Canonical E_∞-operads involved in homotopy colimits of $\mathcal{I}$-chain complexes

Birgit Richter

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Thus, if you don't want to restrict to rational homotopy theory, then you need the full information of the E_∞-structure on the cochains!

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Here, $\mathcal{I}$ is the (skeleton) of the category of finite sets and injective functions.

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What about the other models? So what about differential graded cocommutative coalgebras and Lie-algebras?

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3. A canonical E_∞-operad detecting structure on the homotopy colimit

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The Day convolution product gives $\mathsf{Ch}^{\mathcal{I}}$ a symmetric monoidal structure. Explicitly, for two $\mathcal{I}$-chain complexes X_*, Y_*

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Definition: Commutative $\mathcal{I}$-chain algebras are commutative monoids in $\mathsf{Ch}^{\mathcal{I}}$.

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For any $\mathcal{I}$-chain complex X_*, the free commutative $\mathcal{I}$-chain algebra on X_* is

$$\mathrm{S}^{\mathcal{I}}(X_*) = \bigoplus_{n \geq 0} X_*^{\boxtimes n}/\Sigma_n.$$

The homotopy colimit, $\text{hocolim}_{\mathcal{I}} X_*$, of an $\mathcal{I}$-chain complex X_* is the total complex associated to the bicomplex whose bidegree (p, q)-part is

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If L_* is a Lie-algebra in $\mathrm{Ch}^{\mathcal{I}}$, what can we say about $\text{hocolim}_{\mathcal{I}} L_*$?
In general: not much, because hocolim_I is lax monoidal, but not lax symmetric monoidal!

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If $Z_* \in \mathsf{Ch}^\Sigma$, then

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In this case:
Lemma: For all $n \geq 0$ and all chain complexes C_*:

$$\mathrm{hocolim}_{\mathcal{I}} F_n^{\mathcal{I}}(C_*) \simeq C_*.$$

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on objects as

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The identity $1 \in C(1)$ is then defined to be id_1. □

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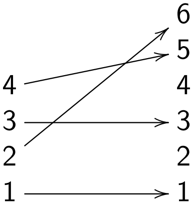

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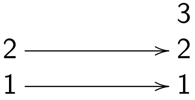

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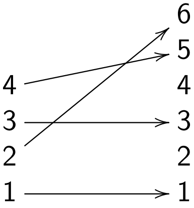

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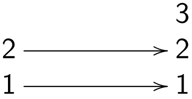
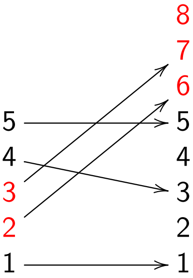


Then $f \circ_2 g \colon \mathcal{I}(5,8)$ is the injection



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Sketch of Proof:
Note that by definition we get $\mathrm{hocolim}_{\mathcal{I}} i_!(C_*)_{p,q} = \bigoplus_{[f_q|\ldots|f_1]\in N\mathcal{I}_q} i_!(C_p)(sf_1) \cong \bigoplus_{[f_q|\ldots|f_1]\in N\mathcal{I}_q} k\{\mathcal{I}(i(-), sf_1)\} \otimes_{\Sigma} C_p$.

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This is isomorphic to

$$\bigoplus_{m \geq 0} k\{N(i(\mathrm{m}) \downarrow \mathcal{I})_q\} \otimes_{\Sigma_m} C_p(\mathrm{m}).$$

□

This now yields:

Theorem:

- For all chain complexes C_*:

$$\mathrm{hocolim}_{\mathcal{I}} \mathsf{S}^{\mathcal{I}}(F_1^{\mathcal{I}}(C_*)) \cong \bigoplus_{m \geq 0} O(m) \otimes_{\Sigma_m} C_*^{\otimes m}.$$

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In particular, $\mathrm{hocolim}_I i_!(\mathrm{Lie}(F_1^{\Sigma}(C_*))$ is a free $O \otimes \mathrm{Lie}$-algebra generated by C_*.

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Here, the structure maps use the diagonal on X_* and the maps from above. As the $\psi_{p,q}$'s use the Alexander-Whitney maps, this coproduct is E_∞-comonoidal.