists such a model because in joint work with Brooke Shipley [4] we proved that there is a chain of Quillen equivalence between E_∞-algebras over k and commutative monoids in the category of I-chain complexes.

Here, I is the category of finite sets and injections with objects $\underline{n} = \{1,\ldots,n\}$ for $n \geq 0$ with $\underline{0} = \varnothing$. Morphisms are injective functions. Note that $\underline{0}$ is an initial object in I and that $I(\underline{n},\underline{n})$ is the symmetric group Σ_n. In addition, I is a permutative category via $\underline{n} \oplus \underline{m} = \underline{n+m}$. An I-chain complex is a functor from the category I to the category Ch of unbounded chain complexes over k and the corresponding category Ch^I has natural transformation as morphisms. An I-chain complex X can be viewed as a coaugmented cosimplicial chain complex with additional symmetries.

For every object $\underline{m}$ of I there is an evaluation functor that sends an I-chain complex X to the chain complex $X(\underline{m})$. This functor has a left adjoint $F^I_m \colon \mathrm{Ch} \to \mathrm{Ch}^I$ with

$$F^I_m(C_*)(\underline{n}) = \bigoplus_{I(\underline{m},\underline{n})} C_*$$

for $C_* \in \mathrm{Ch}$.

The category Ch^I is symmetric monoidal via the Day convolution product, so for X, Y in Ch^I we obtain a product $X \boxtimes Y$ in Ch^I. The unit for this product is

$U^I := F_0^I(S^0)$ where S^0 is the chain complex whose only non-trivial chain module is k in degree zero. We denote by $C(\mathrm{Ch}^I)$ the category of commutative monoids in Ch^I and call its object commutative I-chain complexes.

For I-chain complexes there is a Bousfield-Kan type model of the homotopy colimit: if X is an I-chain complex, then there is a chain complex, $\mathrm{hocolim}_I X$, that is the total complex associated to a simplicial chain complex built out of the nerve of I and the values of X. We show that for every X in $C(\mathrm{Ch}^I)$ the chain complex $\mathrm{hocolim}_I X$ is an E_∞-algebra over k. The proof uses an action of the Barratt-Eccles operad on the nerve of the category I – a fact that was established by Peter May in the 80's.

We construct an I-version of the polynomial forms, $A^I(X)$, for every simplicial set X as an object in $C(\mathrm{Ch}^I)$, by defining

$$A^I_\bullet = B(U_0^I, C(F_1^I(D^0)), U_1^I)$$

as a two-sided bar construction. Here D^0 is the disc complex concentrated in degrees 0 and -1 with value k and with the identity map as the only non-trivial differential and $C(F_1^I(D^0))$ denotes the free commutative I-chain complex generated by $F_1^I(D^0)$. It acts on $U_0^I = U^I$ via the augmentation to zero and on $U_1^I = U^I$ via the augmentation to 1. Placing D^0 in I-level 1 turns $A^I(\underline{n})_{\bullet,q}$ into a contractible simplicial k-module for all $n > 1$ and all chain degrees q [3]. Note that $A^I_\bullet$ is a simplicial object in commutative I-chain complexes.

We then define

$$A^I(X) := \mathrm{sSets}(X, A^I_\bullet).$$

For every commutative ring k and for every simplicial set X this is a commutative I-chain complex.

We show that $\mathrm{hocolim}_I A^I(X)$ is weakly equivalent to Sullivan's $A_{PL}(X)$ if k is a field of characteristic zero and we prove that for every commutative ring k $X \mapsto \mathrm{hocolim}_I A^I(X)$ is a cochain theory in the sense of Mike Mandell [1]. As a corollary we obtain that $\mathrm{hocolim}_I A^I(X)$ is weakly equivalent to $C^*(X;k)$ as an E_∞-algebra [3].

References

[1] M. A. Mandell, *Cochain multiplications*, Amer. J. Math. **124** (2002), 547–566.

[2] M. A. Mandell, *Cochains and homotopy type*, Publ. Math. Inst. Hautes Études Sci. No. **103** (2006), 213–246.

[3] B. Richter, S. Sagave, *A strictly commutative model for the cochain algebra of a space*, arXiv:1801.01060.

[4] B. Richter, B. Shipley, *An algebraic model for commutative $H\mathbb{Z}$-algebras*, Algebr. Geom. Topol. **17** (2017), 2013–2038.