CANONICAL E_∞-OPERADS INVOLVED IN HOMOTOPY COLIMITS OF $\mathcal{I}$-CHAIN COMPLEXES

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Work starting in the late 1960s of Quillen, Sullivan, Bousfield-Gugenheim, Neisendorfer and others enables us to study rational nilpotent spaces of finite type via algebraic models. For instance Quillen developed models in the categories of differential graded cocommutative coalgebras and Lie algebras and Sullivan's differential graded commutative model of the rational cochains on a space allowed for the important concept of minimal models in rational homotopy theory.

Mandell proved in 2006 that two finite type nilpotent spaces are weakly equivalent if and only if their integral singular cochains are quasi-isomorphic as E_∞-algebras. Thus, if you don't want to restrict to *rational* homotopy theory, then you need the full information of the E_∞-structure on the cochains and this is quite an intricate structure.

One can ask whether one can replace the E_∞-algebra of cochains $C^*(X;k)$ on a space X by a strictly commutative model, if k is any commutative ring. Of course this cannot be done in the context of differential graded commutative algebras, because the Steenrod operations for $k = \mathbb{F}_p$ witness that this isn't possible. The existence as a commutative $\mathcal{I}$-chain algebra is guaranteed by [3]. Here, $\mathcal{I}$ is the skeleton of the category of finite sets and injections. In [2] we develop an explicit model $A^{\mathcal{I}}_*(X;k)$ that generalizes Sullivan's model to arbitrary commutative rings k and that detects the homotopy type of nilpotent spaces of finite type.

Can we use $\mathcal{I}$-chains to obtain models of spaces in the setting of differential graded cocommutative coalgebras and Lie-algebras? An obstacle is that the homotopy colimit, that allows us to pass from $\mathcal{I}$-chain complexes to ordinary chain complexes is only lax monoidal, but *not* lax symmetric monoidal or lax symmetric comonoidal. In fact we prove in [2] (modifying a construction from [4, Proposition 6.5] for spaces) that the homotopy colimit sends commutative $\mathcal{I}$-chain algebras to algebras over the Barratt-Eccles operad. So this is one canonical E_∞-operad occurring in this setting.

There is an inclusion of categories $i\colon \Sigma \subset \mathcal{I}$, where Σ is the skeleton of the category of finite sets and bijections. This inclusion and the left Kan extension of symmetric sequences along i already features prominently in the work of Church-Ellenberg-Farb [1]

If Z_* is a symmetric sequence in chain complexes, then the left Kan extension can be explitly described as

$$i_!(Z_*)(\mathbf{m}) = \mathrm{colim}_{i(\mathbf{n})\downarrow \mathbf{m}} Z_*(\mathbf{n}) \cong \bigoplus_{n\geq 0} k\{\mathcal{I}(\mathbf{n},\mathbf{m})\} \otimes_{k[\Sigma_n]} Z_*(\mathbf{n}).$$

We use a canonical operad in the category of small category as the means to describe the homotopy colimits of $\mathcal{I}$-chains of the form $i_!(Z_*)$. The mth arity of the operad is the category of objects under $\mathbf{m}$, $C(m) := \mathbf{m} \downarrow \mathcal{I}$, with $\mathbf{m} = \{1,\dots,m\}$. We can use the nerve functor, the free module functor and the associated chain complex functor to produce an operad O in chain complexes with $O(m) = C_*(k\{N(\mathbf{m}\downarrow \mathcal{I})\})$. This operad is an E_∞-operad in the category of chain

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complexes. Note that if one restricts to bijections, then this corresponds to the Barratt-Eccles operad.

We show that for any symmetric sequence in chain complexes Z_* one has

$$\mathsf{hocolim}_{\mathcal{I}} i_! Z_* \cong \bigoplus_{m\geq 0} O(m) \otimes_{\Sigma_m} Z_*(\mathbf{m}).$$

This yields the main result:

Theorem: For all chain complexes C_* and all operads $(P(m))_{m\geq 0}$ in the category of modules $\mathsf{hocolim}_{\mathcal{I}} i_!(P(F_1^{\Sigma}(C_*))$ is the free $O \otimes P$-algebra generated by C_*.

Here, $F_1^{\Sigma}(C_*)$ denotes the free symmetric sequence on C_* at the object $\mathbf{1} = \{1\}$. In particular, for $P = \mathrm{Lie}$ we get that $\mathsf{hocolim}_{\mathcal{I}} i_!(\mathrm{Lie}(F_1^{\Sigma}(C_*))$ is a free $O \otimes \mathrm{Lie}$-algebra generated by C_*.

For cocommutative comonoids we obtain:

Theorem: If Z_* is a cocommutative comonoid in symmetric sequences of chain complexes, then $i_!(Z_*)$ is a cocommutative monoid in $\mathcal{I}$-chain complexes and $\mathsf{hocolim}_{\mathcal{I}} i_!(Z_*)$ is an E_∞ differential graded coalgebra.

For this structure we use a deconcatenation product on the operad O and this in turn relies on the Alexander-Whitney map.

References

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