(CO-)HOMOLOGY THEORIES FOR COMMUTATIVE (S-)ALGEBRAS

MARIA BASTERRA AND BIRGIT RICHTER

The aim of this paper is to give an overview of some of the existing homology theories for commutative (S-)algebras. We do not claim any originality; nor do we pretend to give a complete account. But the results in that field are widely spread in the literature, so for someone who does not actually work in that subject, it can be difficult to trace all the relationships between the different homology theories. The theories we aim to compare are

- topological André-Quillen homology
- Gamma homology
- stable homotopy of Γ-modules
- stable homotopy of algebraic theories
- the André-Quillen cohomology groups which arise as obstruction groups in the Goerss-Hopkins approach

As a comparison between stable homotopy of Γ-modules and stable homotopy of algebraic theories is not explicitly given in the literature, we will give a proof of Theorem 2.1 which says that both homotopy theories are isomorphic when they are applied to augmented commutative algebras. This result is well-known to experts.

The comparison results provided by Mike Mandell [M] and Basterra-McCarthy [B-McC] can be cobbled together to prove that Gamma cohomology and the André-Quillen cohomology groups in the Goerss-Hopkins approach coincide.

Acknowledgements We are grateful for help we got from Paul Goerss, Mike Mandell and Stefan Schwede: Paul insisted on the isomorphism which is now the subject of Theorem 2.6, Stefan initiated the whole project and the study of the relationship stated in Theorem 2.1; the proof of the isomorphism came out of a discussion with him. And last but not least, Mike patiently answered all our questions concerning André-Quillen cohomology of E_∞ algebras.

1. Different cohomology theories for commutative (S-)algebras

We will briefly describe the definitions of the above mentioned homology theories, their range and the domain, on which they coincide. Here range means, that some of them are defined for genuine commutative algebras whereas others are homology theories for commutative S-algebras à la [EKMM]. These reduce to homology theories for algebras by considering Eilenberg-MacLane spectra of commutative rings.

1.1. **André-Quillen homology, AQ.** To start with, we should mention the algebraic predecessor of these theories, namely André-Quillen homology of commutative algebras. The standard references for this homology theory are [A], [Q2, Q1], and [W]. For a pointed model category, Quillen defined the notion of homology of objects: he considers the subcategory of abelian objects in that model category. If the inclusion of this subcategory in the whole category has a left adjoint – called abelianization– then the homology of an object is the left derived functor of abelianization. That is, one takes an object, considers a cofibrant resolution and applies the abelianization functor to that resolution.

Let k be a commutative ring with unit. For a commutative (simplicial) k-algebra A this means to take a free simplicial resolution $P_* \to A$, to apply the module of Kähler differentials to P_*, and then define André-Quillen homology of A with respect to the ground ring k and coefficients in an A-module M to be

$$\mathsf{AQ}_*(A|k;M) := \pi_*(\Omega^1_{P_*|k} \otimes_{P_*} M).$$

The module $\Omega^1_{P_*|k}$ is called the *cotangent complex* of A over k and is denoted by $\mathbf{L}_{A|k}$; this is well-defined, because the homotopy groups of $\Omega^1_{P_*|k}$ do not depend on the resolution.

For A as above let I denote the kernel of the multiplication map $I := \ker(A \otimes_k A \to A)$. Then the module of Kähler differentials has an alternative description: the ideal I has an induced multiplication and the Kähler differentials $\Omega^1_{A|k}$ are isomorphic to I/I^2. The quotient I/I^2 is the module of indecomposables in I and is often denoted by $Q(I)$.

André-Quillen homology vanishes in positive degrees for smooth algebras: if A is smooth over k, then $\mathsf{AQ}_*(A|k;M) = 0$ for all $* > 0$ and $\mathsf{AQ}_0(A|k;M) \cong \Omega^1_{A|k} \otimes_A M$. In particular, if A is étale over k, then André-Quillen homology vanishes in all degrees.

Let $\Lambda^q(V)$ denote the q-th exterior power on a module V. Quillen [Q2, 8.1] constructs a spectral sequence

$$E^2_{p,q} = H_p(\Lambda^q \mathbf{L}_{A|k}) \Longrightarrow \mathrm{Tor}^{A\otimes_k A}_{p+q}(A,A) \tag{1.1}$$

which starts with André-Quillen homology and its higher versions $H_*(\Lambda^q \mathbf{L}_{A|k})$ and converges to Hochschild homology for commutative algebras A which are k-flat.

The properties, which make André-Quillen homology actually a homology theory, are a transitivity long exact sequence, i.e., for a triple of algebras $A \to B \to C$ the following sequence is long exact:

$$\begin{aligned} \cdots \to\ & \mathsf{AQ}_n(B|A;M) \to \mathsf{AQ}_n(C|A;M) \to \mathsf{AQ}_n(C|B;M) \\ \to\ & \mathsf{AQ}_{n-1}(B|A;M) \to \cdots \end{aligned}$$

In addition, there is a flat-base change property: if A and B are two commutative k-algebras and B is k-flat, then André-Quillen homology does not see the difference between $A \otimes_k B$ relative to B and A relative to k

$$\mathsf{AQ}_*(A \otimes_k B|B; M) \cong \mathsf{AQ}_*(A|k; M)$$

for all $A \otimes_k B$-modules M. Similarly, if $\mathsf{Tor}_*^k(A, B) = 0$ for $* > 0$, then André-Quillen homology of $A \otimes_k B$ relative to k splits as

$$\mathsf{AQ}_*(A \otimes_k B|k; M) \cong \mathsf{AQ}_*(A|k; M) \oplus \mathsf{AQ}_*(B|k; M).$$

The zeroth André-Quillen homology gives the module of Kähler differentials; the zeroth cohomology is therefore the module of derivations. The first André-Quillen cohomology of A with coefficients in M classifies 'infinitesimal extensions'. To be more precise, $\mathsf{AQ}^1(A|k; M)$ classifies surjections of k-algebras $\pi : E \twoheadrightarrow A$ such that the kernel of π is isomorphic to M as an A-module. Here M is considered as a trivial algebra $M^2 = 0$, and the kernel of π gets its A-module structure from the inclusion into E, i.e., $\pi(e)m = em$ for $e \in E$ and m in the kernel of π.

1.2. **Topological André-Quillen homology,** TAQ**.** Several authors (Waldhausen, McClure and Hunter, Kriz, and Robinson among others) initiated the study of a corresponding theory in the category of E_∞ ring spectra before the necessary foundations where in place. The construction of the category of commutative S-algebras in [EKMM], a model category equivalent to the category of E_∞ ring spectra, allowed M. Basterra to mimic the construction of Kähler differentials and define topological André-Quillen (co)-homology. We give a brief account of this theory in the following section. For a more extensive description see [La] or the original account [B].

Let A be a commutative S-algebra and let B be an A-algebra over A. Then one can build the pullback in the category of A-modules

$$\begin{array}{ccc} I_A(B) & \longrightarrow & * \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

and $I_A(B)$ is called the *augmentation ideal of* B. Similarly, for a non-unital A-algebra C (like $I_A(B)$), the multiplication allows us to construct the pushout in the category of A-modules

$$\begin{array}{ccc} C \wedge_A C & \longrightarrow & * \\ \downarrow & & \downarrow \\ C & \longrightarrow & Q(C). \end{array}$$

We call the outcome $Q(C)$ the *module of indecomposables.*

For every commutative A-algebra B, the smash product $B \wedge_A B$ is naturally augmented over B via the multiplication map. The *topological André-Quillen homology of B with respect to A* is defined to be

$$\mathsf{TAQ}(B|A) := (LQ)(RI_B)(B \wedge_A^L B).$$

Here L stands for left and R for right derived functor.

For any B-module M, the topological André-Quillen homology groups of B with coefficients in M are the homotopy groups of the TAQ spectrum

$$\mathsf{TAQ}_*(B|A;M) := \pi_*(\mathsf{TAQ}(B|A) \wedge_B M).$$

Topological André-Quillen homology has properties similar to algebraic André-Quillen homology. For any triple of cofibrant commutative S-algebras we have a transitivity long exact sequence and there is a 'cofibrant base change' property [B, 4.2,4.3 & 4.6].

For a connective commutative S-algebra A, it is shown in [B, Theorem 8.1(Kriz)] that the usual Postnikov tower of A can be refined to a Postnikov towers consisting of commutative S-algebras, such that the k-invariants live in topological André-Quillen cohomology. This result can be used for an obstruction theory for commutative S-algebra structures: assume for a connective S-module A that there is a commutative S-algebra structure on the n-th Postnikov stage. Then this structure can be lifted to a commutative S-algebra structure on the $(n+1)$-st stage, if the k-invariant for that stage can be lifted to a k-invariant in topological André-Quillen cohomology.

Recall that given an E_∞ space X, the suspension spectrum of X_+, the space obtained by adjoining a disjoint point to X, is an S-module with an E_∞-ring structure coming from the H-space structure. Hence, $S \wedge (X_+) = \Sigma^\infty(X_+)$ is a commutative S-algebra. In work in progress, M. Basterra and M. Mandell show that its cotangent complex is equivalent to $\underline{X}$, the S-module associated to the spectrum obtained from X using an *infinite loop space machine* (see [EKMM, VII.3]).

More generally, given an augmented commutative A-algebra B there is a reduced version of TAQ with $\widetilde{\mathsf{TAQ}}(B|A) = (LQ_A)(RI_A)(B)$. Then, for an E_∞ space X,

$$\widetilde{\mathsf{TAQ}}(A \wedge X_+|A) \cong A \wedge \underline{X}.$$

The authors use this fact and the weak equivalence of E_∞-ring spectra $MU \wedge MU \to MU \wedge BU_+$ provided by the Thom isomorphism to calculate:

$$\mathsf{TAQ}(MU|S) \cong \widetilde{\mathsf{TAQ}}(MU \wedge BU_+|MU) \cong MU \wedge bu$$

i.e., the cotangent complex of MU, the complex cobordism S-algebra, is the connective complex K-theory module.

In [Mi], Minasian constructed a spectral sequence similar to the spectral sequence (1.1) in the algebraic setting. It is of the form

$$E_1^{s,t} \cong \pi_{t-s}\left(\left(\bigwedge_{i=1}^{s-1} \Sigma\mathsf{TAQ}(A|S)\right)_{h\Sigma_{s-1}}\right) \quad \text{for} \quad t \geq s \geq 0.$$

This spectral sequence converges to the reduced topological Hochschild homology of A. With the help of this spectral sequence, Minasian could prove [Mi, Corollary 2.8] that for a connective cofibrant S-algebra A topological André-Quillen homology vanishes, if and only if the reduced topological Hochschild homology of A is trivial. Here, it is crucial to assume, that A is connective (compare with the discussion at the end of the paper).

In [McC-Mi] McCarthy and Minasian developed a notion of TAQ-*smooth* and THH-*smooth* commutative S-algebras and they could prove an analogue of the Hochschild-Kostant-Rosenberg theorem for usual Hochschild-homology, which states that Hochschild homology of smooth algebras is isomorphic to the exterior powers of the modules of Kähler differentials.

The counterpart in the context of S-algebras of this theorem [McC-Mi, Theorem 6.1] says, that for a THH-smooth R-algebra A in the category of connective S-algebras, there is a natural equivalence of A-algebras

$$\mathbf{P}_A(\Sigma\mathsf{TAQ}(A|R)) \simeq \mathsf{THH}(A|R)$$

between the free commutative A-algebra on $\Sigma\mathsf{TAQ}(A|R)$ and topological Hochschild homology of A.

With the help of TAQ, one can distinguish certain classes of commutative S-algebras. For instance, the algebraic notion of étaleness can be transferred to étaleness for commutative S-algebras. John Rognes, Randy McCarthy and others use the notion of TAQ-étale maps of S-algebras – maps $A \to B$ of S-algebras with the property that $\mathsf{TAQ}(B|A;B) \sim *$ – andTHH-étale maps of S-algebras – maps, such that the reduced topological Hochschild homology $\widetilde{\mathsf{THH}}(B|A;B)$ is trivial – to transfer statements of classical algebra to the theory of commutative spectra. In particular, Rognes applies these and other notions in his work on Galois theory of commutative S-algebras.

1.3. **Gamma homology,** $\mathbf{H\Gamma}$. In the mid 90's, Alan Robinson and Sarah Whitehouse developed a homology theory for E_∞ algebras, called Gamma homology ($\mathrm{H\Gamma}$). A published account of this work is [Ro-Wh]. The general definition of Gamma homology is quite involved: they construct an analog of the cotangent complex in the case of E_∞-algebras: if A is a k-algebra over some E_∞ operad $\mathcal{C}$ and M is an A-module, then the *realization* of these data is defined as the cofibre $\mathcal{K}(A|k;M)$ of $|\mathcal{M}(A|k;M)|' \to \mathcal{M}_2$ where $|\mathcal{M}(A|k;M)|'$ is a quotient of $\bigoplus_{n\geq 2}\mathcal{C}_{n+1}\otimes_{\Sigma_n}A^{\otimes n}\otimes M$; $|\mathcal{M}|'$ has a natural filtration by taking the k-th filtration to be everything that is the quotient of $\bigoplus_{2\leq n\leq k}\mathcal{C}_{n+1}\otimes A^{\otimes n}\otimes$

M under the action of the symmetric groups and the other identifications defined in [Ro-Wh, 2.8 (1),(2)]. The part $\mathcal{M}_2$ is the bottom filtration piece.

For an E_∞ subalgebra A of B the cotangent complex is defined to be the cofibre of

$$\mathcal{K}(A|k;M) \to \mathcal{K}(B|k;M)$$

and Gamma homology is the homology of the cotangent complex. They provide a transitivity long exact sequence [Ro-Wh, 3.4] for a triple of inclusions of E_∞ algebras $A \hookrightarrow B \hookrightarrow C$ and there is also a variant of Gamma homology for cyclic E_∞-algebras [Ro-Wh, 2.9]. In the special example of commutative algebras (viewed as E_∞ algebras) there are several concrete chain complex models for Gamma homology. Sarah Whitehouse gave one model in her thesis [Wh], and Alan Robinson uses a quasi-isomorphic one in [Ro1]. We will briefly give the description of the latter (compare [Ro1, 2.5]).

For a commutative k-algebra A and an A-module M the complex for Gamma homology, $C\Gamma$ is the total complex of a bicomplex $\Xi_{*,*}$, which in bidegree (p,q) consists of

$$\Xi_{p,q} = \mathrm{Lie}^*_{q+1} \otimes k[\Sigma_{q+1}]^{\otimes p} \otimes M \otimes A^{\otimes q+1}.$$

Here, all tensor products are taken with respect to the ground ring k. The k-module Lie^*_n is the dual of the n-th part of the operad for Lie-algebras, i.e., Lie_n (without the dualization) is the free k-module on all Lie words on n generators $x_1,\ldots,x_n$ which contain each x_i exactly once; this is a left-Σ_n-module, where the action of $\sigma \in \Sigma_n$ on a word of length n is given by the sign-action and the permutation of the variables $x_1,\ldots,x_n$.

The horizontal differential is just the differential in the two-sided bar construction of the symmetric group, using the right action of Σ_{q+1} on Lie^*_{q+1} and the left-action on $M \otimes A^{\otimes q+1}$ by permuting the tensor factors in $A^{\otimes q+1}$. The vertical differential uses an action of certain standard surjection on Lie^*_{q+1}. For the precise definition see [Ro1, 2.2–2.5]. In order to get a homotopy invariant definition one should either insist that the algebra A is k-flat or assume that A is replaced by a simplicial flat resolution and the complex $\Xi_{*,*}$ is applied to that.

In the case of commutative algebras, Gamma homology vanishes on étale extensions. There is a transitivity long exact sequence for a triple $A \to B \to C$ of algebras and there is a flat-base change theorem. Gamma homology agrees with André-Quillen homology for algebras over the rational numbers and in general, Gamma homology in degree zero gives the first Hochschild homology group. The zeroth Gamma cohomology is the module of derivations and the first Gamma cohomology group is the module of 'infinitesimal extensions', i.e., it is isomorphic to the first André-Quillen cohomology group.

Some calculations of Gamma homology can be found in [Ri-Ro]. In particular, for smooth algebras, for group rings and for truncated polynomial algebras, there are explicit formulae for Gamma homology.

For a commutative ring spectrum E, Gamma cohomology groups of the algebra of cooperations E_*E contain information about the obstructions for refining the given multipication on the ring spectrum to an E_∞ ring structure. Alan Robinson established this obstruction theory in [Ro1]; an overview can be found in this volume [Ro2]. For instance, the existence of the unique E_∞ structures on the Lubin-Tate spectra E_n [Ro1, Ri-Ro], on real and complex K-theory, on the Adams summand and on the I_n-adic completion of the Johnson-Wilson spectra $\widehat{E(n)}$ [B-R] can be proven this way.

1.4. **Stable homotopy of Γ-modules, π_*^{st}.** Let Γ denote the skeleton of the category of finite pointed sets with set of objects $[n] = \{0, \dots, n\}$ with 0 as base point.

There is a well-known way to associate a spectrum to any covariant functor F from Γ to some pointed category $\mathcal{C}$ which has a forgetful functor the category Sets$_*$ of pointed sets. Let us call such an F a left Γ-object in $\mathcal{C}$. Such a functor F can be prolonged to a functor from pointed simplicial sets to simplicial $\mathcal{C}$-objects by approximating an arbitrary pointed set by finite pointed sets and by applying F degreewise: for a pointed simplicial set X_* let $F(X_*)$ be $F(X_n)$ in simplicial degree n.

For two pointed simplicial spaces X_* and Y_*, and for a left Γ-object F in $\mathcal{C}$ we obtain a map $X_* \wedge F(Y_*) \to F(X_* \wedge Y_*)$: each element $x \in X_n$ defines a morphism $x : Y_n \to (X_* \wedge Y_*)_n$ by sending an element y in Y_n to $x(y) := [(x, y)]$, i.e., to the equivalence class of (x, y) in the smash-product. This yields the desired transformation $X_* \wedge F(Y_*) \to F(X_* \wedge Y_*)$ by naturality of F. In particular, we obtain maps

$$\mathbb{S}^1 \wedge F(\mathbb{S}^n) \to F(\mathbb{S}^{n+1})$$

such that the sequence $(F(\mathbb{S}^n))_{n\geqslant 0}$ becomes a spectrum and we denote the stable homotopy groups of that spectrum by $\pi_*^{st}(F)$.

For a commutative ring with unit k, a left Γ-module is a functor from Γ to the category of k-modules. Teimuraz Pirashvili showed in [P, Prop.2.2], that the groups $\pi_*^{st}(F)$ are isomorphic to the derived functors $\mathsf{Tor}_*^{\Gamma}(t, F)$ of the tensor product $t \otimes_\Gamma F$. Here t is a contravariant functor from Γ to k-modules, which is given by $t[n] = \mathrm{Hom}_{\mathrm{Sets}_*}([n], k)$. The Tor-groups in turn have been identified with the homology groups of the first layers in the Goodwillie tower for F in [Ri2, Theorem 4.5].

If one considers the particular case of the left Γ-module $\mathcal{L}(A|k; M)$ which is given by $[n] \mapsto M \otimes A^{\otimes n}$, for any commutative k-algebra A and any A-module M, then we obtain stable homotopy groups associated to an algebra and a module. Here a map of finite pointed sets $f : [n] \to [m]$ induces multiplication in A, insertion of the unit or the action of A on M: the map f sends an element $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in M \otimes A^{\otimes n}$ to $b_0 \otimes b_1 \otimes \cdots \otimes b_m \in M \otimes A^{\otimes m}$ where each b_i is a product $\prod_{f(j)=i} a_j$, where we interpret this to be the unit of A whenever $f^{-1}(i) = \varnothing$.

There is a visible relationship to Hochschild homology: recall the usual specific chain complex for Hochschild homology. The underlying simplicial set looks like

$$M \rightleftarrows M \otimes A \ \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow \\ \longleftarrow} \ M \otimes A \otimes A \quad \cdots$$

Here the face maps induce the multiplication in the algebra A respectively the action of A on M.

Taking the simplicial model for the 1-sphere which consists of $n+1$ elements in degree n it is visible, that Hochschild homology of A with coefficients in M is the homotopy of $\mathcal{L}(A|k;M)$ evaluated at $\mathbb{S}^1$. This gives a stabilization map from Hochschild homology to stable homotopy of $\mathcal{L}(A|k;M)$

$$\pi_*\mathcal{L}(A|k;M)(\mathbb{S}^1) \to \mathrm{colim}_n \pi_{*+n}\mathcal{L}(A|k;M)(\mathbb{S}^{n+1}) = \pi^{st}_{*-1}(\mathcal{L}(A|k;M)).$$

As stable homotopy splits tensor products of Γ-modules into sums (see [P, 4.2]) in the following way

$$\pi^{st}_*(F \otimes G) \cong \pi^{st}_*(F) \otimes G[0] \oplus F[0] \otimes \pi^{st}_*(G)$$

we obtain, that stable homotopy of $\mathcal{L}(k[x_i, i \in I]|k;k)$ for an arbitrary indexing set I is isomorphic to $\bigoplus_I \pi^{st}_*(\mathcal{L}(k[x]|k;k))$; therefore, stable homotopy of a free simplicial resolution of an algebra gives as many copies of $\pi^{st}_*(\mathcal{L}(k[x]|k;k))$ as there are generators in the resolving algebra. This additivity property leads to an Atiyah-Hirzebruch spectral sequence (compare [Ri1]) for stable homotopy of augmented commutative algebra, which is of the form

$$E^2_{*,*} = \mathsf{AQ}_*(A|k;\pi^{st}_*(\mathcal{L}(k[x]|k;k))) \Rightarrow \pi^{st}_*(\mathcal{L}(A|k;k)).$$

The algebra $k[x]$ is the free commutative algebra on one generator and might be interpreted as the 'base point' in this context.

We will meet this spectral sequence again in Schwede's [Sch2] stable homotopy of the algebraic theory of augmented commutative algebras.

1.5. **Stable homotopy of algebraic theories, $\pi^{\mathcal{T}}_*$.** We will describe this approach by Stefan Schwede in some detail, because we will later give a proof of Theorem 2.1, which compares stable homotopy of the algebraic theory of augmented commutative k-algebras to stable homotopy of the functor $\mathcal{L}(-|k;k)$. Note that our category Γ is denoted by Γ^{op} in [Sch2].

The model for the category of connective spectra used in this approach is the symmetric monoidal category of Γ-spaces, i.e., functors from Γ to the category sSets of simplicial sets which send $[0]$ to a one-point simplicial set. The monoidal structure is given by a smash-product whose definition and properties can be found in [Ly].

Start with a pointed simplicial algebraic theory. This is a pointed simplicial category $\mathcal{T}$ which has the same objects as the category of finite pointed sets Γ and which has a functor from Γ^{op} to $\mathcal{T}$ which preserves products and is the identity on objects. Note that the object $[n]$ is the n-fold product of the object $[1]$ in the category Γ^{op}.

If you do not feel comfortable with algebraic theories, then think of the morphisms from $[n]$ to $[1]$ as all possible n-ary operations in the theory, i.e., in our example of augmented commutative k-algebras. Such a morphism gives an operation from A^n to A for every such algebra A. For any theory $\mathcal{T}$, $\mathcal{T}$-algebras are product-preserving simplicial functors from $\mathcal{T}$ to the category sSets$_*$ of pointed simplicial sets. Therefore these functors are determined by their value on $[1]$. For the theory of augmented commutative k-algebras, a functor $G : \mathcal{T} \to \mathrm{sSets}_*$ corresponds to an algebra A as above by $G[1] \cong A$.

Schwede establishes in Theorem [Sch2, 3.1] a simplicial model category of $\mathcal{T}$-algebras. The simplicial structure allows to talk about suspensions of objects: for any $\mathcal{T}$-algebra A, the suspension ΣA is the geometric realization of the simplicial object that sends the simplicial object $\{0 < \ldots < m\}$ to the m-fold coproduct $\coprod_m A$ of A.

Spectra of $\mathcal{T}$-algebras can now be defined by the suspension functor as sequences of $\mathcal{T}$-algebras (A_n) together with maps $\rho_n^A : \Sigma A_n \to A_{n+1}$. Maps of spectra $f : (A_n) \to (B_n)$ are strict maps in the sense that $\rho_n^B \circ \Sigma(f_n) = f_{n+1} \circ \rho_n^A$.

Theorem [Sch2, 4.3] states that the category of spectra of $\mathcal{T}$-algebras, called $\mathcal{S}p(\mathcal{T})$, is a closed simplicial model category. To any theory $\mathcal{T}$, one can associate a monoid in the symmetric monoidal category of Γ-spaces, T^s, such that there is an equivalence between the homotopy category of modules over T^s and the homotopy category of connective spectra (cf. [Sch2, 4.4]).

Stable homotopy of $\mathcal{T}$-algebras can be defined as the homotopy groups of the suspension spectrum of any $\mathcal{T}$-algebra

$$\pi_*^{\mathcal{T}}(A) := \pi_*^{st}(\Sigma^\infty(A)).$$

Having a nice model category around, it makes also sense to talk about Quillen homology which is defined ([Sch2, 5.1]) as:

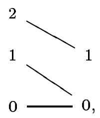
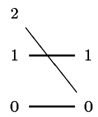
$$H_*(A) := \pi_*(X_{ab}^c); \qquad H_*(A; M) := \pi_*(M \otimes_{T_{ab}} X_{ab}^c).$$

Here $(-)^c$ is the cofibrant replacement, $(-)_{ab}$ denotes the abelianization of a $\mathcal{T}$-algebra, and M is a right simplicial module over a certain simplicial ring T_{ab}. In a similar way as connective spectra of $\mathcal{T}$-algebras are equivalent to T^s-modules, the category of abelian objects in $\mathcal{T}$-algebras is equivalent to modules over T_{ab}.

This simplicial ring T_{ab} can be described in a more explicit way: there is a *linearization functor* L (see [Sch2, 5.2]) from Γ-spaces to simplicial abelian groups. Let $\overline{\mathbb{Z}}[S_*]$ denote the free abelian group of the pointed simplicial set S_* with the relation that the base point is equivalent to zero. The linearization takes a Γ-space F and assigns

$$L(F) = \mathrm{coker}((p_1)_* + (p_2)_* - \nabla_* : \overline{\mathbb{Z}}[F[2]] \to \overline{\mathbb{Z}}[F[1]])$$

to it. Here the p_1 and p_2 are the projections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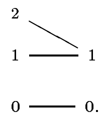

The simplicial ring T_{ab} is isomorphic to $L(T^s)$; in particular, if the theory is discrete, then the description of T_{ab} as the linearization of T^s shows that T_{ab} reduces to $\pi_0(T^s)$ in that case (cf. [Sch2, 5.2]).

The suspension spectrum of a $\mathcal{T}$-algebra can be identified with a different spectrum, which is closer related to the stabilization process for Γ-spaces. In [Sch2, 5.1] an alternative to the suspension spectrum is described: define the functor

$$\widetilde{\Sigma^\infty}(A) : \Gamma \longrightarrow \mathcal{T}-\text{algebras}$$

by $\widetilde{\Sigma^\infty}(A)[n] := \coprod_n A$. Surjective maps of finite pointed sets induce folding maps or the projection of components and injective maps of finite pointed sets induce inclusion maps on the coproduct. This functor has the following properties

- The spectrum associated to the Γ-space $\widetilde{\Sigma^\infty}(A)$ is equivalent to the suspension spectrum of A.
- The abelianization of an arbitrary $\mathcal{T}$-algebra is isomorphic to the linearization $L(\widetilde{\Sigma^\infty}(A))$.

Schwede constructs a universal coefficient spectral sequence and an Atiyah-Hirzebruch spectral sequence. The latter has the following shape:

$$E^2_{p,q} = H_p(A; \pi_q W) \Rightarrow W_{p+q}(A).$$

Here W is a right T^s-module and W-homology is defined to be the homotopy of the derived smash product of W with the suspension spectrum of A, $W \wedge^L_{T^s} \Sigma^\infty A$.

In particular, for $W = T^s$ we obtain a spectral sequence which starts with André-Quillen homology of A with coefficients in the homotopy groups of T^s converging to the stable homotopy of A

$$E^2_{p,q} = H_p(A, \pi^{\mathcal{T}}_q(T^s)) \Rightarrow \pi^{\mathcal{T}}_{p+q}(A).$$

2. Comparison results

As promised, we will describe the relationship between the different homology theories for commutative (S-)algebras. Except for the first theorem, we will not give proofs of the comparison results, because these can be found in the literature.

We will start with the two homology theories arising from Γ-spaces which have their range of definition in purely algebraic objects:

Theorem 2.1. *Stable homotopy of the* Γ*-module* $\mathcal{L}(A|k;k)$ *of an augmented unital commutative* k*-algebra* A *is isomorphic to stable homotopy of* A*,* $\pi_*^{\mathcal{T}}(A)$ *for the theory* $\mathcal{T}$ *of commutative augmented* k*-algebras.*

Proof. We will identify the two Γ-spaces which give stable homotopy of algebraic theories on the one hand and stable homotopy of $\mathcal{L}(A|k;k)$ on the other hand. So let A be an arbitrary augmented commutative k-algebra. The model $\widetilde{\Sigma^\infty}(A)$ of the suspension spectrum looks as follows: the object $[n] \in \Gamma$ is sent to the n-fold coproduct $\coprod_n A$ of A. In the category of commutative algebras, this is the same as the n-fold tensor product of A with itself, $A^{\otimes n}$.

Order-preserving injective maps of finite pointed sets induce the insertion of units on both functors. Let us distinguish surjective maps of finite pointed sets with the property that the preimage of the basepoint zero is only zero from all other surjective maps. Maps of the first kind induce the folding map on the coproduct (which is multiplication), and maps of the second kind involve the projection of components in the coproduct to the basepoint, which is the ring k. Consequently, in the first case elements in A are just multiplied whereas in the other case there is an additional action of A on k by the augmentation.

The Γ-module $\mathcal{L}(A|k;k)$ sends the object $[n]$ to the n-fold tensor product $A^{\otimes n}$ and from the definition of $\mathcal{L}$ in part 1.4 it follows that maps of finite pointed sets induce the same maps on this Γ-module. Therefore the two Γ-spaces are isomorphic and the defining spectra for stable homotopy in both cases agree. □

Corollary 2.2. *The Atiyah-Hirzebruch spectral sequence for stable homotopy of the algebraic theory of augmented commutative* k*-algebras coincides with the one for stable homotopy of the functor* $\mathcal{L}(-|k;k)$.

Proof. Stable homotopy of T^s for the theory of augmented commutative k-algebras is isomorphic to the singular k-homology of the Eilenberg-MacLane spectrum of the integers, because T^s is stably equivalent to $Hk \wedge^L H\mathbb{Z}$ (see [Sch2, 7.9]). The result [Ri1, 3.1] (or [Ri-Ro, 3.2]) identifies $Hk_*H\mathbb{Z}$ with stable homotopy of $\mathcal{L}(k[x]|k;k)$, so there is an isomorphism on the level of E^2-terms.

This is not only an additive isomorphism but will lead to an isomorphism of spectral sequences. Let us denote the linearization functor from the category Γ to the category of k-modules which sends a set $[n]$ to the free module k^n by ℓ (in order to distinguish it from the functor L used above). The identification of $\pi_*^{st}(\mathcal{L}(k[x]|k;k))$ with $Hk_*H\mathbb{Z}$ in [Ri1] uses the fact, that the functor $\mathcal{L}(k[x]|k;k)$ can be identified with the linearization functor composed with the infinite symmetric product functor Sym^* from k-modules to k-modules. Stable homotopy of any such composed functor $G \circ \ell$ is isomorphic to the stable derived functors L_*^{st} of Eilenberg and MacLane. See for instance Betley's paper [Be] for a proof of this last claim.

Schwede proves in [Sch2, 7.9] an equivalence between T^s and the composite functor $Hk \circ \mathsf{Sym}$, where Sym is the infinite symmetric product functor on pointed spaces. Therefore we get a natural stable weak equivalence of Γ-spaces and the claim follows. □

The second comparison result relates Gamma homology, a homology theory for commutative rings, which at first sight has nothing to do with functors from finite pointed sets to modules, to stable homotopy of Γ-modules. The proof of Theorem 1 in [P-R] uses an enlargement of the domain of definition for Gamma homology to all Γ-modules. See also [Ro2] for a proof.

Theorem 2.3. [P-R] *Gamma homology of any commutative k-algebra A with coefficients in an A-module M is isomorphic to stable homotopy of the Γ-module $\mathcal{L}(A|k; M)$.*

The second result obtained by Basterra and McCarthy compares topological André-Quillen homology – a homology theory for genuine S-algebras – with Gamma homology – a homology theory for algebras.

Theorem 2.4. [B-McC, 4.2] *Gamma homology is isomorphic to* TAQ *of the corresponding Eilenberg-MacLane spectra for flat algebras, i.e., if A is k-flat, then*

$$\mathsf{TAQ}_*(H(A)|H(k); H(A)) \cong \mathrm{H}\Gamma_*(A|k; A).$$

Using the 'hyperhomology' spectral sequence from [EKMM, 4.1] for the $H(A)$-modules $\mathsf{TAQ}(H(A)|H(k); H(A))$ and $H(M)$ for an A-module M

$$\begin{aligned} &E^2_{p,q} = \mathsf{Tor}^A_{p,q}(\mathsf{TAQ}_*(H(A)|H(k); H(A)), M) \\ &\Rightarrow \mathsf{Tor}^{H(A)}_{p+q}(\mathsf{TAQ}(H(A)|H(k); H(A)), H(M)) = \mathsf{TAQ}_{p+q}(H(A)|H(k); H(M)) \end{aligned}$$

on the one hand and the corresponding spectral sequence for modules on the other hand for the chain complex $C\Gamma_*(A|k; M) = C\Gamma_*(A|k; A) \otimes_A M$, we can extend this isomorphism. The theorem above yields an isomorphism on the level of spectral sequences and we obtain that

$$\mathsf{TAQ}_*(H(A)|H(k); H(M)) \cong \mathrm{H}\Gamma_*(A|k; M) \tag{2.1}$$

for k-flat A. Similarly, the corresponding spectral sequences for Ext-groups ensure, that

$$\mathsf{TAQ}^*(H(A)|H(k); H(M)) \cong \mathrm{H}\Gamma^*(A|k; M) \tag{2.2}$$

for k-projective A.

In the flat case we obtain an equivalence of all these theories

$$\mathsf{TAQ}_*(H(A)|H(k); H(A)) \cong \mathrm{H}\Gamma_*(A|k; A) \cong \pi_*^{st}(\mathcal{L}(A|k; A))$$

and for A an augmented k-flat algebra we get isomorphisms between all these homology theories:

$$\begin{array}{ccccc} & & \mathsf{TAQ}_*(H(A)|H(k);H(k)) & & \\ & \diagup & & \diagdown & \\ \mathrm{H}\Gamma_*(A|k;k) & & & & \pi_*^{st}(\mathcal{L}(A|k;k)) \\ & \diagdown & & \diagup & \\ & & \pi_*^{\mathcal{T}}(A) & & \end{array}$$

The last comparison theorem which we will mention is a result by Mike Mandell. He relates topological André-Quillen cohomology of spectra to TAQ in a differential graded resp. simplicial setting of E_∞-algebras.

Let k be again an arbitrary commutative ring with unit. Mandell defines in [M, 1.1] André-Quillen (co)homology for E_∞-differential graded k-algebras – which we will call $\mathsf{AQ}^*_{\mathrm{dg}E_\infty}$ – and for simplicial E_∞-algebras – here denoted by $\mathsf{AQ}^*_{\mathrm{s}E_\infty}$.

Theorem 2.5.

(1) [M, 1.8] *The normalization functor N from simplicial k-modules to differential graded k-modules transforms $\mathsf{AQ}^*_{\mathrm{s}E_\infty}$ into André-Quillen homology of differential graded E_∞-algebras: for any simplicial E_∞ k-algebra A and any A-module M there is a natural isomorphism*

$$\mathsf{AQ}^*_{\mathrm{s}E_\infty}(A|k;M) \cong \mathsf{AQ}^*_{\mathrm{dg}E_\infty}(N(A)|k;N(M)).$$

This isomorphism can be extended to simplicial E_∞-algebras relative to another algebra: if $f: A \to B$ is a map of simplicial E_∞-algebras, then

$$\mathsf{AQ}^*_{\mathrm{s}E_\infty}(B|A;M) \cong \mathsf{AQ}^*_{\mathrm{dg}E_\infty}(N(B)|N(A);N(M)).$$

If the homotopy groups of the module M are concentrated in non-positive degrees then André-Quillen cohomology with coefficients in M resp. in $N(M)$ is concentrated in non-negative degrees.

(2) [M, 7.8–7.10] *Let R be a connective and cofibrant commutative S-algebra. There is a functor Ξ from the category of E_∞ R-algebras to differential graded E_∞-algebras and there is a functor $\mathbf{R}$ from the homotopy category of modules over $\Xi(A)$, for A an E_∞-algebra over R, to the homotopy category of R-modules such that*

$$\mathsf{TAQ}^*(A|R;\mathbf{R}(M)) \cong \mathsf{AQ}^*_{\mathrm{dg}E_\infty}(\Xi(A)|\Xi(R);M).$$

A similar result applies to any map $f: A \to B$ of E_∞-R-algebras:

$$\mathsf{TAQ}^*(B|A;\mathbf{R}(M)) \cong \mathsf{AQ}^*_{\mathrm{dg}E_\infty}(\Xi(B)|\Xi(A);M).$$

In the cases of coefficients in an Eilenberg-MacLane spectrum, the isomorphism specializes to something very concrete: let A be a connective E_∞-algebra over R and let N be a module over $\pi_0(A)$. Then for any

$\Xi(A)$-module M with $H_0(M) \cong N$ and trivial other homology groups we obtain

$$\mathsf{TAQ}^*(A|R; H(N)) \cong \mathsf{AQ}^*_{\mathrm{dg}E_\infty}(\Xi(A)|\Xi(R); M).$$

The above isomorphisms preserves more structure than the mere additive module structure: all three kinds of André-Quillen cohomology mentioned in the theorem possess transitivity sequences and long exact sequences for short exact and the isomorphisms respects them ([M, 1.9 and 13.2]).

The identification of topological André-Quillen cohomology of spectra with a cohomology theory for differential graded objects made it for instance possible to find a concrete example for an S-algebra, which is TAQ-étale but not THH-étale (see [McC-Mi]). This example (and its chain model – which is just the cochain algebra on the n-th Eilenberg-MacLane space on the field with p elements for $n > 1$) are necessarily not connective, because Minasian's work in [Mi] proves that both notions coincide for connective commutative S-algebras.

Paul Goerss and Mike Hopkins develop an obstruction theory for the existence of E_∞-structures on ring spectra (see [GH2]). The obstruction groups that arise in that context are André-Quillen cohomology groups of algebras over simplicial E_∞ algebras. More precisely, the obstructions for E_∞ structures on a commutative ring spectrum E live in

$$\mathsf{AQ}^*(E_*E|E_*; E_*)$$

where AQ means that one views the graded commutative commutative algebra E_*E of cooperations as a constant simplicial E_∞ algebra.

It is a natural question to ask, what the relationship is between these obstruction groups and the ones developed by Alan Robinson (see [Ro2] and [Ro1]). In his approach, the obstruction groups live in Gamma cohomology of the algebra of cooperations

$$\mathrm{H}\Gamma^*(E_*E|E_*; E_*).$$

In the following we sketch an argument, why the obstruction groups in the two approaches are actually isomorphic. Let k be a commutative ring with unit, let A be a unital commutative k-algebra which is projective as a k-module and let M be an A-module. The rough idea of the proof is to combine Mike Mandell's results [M] with the comparison result in [B-McC] to obtain the desired isomorphism.

The Goerss-Hopkins groups do not actually depend on the choice of a simplicial E_∞ operad, neither do the simplicial André-Quillen groups in Mandell's work. André-Quillen cohomology in both contexts is defined via a cofibrant resolution in the category of simplicial E_∞ algebras. Here the used model categories (in [M, 3.3] resp. [GH2, 4.1]) agree: the weak equivalences are given by maps which induce an isomorphism on homotopy groups and the fibrations are maps which lead to surjective maps in positive degrees after normalization.

Therefore we obtain

$$\mathsf{AQ}^*(A|k;M) \cong \mathsf{AQ}^*_{\mathrm{s}E_\infty}(A|k;M)$$

where the first groups denote the Goerss-Hopkins groups and the latter Mandell's groups. As A, k and M are viewed as constant simplicial objects, the result 2.5 yields an isomorphism of these groups with André-Quillen cohomology groups in the category of differential graded E_∞ algebras:

$$\mathsf{AQ}^*_{\mathrm{s}E_\infty}(A|k;M) \cong \mathsf{AQ}^*_{\mathrm{dg}E_\infty}(A|k;M).$$

These cohomology groups have a relationship with topological André-Quillen cohomology of Eilenberg-MacLane spectra in the following way.

First of all, in the case of constant coefficients M, the functor $\mathbf{R}$ from differential modules over A into modules over HA reduces to

$$\mathbf{R}(M) \simeq H(M).$$

An argument for this can be found in [M, 7.10]. So the cohomology groups on the level of E_∞ ring spectra

$$\mathsf{TAQ}^*(H(A)|H(k);H(M))$$

are isomorphic to $\mathsf{AQ}^*_{\mathrm{dg}E_\infty}(\Xi(H(A))|\Xi(H(k));M)$ and we have to compare these groups to $\mathsf{AQ}^*_{\mathrm{dg}E_\infty}(A|k;M)$.

As the algebra $\Xi(H(A))$ is connected, there is a natural map to $H_0(\Xi(H(A))$ $= A$. This map

$$\varphi \colon \Xi(H(A)) \longrightarrow A$$

is a map of differential graded E_∞ algebras and is the unique map which gives the inverse of the isomorphism

$$A = \pi_0 H(A) \cong H_0(H(A)) = H_0(\Xi(H(A)))$$

on homology. The functor Ξ is a composition $C_* \circ \Gamma$, where Γ is a CW approximation functor in the category of E_∞ $H(k)$-algebras and C_* is a cellular chains functor. By construction [M, 10.3] there is a canonical weak equivalence

$$\gamma \colon \Gamma(H(A)) \longrightarrow H(A).$$

The cellular chain functor constructed in [M, §9] does not change the homology which for Eilenberg-MacLane spectra gives the ordinary homotopy groups. Therefore φ is a weak equivalence of E_∞ algebras:

$$H_*\Xi(H(A)) = \pi_*\Gamma(H(A)) \xrightarrow{\simeq} \pi_* H(A) = A.$$

Topological André-Quillen cohomology of commutative $H(k)$-algebras in the category of E_∞ $H(k)$-algebras is isomorphic to usual topological André-Quillen cohomology of commutative $H(k)$-algebras. Taking all these steps together, the Goerss-Hopkins groups $\mathsf{AQ}^*(A|k;M)$ are isomorphic to $\mathsf{TAQ}^*(H(A)|H(k);H(M))$.

Using the comparison result from [B-McC] and adapting it to cohomology as above (2.2) we get an isomorphism of the latter to Gamma cohomology. In fact, for the comparison result we do not need A to be projective over k. The comparison of Gamma homology and topological André-Quillen homology works for flat algebras. To transfer this to cohomology, we just have to have that the universal coefficient spectral sequence collapses. So for such commutative k-algebras A and A-modules M we obtain:

Theorem 2.6. *The Goerss-Hopkins André-Quillen cohomology groups* $\mathsf{AQ}^*(A|k;M)$ *are isomorphic to Alan Robinson's Gamma cohomology groups* $\mathrm{H}\Gamma^*(A|k;M)$.

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Department of Mathematics, University of New Hampshire in Durham, Kingsbury Hall, Durham, New Hampshire 03824, USA
email-address: basterra@cisunix.unh.edu

Mathematisches Institut der Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany *email-address:* richter@math.uni-bonn.de