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Homotopy algebras and the inverse of the normalization functor

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Abstract

In this paper, we investigate multiplicative properties of the classical Dold–Kan correspondence. The inverse of the normalization functor maps commutative differential graded algebras to E_∞-algebras. We prove that it in fact sends algebras over arbitrary differential graded E_∞-operads to E_∞-algebras in simplicial modules and is part of a Quillen adjunction. More generally, this inverse maps homotopy algebras to weak homotopy algebras. We prove the corresponding dual results for algebras under the conormalization, and for coalgebra structures under the normalization resp. the inverse of the conormalization.

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1. Introduction

The Dold–Kan correspondence [5, Theorem 1.9] states, that the normalization functor N from the category of simplicial modules to non-negative chain complexes is part of an equivalence of categories; we denote its inverse by D. The pair (N, D) gives rise to a Quillen equivalence between the corresponding model categories. Shipley and Schwede proved in [23, Theorem 1.1.(3)] that this equivalence passes to the subcategories of associative monoids. The subject of this paper is to investigate to what extent commutative structures are preserved by the functor D.

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The normalization functor is lax symmetric monoidal. In particular, it sends commutative simplicial algebras to differential graded commutative algebras, and more generally it preserves all algebra structures over operads: if a simplicial module X is an $\mathcal{O}$-algebra, then $N\mathcal{O}$ is an operad in chain complexes and NX is an $N\mathcal{O}$-algebra. In [21] we proved that the functor D sends differential graded commutative algebras to simplicial E_∞-algebras. Conversely it is clear that the normalization functor N maps an E_∞-algebra to an E_∞-algebra.

In positive characteristic there is no 'reasonable' model category structure on differential graded commutative algebras: Don Stanley proved in [25, Section 9], that the category of differential graded commutative algebras over an arbitrary commutative ring possesses a model category structure, where the weak equivalences are the homology isomorphisms and the cofibrant objects are 'semi-free' algebras (*á la* Quillen [19, II.4.11]). The fibrations are then determined and it turns out that fibrations are not necessarily surjective in positive degrees, i.e., the weak equivalences and fibrations are not determined by the forgetful functor from differential graded commutative algebras to chain complexes alias differential graded modules.

In order to avoid such problems it is advisable to replace the category of commutative algebras with a homotopically invariant analog, i.e., to pass to the category of differential graded E_∞-algebras. But it does *not* immediately follow from the results in [21] that D maps differential graded E_∞-algebras to simplicial E_∞-algebras. The aim of this paper is to provide this result.

Mandell [18, Theorem 1.3] proved that there is a Quillen equivalence between the model category of simplicial E_∞-algebras and the model category of differential graded E_∞-algebras. As the homotopy categories in the E_∞-context do not depend on the chosen operad, Mandell chose operads which arise from the linear isometries operad $\mathcal{L}$: in the simplicial case he uses the free k-module on the singular simplicial set on the linear isometries operad in topological spaces and in the differential graded context the normalized chains on this simplicial operad [18, 2.1].

Mandell starts, however, with the normalization functor and he constructs an adjoint to it. If we want to keep control over differential graded E_∞-algebras while transferring them to the simplicial setting, we should look out for a correspondence which takes the inverse D as a starting point.

We develop a general operadic approach and define *generalized parametrized endomorphism operads* for any functor F between closed symmetric monoidal categories. One important feature of the operads that arise in this way is that they preserve associativity: if the functor F is lax monoidal then there is a map of operads from the operad of associative monoids to the generalized endomorphism operad associated to F (see Theorem 4.4.1).

Using this set-up, we prove that the functor D sends E_∞-algebras in the category of differential graded modules to simplicial E_∞-algebras and more generally, it preserves homotopy algebra structure. We prove that D possesses a left adjoint which can be seen to build a Quillen adjunction. If we start with strictly associative E_∞-algebras then D preserves this structure; therefore we get a Quillen adjunction on the level of strictly associative E_∞-algebras.

Prolonging D with the functor which associates the symmetric Eilenberg–MacLane spectrum to a simplicial module yields canonical E_∞-monoids in the category of symmetric spectra.

As a second application we investigate to what extend the conormalization functor preserves operad actions. In [9], Hinich and Schechtman studied the multiplicative behaviour of the conormalization functor N^* from cosimplicial abelian groups to cochain complexes. They proved, that the conormalization of a commutative cosimplicial abelian group is a *May-algebra*, i.e., possesses an operad action of an acyclic operad. In particular, it has a structure of an E_∞-algebra in the category of cochain complexes. Using generalized parametrized endomorphism operads allows us to generalize this result and show, that N^* behaves similar to D: it sends algebras over an operad $\mathcal{O}$ in the category of modules to weak homotopy $N^*\mathcal{O}$-algebras.

Having achieved some understanding of algebraic structures, we apply our methods to coalgebra structures and their preservation under the functors D and N^*.

The structure of the paper is as follows: We start with providing the general set-up in Sections 2 and 3 by constructing generalized endomorphism operads and parametrized endomorphism operads for an arbitrary functor $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories. Parametrized endomorphism operads are generalized endomorphism operads into which another operad is implanted. We hope that these general constructions will be of independent interest. If $\mathcal{C}$ and $\mathcal{D}$ have appropriate model category structures, then Theorem 4.2.1 ensures that a Quillen adjunction with F as a right adjoint passes to a Quillen adjunction on the level of algebras over operads.

From Section 5 we turn on to the case of the Dold–Kan correspondence. We mention the standard construction of a left adjoint for D on the level of algebras. In Section 5.2 we apply the concepts from Sections 2 and 3 to the functors involved in the Dold–Kan correspondence. We use parametrized endomorphism operads of the functor D to provide concrete acyclic operads, which ensure that the functor D sends E_∞-algebras to E_∞-algebras. In addition we prove in Theorem 5.5.5 that D maps general homotopy algebras to *weak homotopy algebras*: these are algebras over an operad which is weakly equivalent to the original operad but not necessarily cofibrant.

It is straightforward to see, that D possesses a left adjoint functor on the level of E_∞-algebras and we show in Theorem 5.4.2 that this passes to the level of homotopy categories, i.e., that the corresponding adjoint pair is a Quillen adjunction. In Theorem 5.5.5, we generalize this result and show, that D induces a Quillen adjunction on the level of homotopy-$\mathcal{O}$-algebras, where $\mathcal{O}$ is an arbitrary operad in the category of modules. At the moment, we are unable to prove that this Quillen adjunction is a Quillen equivalence.

Section 6 discusses the dual situation of the conormalization functor. We give an explicit construction of the generalized (parametrized) endomorphism operad in these cases and use it to prove that N^* maps homotopy algebras to weak homotopy algebras.

Section 7 deals with our results for coalgebra structures: if an E_∞-cooperad coacts on a simplicial module $A_\bullet$, then there is an E_∞-operad parametrizing a coalgebra structure on the normalization of $A_\bullet$ (cf. Theorem 7.3.2).

In order to assure, that our construction of homotopy algebra structures is homotopically well-behaved, we use Markus Spitzweck's notion of semi-model categories and the model structures on operads and their algebras provided by Berger and Moerdijk. We will give a short overview over these results in Section 8.

Notation: We will make frequent use of several categories and therefore we fix notation for these. Let k be a fixed commutative ring with unit and let smod, resp. dgmod, be the category of simplicial k-modules, resp. differential graded k-modules which are concentrated in non-negative degrees. Dually, cmod denotes the category of cosimplicial k-modules and $\delta\mathsf{mod}$ is the category of cochain complexes which are concentrated in non-negative degrees.

We abbreviate the category of simplicial E_∞-algebras to sE_∞, its differential graded analog is denoted by dgE_∞. In the dual case cE_∞ and $\delta\mathsf{E}_\infty$ stand for the category of cosimplicial, respectively, cochain E_∞-algebras. We use dgca for the category of differential graded commutative algebras.

Throughout the paper we use the notion of model categories and operads. Standard references are the book by Hovey [10] for the first and the monograph by Kriz and May [13] for the latter.

2. Generalized endomorphism operads

Let us consider two symmetric monoidal closed categories $(\mathscr{C}, \otimes, \mathbf{1}_\mathscr{C})$ and $(\mathscr{D}, \hat{\otimes}, \mathbf{1}_\mathscr{D})$ and a functor $F : \mathscr{C} \to \mathscr{D}$ which we do not assume to be monoidal, but F should be coherent with the units in the two monoidal structures, so we assume either that F applied to the unit of $\mathscr{C}$ is isomorphic to the unit of $\mathscr{D}$

$$F(\mathbf{1}_\mathscr{C}) \cong \mathbf{1}_\mathscr{D} \tag{2.1}$$

or at least that F allows a map

$$\mathbf{1}_\mathscr{D} \to F(\mathbf{1}_\mathscr{C}). \tag{2.2}$$

If hom denotes the internal homomorphism object in $\mathscr{D}$ then for any object $X \in \mathscr{D}$ one can build the endomorphism operad $\mathsf{End}(n) = \mathsf{hom}(X^{\hat{\otimes} n}, X)$. The following is a slight variant of this operad.

2.1. *The definition of* End_F

Let us assume that the bifunctor

$$((C_1, \ldots, C_n), (C'_1, \ldots, C'_n)) \mapsto \mathsf{hom}(F(C_1)\hat{\otimes} \cdots \hat{\otimes} F(C_n), F(C'_1 \otimes \cdots \otimes C'_n))$$

from $(\mathscr{C}^n)^{op} \times \mathscr{C}^n$ to $\mathscr{D}$ possesses a categorical end $\int_{\mathscr{C}^n} \mathsf{hom}(F^{\hat{\otimes} n}, F^{\otimes n})$ in $\mathscr{D}$ for every n, and let us denote this end by $\mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n})$. Following [15, IX.5] let $w_{(C_1,\ldots,C_n)}$ (or w^n for short) be the binatural transformation from the end $\mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n})$ to $\mathsf{hom}(F(C_1)\hat{\otimes} \cdots \hat{\otimes} F(C_n), F(C_1 \otimes \cdots \otimes C_n))$.

Definition 2.1.1. The *generalized endomorphism operad with respect to the functor F* is defined as

$$\mathsf{End}_F(n) := \mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n}) = \int_{\mathscr{C}^n} \mathsf{hom}(F(C_1)\hat{\otimes} \cdots \hat{\otimes} F(C_n), F(C_1 \otimes \cdots \otimes C_n)).$$

We define operad term in degree zero, $\mathsf{End}_F(0)$, to be $F(\mathbf{1}_\mathscr{C})$. Thus if the functor F satisfies the strong unit condition 2.1, then $\mathsf{End}_F(0)$ is isomorphic to $\mathbf{1}_\mathscr{D}$, which in turn is isomorphic to the internal homomorphism object $\mathsf{hom}(\mathbf{1}_\mathscr{D}, \mathbf{1}_\mathscr{D})$ on the unit $\mathbf{1}_\mathscr{D}$.

Morphisms in $\mathscr{D}$ from the unit $\mathbf{1}_{\mathscr{D}}$ to $\mathsf{hom}(F(C), F(C))$ correspond uniquely to elements in the morphism set $\mathrm{Hom}_{\mathscr{D}}(F(C), F(C))$. We define the unit $\eta : \mathbf{1}_{\mathscr{D}} \to \mathsf{End}_F(1)$ to be the unique morphism corresponding to the family of maps $\mathbf{1}_{\mathscr{D}} \to \mathsf{hom}(F(C), F(C))$ which are induced from the identity map on $F(C)$ for every C in $\mathscr{C}$.

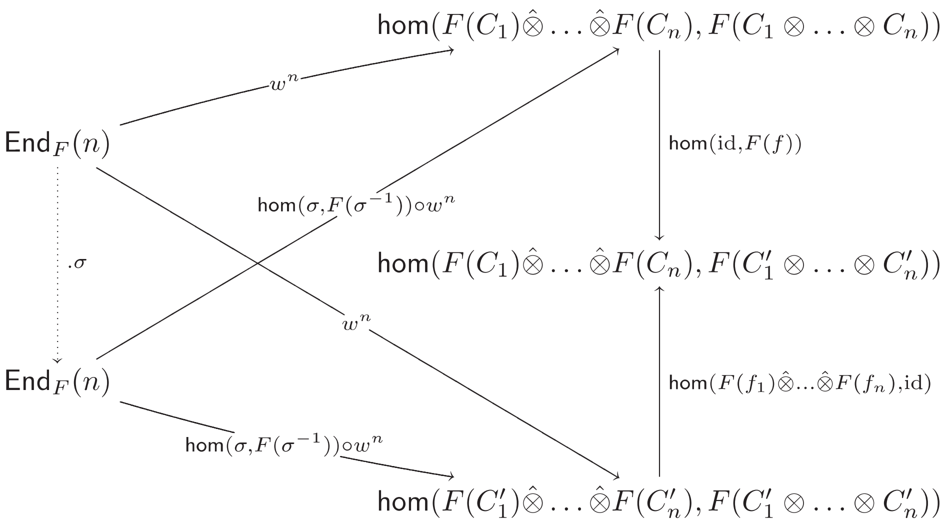
The action of the symmetric group on n letters, Σ_n, on $\mathsf{End}_F(n)$ is defined via the universal property of ends. For any $\sigma \in \Sigma_n$ we define twisted binatural transformations $\mathsf{hom}(\sigma, F(\sigma^{-1})) \circ w_{(C_{\sigma^{-1}(1)},\dots,C_{\sigma^{-1}(n)})}$ where the map $w_{(C_{\sigma^{-1}(1)},\dots,C_{\sigma^{-1}(n)})}$ is the given binatural transformation from $\mathsf{End}_F(n)$ to

$$\mathsf{hom}(F(C_{\sigma^{-1}(1)})\hat{\otimes}\cdots\hat{\otimes}F(C_{\sigma^{-1}(n)}), F(C_{\sigma^{-1}(1)} \otimes \cdots \otimes C_{\sigma^{-1}(n)}).$$

Note that the twisted transformations are maps from $\mathsf{End}_F(n)$ to $\mathsf{hom}(F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n), F(C_1 \otimes \cdots \otimes C_n))$.

In order to check that this gives a coherent family of transformations, we consider a morphism $f : (C_1, \dots, C_n) \to (C'_1, \dots, C'_n)$ in $\mathscr{C}^n$, i.e., an n-tuple of morphisms $f_i : C_i \to C'_i$ in $\mathscr{C}$. We have to show that

$$\begin{aligned}&\mathsf{hom}(f^*, \mathrm{id}) \circ \mathsf{hom}(\sigma, F(\sigma^{-1})) \circ w_{(C_{\sigma^{-1}(1)},\dots,C_{\sigma^{-1}(n)})}\\ &\quad = \mathsf{hom}(\mathrm{id}, F(f)) \circ \mathsf{hom}(\sigma, F(\sigma^{-1})) \circ w_{(C_{\sigma^{-1}(1)},\dots,C_{\sigma^{-1}(n)})}.\end{aligned}$$



But on the one hand we have that

$$\begin{aligned}&\mathsf{hom}(f^*, \mathrm{id}) \circ \mathsf{hom}(\sigma, F(\sigma^{-1}))\\ &\quad = \mathsf{hom}(\sigma, F(\sigma^{-1})) \circ \mathsf{hom}(F(f_{\sigma^{-1}(1)})\hat{\otimes}\cdots\hat{\otimes}F(f_{\sigma^{-1}(n)}), \mathrm{id})\end{aligned}$$

and on the other hand

$$\begin{aligned}&\mathsf{hom}(\mathrm{id}, F(f)) \circ \mathsf{hom}(\sigma, F(\sigma^{-1}))\\ &\quad = \mathsf{hom}(\sigma, F(\sigma^{-1})) \circ \mathsf{hom}(\mathrm{id}, F(f_{\sigma^{-1}(1)}) \otimes \cdots \otimes F(f_{\sigma^{-1}(n)})).\end{aligned}$$

The claim is then straightforward, because the transformations w^n were coherent.

Therefore the universal property of ends (see for instance [15, IX,5]) ensures that there is a unique map from $\mathsf{End}_F(n)$ to $\mathsf{End}_F(n)$ given by the above twisted transformations and we define this to be the action of σ on $\mathsf{End}_F(n)$.

Lemma 2.1.2. *The sequence* $(\mathsf{End}_F(n), n \geqslant 0)$ *with symmetric group action and units as above defines an operad in* $\mathscr{D}$.

Proof. We have to give $\mathsf{End}_F(n)$, $n \geqslant 0$ an operad composition

$$\gamma : \mathsf{End}_F(n) \hat{\otimes} \mathsf{End}_F(k_1) \hat{\otimes} \cdots \hat{\otimes} \mathsf{End}_F(k_n) \longrightarrow \mathsf{End}_F\left(\sum_{i=1}^{n} k_i\right).$$

The fact that each single $\mathsf{End}_F(n)$ is an end allows us to take the binatural transformation w^n from $\mathsf{End}_F(n)$ to

$$\mathsf{hom}\left(F(C_1 \otimes \cdots \otimes C_{k_1}) \hat{\otimes} \cdots \hat{\otimes} F(C_{k_{N(n)}} \otimes \cdots \otimes C_{\sum k_i}), F(C_1 \otimes \cdots \otimes C_{\sum k_i})\right)$$

with $k_{N(i)} = (\sum_{j=1}^{i-1} k_j) + 1$ and appropriate w^{k_i} from $\mathsf{End}_F(k_i)$ to

$$\mathsf{hom}(F(C_{k_{N(i-1)}}) \hat{\otimes} \cdots \hat{\otimes} F(C_{k_{N(i)}-1})), \quad F(C_{k_{N(i-1)}} \otimes \cdots \otimes C_{k_{N(i)}-1})).$$

Using the composition morphism

$$\mathsf{hom}(D_1, D_2) \hat{\otimes} \mathsf{hom}(D_2, D_3) \to \mathsf{hom}(D_1, D_3)$$

the morphisms

$$\mathsf{hom}(D_1, D_2) \hat{\otimes} \mathsf{hom}(D_3, D_4) \to \mathsf{hom}(D_1 \hat{\otimes} D_3, D_2 \hat{\otimes} D_4)$$

and the evaluation maps $\mathsf{hom}(D_1, D_2) \hat{\otimes} D_1 \to D_2$ in the symmetric monoidal category $\mathscr{D}$ gives binatural transformations from $\mathsf{End}_F(n) \hat{\otimes} \mathsf{End}_F(k_1) \hat{\otimes} \cdots \hat{\otimes} \mathsf{End}_F(k_n)$ to

$$\mathsf{hom}\left(F(C_1) \hat{\otimes} \cdots \hat{\otimes} F\left(C_{\sum_{i=1}^n k_i}\right), F\left(C_1 \otimes \cdots \otimes C_{\sum_{i=1}^n k_i}\right)\right).$$

Due to the universality of $\mathsf{End}_F(\sum k_i)$ this yields the desired composition map to $\mathsf{End}_F\left(\sum k_i\right)$ in a unique way. The associativity of these compositions γ follows from the associativity of the corresponding composition and evaluation maps on the internal morphism objects hom in $\mathscr{D}$.

The equivariance of the composition maps γ with respect to the action of the symmetric groups Σ_n, $n \geqslant 1$ and the unit condition are straightforward to check. □

We feel obliged to warn the reader that the assumption that we made at the beginning of this section about the existence of ends is crucial. If the functor F does not start from a small category, then in general the categorical end of natural transformations does not have to exist in the category $\mathscr{D}$, because one would actually deal with proper classes and not sets. In the cases which we will consider, the functor F will be representable and this will guarantee that the natural transformations $\mathsf{End}_F(n)$ are sets and in fact objects in $\mathscr{D}$.

2.2. Examples

Before we generalize the concept of generalized endomorphism operads to such an extent that we can transfer operadic algebra structures, we want to mention some typical examples of generalized endomorphism operads.

Example 2.2.1. In [20] we proved that the cubical construction of Eilenberg and MacLane on a commutative ring is a differential graded E_∞-algebra. The E_∞-operad used in the proof for this fact is built out of a generalized endomorphism operad.

Example 2.2.2. The starting point of the investigations of this paper is the property of the inverse of the Dold–Kan-correspondence D to transform commutative differential graded algebras into E_∞-simplicial algebras (see [21]). In this case, the generalized endomorphism operad of D is used to obtain that result. We defer details to Section 5.

Example 2.2.3. Using Satz 1.6 from Dold's article [6] one can read off that the unnormalized chain complex functor from simplicial abelian groups to chain complexes possesses a comonoidal analog of a generalized endomorphism operad which is acyclic. This operad is not an E_∞ operad but receives a map from one. Therefore, it yields an E_∞-comonoidal structure on every chain complex associated to a cocommutative simplicial module.

Example 2.2.4. In Section 6 we will investigate the multiplicative behaviour of the conormalization functor N^* from cosimplicial modules to cochain complexes with the help of generalized parametrized endomorphism operads.

Example 2.2.5. An example close to the classical endomorphism operad of an object is the following: Consider the full, though not closed, subcategory of powers of an object $C \in \mathscr{C}$, i.e., $C^{\otimes 0} = \mathbf{1}_{\mathscr{C}}, C, C^{\otimes 2}, \ldots$. Then we can build the generalized endomorphism operad which is built out of natural transformations from $F^{\hat{\otimes} n}$ to $F^{\otimes n}$ on that subcategory.

2.3. Augmentations

The operad End_F comes with a canonical augmentation map. We obtain a morphism

$$\begin{array}{c} \mathsf{End}_F(n)\hat{\otimes}\mathsf{End}_F(0)^{\hat{\otimes} n} \\ \downarrow \\ \mathsf{hom}(F(\mathbf{1}_{\mathscr{C}})\hat{\otimes}\cdots\hat{\otimes}F(\mathbf{1}_{\mathscr{C}}), F(\mathbf{1}_{\mathscr{C}}\otimes\cdots\otimes\mathbf{1}_{\mathscr{C}}))\hat{\otimes}F(\mathbf{1}_{\mathscr{C}}))^{\hat{\otimes} n} \\ \downarrow \\ F(\mathbf{1}_{\mathscr{C}}\otimes\cdots\otimes\mathbf{1}_{\mathscr{C}})\cong F(\mathbf{1}_{\mathscr{C}}). \end{array}$$

Note, that we do *not* obtain a map to the unit $\mathbf{1}_{\mathscr{D}}$ in general. If F satisfies, however, the stronger condition $F(\mathbf{1}_{\mathscr{C}})\cong\mathbf{1}_{\mathscr{D}}$ and $\mathsf{End}_F(0)$ is isomorphic to $\mathbf{1}_{\mathscr{D}}$, then we get an augmentation to the unit $\mathbf{1}_{\mathscr{D}}$, which is nothing but the nth term of the operad $\mathscr{C}$om of commutative monoids in $\mathscr{D}$.

3. Parametrized operads

3.1. The definition of parametrized operads

If one assumes that in addition to the functor F there is an operad $\mathscr{O}$ in $\mathscr{C}$, then we can construct an amalgamation of the operad End_F and the given operad $\mathscr{O}$ by implanting the operad into the generalized endomorphism operad. Again, we assume that all mentioned bifunctors posses ends in $\mathscr{D}$.

Definition 3.1.1. A *parametrized endomorphism operad* with parameters F and $\mathscr{O}$ is the end of the bifunctor from $(\mathscr{C}^n)^{op}\times\mathscr{C}^n$ to $\mathscr{D}$ which maps a pair $((C_1,\ldots,C_n),(C'_1,\ldots,C'_n))$ to

$$\mathsf{hom}(F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n), F^{\otimes n}(\mathscr{O}(n)\otimes C'_1\otimes\cdots\otimes C'_n)).$$

We will denote this operad by

$$\begin{aligned}\mathsf{End}_F^{\mathscr{O}}(n) &:= \mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n}(\mathscr{O}(n)\otimes -)) \\ &= \int_{\mathscr{C}^n}\mathsf{hom}(F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n), F(\mathscr{O}(n)\otimes C_1\otimes\cdots\otimes C_n)).\end{aligned}$$

For $n=0$ we set $\mathsf{End}_F^{\mathscr{O}}(0)$ to be $F(\mathscr{O}(0))\cong F(\mathscr{O}(0)\otimes\mathbf{1}_{\mathscr{C}})$.

Similarly to the unparametrized case, the sequences $(\mathsf{End}_F^{\mathscr{O}}(n))_{n\in\mathbb{N}}$ have canonical composition maps. We consider the binatural transformations to obtain maps from

$\mathsf{End}_F^{\mathcal{O}}(n)\hat{\otimes}\ \mathsf{End}_F^{\mathcal{O}}(k_1)\ \hat{\otimes}\cdots\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(k_n)$ to the internal morphism object with domain $F(C_{1,1})\hat{\otimes}\cdots\hat{\otimes}F(C_{n,k_n})$ and codomain

$$F(\mathcal{O}(n)\otimes(\mathcal{O}(k_1)\otimes C_{1,1}\otimes\cdots\otimes C_{1,k_1})\otimes\cdots\otimes(\mathcal{O}(k_n)\otimes C_{n,1}\otimes\cdots\otimes C_{n,k_n}))$$

for any $\sum_{i=1}^n k_i$-tuple of objects $(C_{1,1},\ldots,C_{n,k_n})$ in $\mathscr{C}$. We use the natural symmetry-isomorphism in the category $\mathscr{C}$ to collect the operad pieces $\mathcal{O}(n),\mathcal{O}(k_1),\ldots,\mathcal{O}(k_n)$ together. As the given composition in the operad $\mathcal{O}$ is natural with respect to the entries from $\mathscr{C}^n$, we can use it to define the desired map to

$$\mathsf{hom}\left(F(C_{1,1})\hat{\otimes}\cdots\hat{\otimes}F(C_{n,k_n}),\ F\left(\left(\mathcal{O}\left(\sum_{i=1}^n k_i\right)\otimes C_{1,1}\otimes\cdots\otimes C_{n,k_n}\right)\right)\right).$$

These maps are clearly binatural and hence give a composition map

$$\gamma:\mathsf{End}_F^{\mathcal{O}}(n)\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(k_1)\hat{\otimes}\cdots\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(k_n)\longrightarrow\mathsf{End}_F^{\mathcal{O}}\left(\sum_{i=1}^n k_i\right).$$

The action of the symmetric groups is defined as follows: As in the unparametrized case, we will specify the corresponding twisted binatural transformations. On an n-tuple $(C_1,\ldots,C_n)$ an element $\sigma\in\Sigma_n$ permutes the incoming entries $\sigma: F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n)\to F(C_{\sigma^{-1}(1)})\hat{\otimes}\cdots\hat{\otimes}F(C_{\sigma^{-1}(n)})$; and on $F(\mathcal{O}(n)\otimes C_{\sigma^{-1}(1)}\otimes\cdots\otimes C_{\sigma^{-1}(n)})$ we have a natural action given by $F(\sigma\otimes\sigma^{-1})$. Taking these together, we define the twisted structure maps as $\mathsf{hom}(\sigma, F(\sigma\otimes\sigma^{-1}))\circ w_{C_{\sigma^{-1}(1)},\ldots,C_{\sigma^{-1}(n)}}$ from $\mathsf{End}_F^{\mathcal{O}}(n)$ to $\mathsf{hom}(F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n), F(\mathcal{O}(n)\otimes C_1\otimes\cdots\otimes C_n))$.

The unit of the operads is easily defined. A morphism from the unit $\mathbf{1}_{\mathscr{D}}$ in $\mathscr{D}$ to $\mathsf{hom}(F(C), F(\mathcal{O}(1)\otimes C))$ corresponds by adjunction to a morphism in $\mathrm{Hom}_{\mathscr{D}}(F(C), F(\mathcal{O}(1)\otimes C))$. We define the unit in the parametrized case $\tilde{\eta}:\mathbf{1}_{\mathscr{D}}\to\mathsf{End}_F^{\mathcal{O}}(1)$ to be the unique map that is determined by the family of morphisms

$$\mathbf{1}_{\mathscr{D}}\longrightarrow\mathrm{Hom}_{\mathscr{D}}(F(C), F(\mathcal{O}(1)\otimes C)),$$

where the maps are induced by the identity map on the objects $F(C)$ decorated with the unit $\eta_{\mathcal{O}}$ of the operad $\mathcal{O}$, i.e.,

$$w_C^1\circ\tilde{\eta}: F(C)\cong F(\mathbf{1}_{\mathscr{C}}\otimes C)\xrightarrow{F(\eta_{\mathcal{O}}\otimes\mathrm{id}_C)}F(\mathcal{O}(1)\otimes C).$$

3.2. Verification of the operad property

The proof that $\mathsf{End}_F^{\mathcal{O}}$ is actually an operad, is quite ugly. It is obvious that the action of the symmetric groups interacts nicely with the composition and that the unit is actually a unit. The tricky point is the associativity of the composition. In the following, we denote the composition in $\mathsf{End}_F^{\mathcal{O}}$ by γ and the operad composition in $\mathcal{O}$ by $\gamma_{\mathcal{O}}$.

Fact 3.2.1. *The composition in the collection* $\{\mathsf{End}_F^{\mathcal{O}}(n)\}_{n\geqslant 0}$ *is associative.*

Proof. We have to prove, that the two possible ways of composition in $\mathsf{End}_F^{\mathcal{O}}$, $\gamma(\mathrm{id}; \gamma, \ldots, \gamma)$ and $\gamma(\gamma; \mathrm{id} \ldots, \mathrm{id})$, coincide as maps

$$\begin{aligned}&\mathsf{End}_F^{\mathcal{O}}(n)\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(m_1)\hat{\otimes}\cdots\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(m_n)\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(\ell_{1,1})\hat{\otimes}\cdots\hat{\otimes}\mathsf{End}_F^{\mathcal{O}}(\ell_{n,m_n})\\ &\quad\longrightarrow \mathsf{End}_F^{\mathcal{O}}\left(\sum \ell_{i,j}\right).\end{aligned}$$

In the following we will need various kinds of binatural transformations w^i from the ends $\mathsf{End}_F^{\mathcal{O}}(i)$ to the internal morphism objects hom in $\mathscr{D}$. Let $(C_{1,1,1}, \ldots, C_{n,m_n,\ell_{n,m_n}})$ be an arbitrary $\sum_{i,j}\ell_{i,j}$-tuple of objects in $\mathscr{C}$.

For the first operad composition $\gamma(\mathrm{id}; \gamma, \ldots, \gamma)$, the transformations involved are

(1) $w_{(C_{i,j,1},\ldots,C_{i,j,\ell_{i,j}})}$ from $\mathsf{End}_F^{\mathcal{O}}(\ell_{i,j})$ to

$$\mathsf{hom}(F(C_{i,j,1})\hat{\otimes}\cdots\hat{\otimes}F(C_{i,j,\ell_{i,j}}), F(\mathcal{O}(\ell_{i,j})\otimes C_{i,j,1}\otimes\cdots\otimes C_{i,j,\ell_{i,j}})).$$

(2) $w_{(\mathcal{O}(\ell_{i,1})\otimes C_{i,1,1}\otimes\cdots\otimes C_{i,1,\ell_{i,1}},\ldots,\mathcal{O}(\ell_{i,m_i})\otimes C_{i,m_i,1}\otimes\cdots\otimes C_{i,m_i,\ell_{i,m_i}})}$ which ends in the internal morphism object with target

$$F\left(\mathcal{O}(m_i)\otimes\mathcal{O}(\ell_{i,1})\otimes\left(\bigotimes_{j=1}^{\ell_{i,1}} C_{i,1,j}\right)\otimes\cdots\otimes\mathcal{O}(\ell_{i,m_i})\otimes\left(\bigotimes_{j=1}^{\ell_{i,m_i}} C_{i,m_i,j}\right)\right).$$

Then the operad composition shuffles the operad entries to the front and uses the operad composition $\gamma_{\mathcal{O}}$ in $\mathcal{O}$ to end up in terms like

$$F\left(\mathcal{O}\left(\sum_{j=1}^{m_i}\ell_{i,j}\right)\otimes C_{i,1,1}\otimes\cdots\otimes C_{i,m_i,\ell_{i,m_i}}\right).$$

(3) The final transformation from $\mathsf{End}_F^{\mathcal{O}}(n)$ in this case is

$$w_{\left(\mathcal{O}\left(\sum_{j=1}^{m_1}\ell_{1,j}\right)\otimes C_{1,1,1}\otimes\cdots\otimes C_{1,m_1,\ell_{1,m_1}},\ldots,\mathcal{O}\left(\sum_{j=1}^{m_n}\ell_{n,j}\right)\otimes C_{n,1,1}\otimes\cdots\otimes C_{n,m_n,\ell_{n,m_n}}\right)}.$$

This transformation is followed again by shuffle maps and the operad composition $\gamma_{\mathcal{O}}$ to end up in the internal morphism object

$$\mathsf{hom}\left(F(C_{1,1,1})\hat{\otimes}\cdots\hat{\otimes}F(C_{n,m_n,\ell_{n,m_n}}), F\left(\mathcal{O}\left(\sum\ell_{i,j}\right)\otimes C_{1,1,1}\otimes\cdots\otimes C_{n,m_n,\ell_{n,m_n}}\right)\right).$$

For the operad compositions which are used between the second and third step we use shuffle maps π_i on every single entry

$$F\left(\mathcal{O}(m_i)\otimes\mathcal{O}(\ell_{i,1})\otimes\bigotimes_{j=1}^{\ell_{i,1}} C_{i,1,j}\otimes\cdots\otimes\mathcal{O}(\ell_{i,m_i})\otimes\bigotimes_{j=1}^{\ell_{i,m_i}} C_{i,m_i,j}\right)$$

in order to bring the operad parts $\mathcal{O}(\ell_{i,j})$ next to $\mathcal{O}(m_i)$. The operad composition $\gamma_{\mathcal{O}}$ is then applied in every single entry as well.

The second operad composition $\gamma(\gamma; \mathrm{id} \ldots, \mathrm{id})$ uses

(1) the same binatural transformations from $\mathsf{End}_F^{\mathcal{O}}(\ell_{i,j})$ to

$$\mathsf{hom}(F(C_{i,j,1})\hat{\otimes} \cdots \hat{\otimes} F(C_{i,j,\ell_{i,j}}), F(\mathcal{O}(\ell_{i,j}) \otimes C_{i,j,1} \otimes \cdots \otimes C_{i,j,\ell_{i,j}}).$$

(2) From $\mathsf{End}_F^{\mathcal{O}}(m_i)$ to the internal morphism operad we get as well

$$w_{(F(\mathcal{O}(\ell_{i,1})\otimes C_{i,1,1}\otimes\cdots\otimes C_{i,1,\ell_{i,1}}),\ldots,F(\mathcal{O}(\ell_{i,m_i})\otimes C_{i,m_i,1}\otimes\cdots\otimes C_{i,m_i,\ell_{i,m_i}}))}.$$

But here the operad product is deferred to the third step.

(3) For this product we use the binatural transformation

$$w_{(\mathcal{O}(m_i)\otimes\mathcal{O}(\ell_{i,1})\otimes C_{i,1,1}\otimes\cdots\otimes C_{i,1,\ell_{i,1}}\otimes\mathcal{O}(\ell_{i,m_i})\otimes C_{i,m_i,1}\otimes\cdots\otimes C_{i,m_i,\ell_{i,m_i}}|i=1,\ldots,n)}$$

and afterwards the operad parts $\mathcal{O}(m_i)$ are shuffled to the front and

$$\gamma_{\mathcal{O}} : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \longrightarrow \mathcal{O}(m_1 + \cdots + m_n)$$

is applied. Finally a second shuffle map brings the parts $\mathcal{O}(\ell_{i,j})$ next to $\mathcal{O}(m_1+\cdots+m_n)$ and we apply $\gamma_{\mathcal{O}}$ to $\mathcal{O}(m_1 + \cdots + m_n)$ and all the $\mathcal{O}(\ell_{i,j})$.

Let w^n_{**} denote the binatural transformation from the second way of composition, i.e.,

$$w_{(\mathcal{O}(m_i)\otimes\mathcal{O}(\ell_{i,1})\otimes C_{i,1,1}\otimes\cdots\otimes C_{i,1,\ell_{i,1}}\otimes\mathcal{O}(\ell_{i,m_i})\otimes C_{i,m_i,1}\otimes\cdots\otimes C_{i,m_i,\ell_{i,m_i}}|i=1,\ldots,n)}.$$

Similarly, we denote the binatural transformation from the first way of composition by w^n_*. The defining property of an end ensures that

$$\begin{aligned}&\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}} \circ \pi_1 \otimes \cdots \otimes \gamma_{\mathcal{O}} \circ \pi_j)) \circ w^n_{**}\\ &\quad = \mathsf{hom}(F(\gamma_{\mathcal{O}} \circ \pi_1)\hat{\otimes} \cdots \hat{\otimes} F(\gamma_{\mathcal{O}} \circ \pi_j), \mathrm{id}) \circ w^n_*.\end{aligned} \tag{3.1}$$

The term on the right-hand side is precisely the part of the first composition where the first shuffles and compositions $\gamma_{\mathcal{O}}$ appear. Up to that stage, both compositions agree. But after that stage, the only difference between $\gamma(\mathrm{id}; \gamma, \ldots, \gamma)$ and $\gamma(\gamma; \mathrm{id} \ldots, \mathrm{id})$ can be described as the evaluation applied to $\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}}(\mathrm{id}; \gamma_{\mathcal{O}}, \ldots, \gamma_{\mathcal{O}})))$ on the one hand and $\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}}(\gamma_{\mathcal{O}}; \mathrm{id}, \ldots, \mathrm{id})))$ on the other hand as follows: Let π denote the final shuffle permutation in the composition, so that

$$\begin{aligned}&\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}} \circ \pi)) \circ \mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}} \circ \pi_1 \otimes \cdots \otimes \gamma_{\mathcal{O}} \circ \pi_j))\\ &\quad = \mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}}(\mathrm{id}; \gamma_{\mathcal{O}}, \ldots, \gamma_{\mathcal{O}}))).\end{aligned}$$

Therefore if we compose everything in Eq. (3.1) with $\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}} \circ \pi)$ we obtain that

$$\begin{aligned}&\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}}(\mathrm{id}; \gamma_{\mathcal{O}}, \ldots, \gamma_{\mathcal{O}})) \circ w^n_{*,*}\\ &\quad = \mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}} \circ \pi)) \circ \mathsf{hom}(F(\gamma_{\mathcal{O}} \circ \pi_1)\hat{\otimes} \cdots \hat{\otimes} F(\gamma_{\mathcal{O}} \circ \pi_j), \mathrm{id}) \circ w^n_{*}.\end{aligned}$$

Here, the right-hand side of the equation agrees with the first composition, and as the operad composition $\gamma_{\mathcal{O}}$ is associative the left-hand side equals

$$\mathsf{hom}(\mathrm{id}, F(\gamma_{\mathcal{O}}(\gamma_{\mathcal{O}}; \mathrm{id}, \ldots, \mathrm{id}))) \circ w^n_{*,*}$$

and this is precisely $\gamma(\gamma; \mathrm{id}, \ldots, \mathrm{id})$.

Therefore, both ways of composition give the same map from

$$\mathsf{End}^{\mathcal{O}}_F(n)\hat{\otimes}\mathsf{End}^{\mathcal{O}}_F(m_1)\hat{\otimes} \cdots \hat{\otimes}\mathsf{End}^{\mathcal{O}}_F(m_n)\hat{\otimes}\mathsf{End}^{\mathcal{O}}_F(\ell_{1,1})\hat{\otimes} \cdots \hat{\otimes}\mathsf{End}^{\mathcal{O}}_F(\ell_{n,m_n})$$

to $\mathsf{hom}(F(C_{1,1,1})\hat{\otimes} \cdots \hat{\otimes} F(C_{n,m_n,\ell_{n,m_n}}), F(\mathcal{O}(\sum \ell_{i,j}) \otimes C_{1,1,1} \otimes \cdots \otimes C_{n,m_n,\ell_{n,m_n}}))$ and this map is easily seen to be binatural. Therefore both compositions agree and give a well-defined map to $\mathsf{End}^{\mathcal{O}}_F\left(\sum \ell_{i,j}\right)$. □

Remark 3.2.2. Note that the operad composition in $\mathsf{End}^{\mathcal{O}}_F$ gives rise to an augmentation map $\mathsf{End}^{\mathcal{O}}_F(n) \to \mathsf{End}^{\mathcal{O}}_F(0) = F(\mathcal{O}(0))$. If F satisfies the strong unit condition 2.1 we can use the evaluation at $\mathbf{1}_{\mathcal{D}} \cong \mathbf{1}_{\mathcal{D}}^{\hat{\otimes} n} \cong F(\mathbf{1}_{\mathcal{C}})^{\hat{\otimes} n}$ to obtain a map

$$\mathsf{End}^{\mathcal{O}}_F(n) \cong \mathsf{End}^{\mathcal{O}}_F(n)\hat{\otimes}\mathbf{1}_{\mathcal{D}}^{\hat{\otimes} n} \longrightarrow F(\mathcal{O}(n) \otimes \mathbf{1}_{\mathcal{C}}^{\otimes n}) \cong F(\mathcal{O}(n)).$$

However, as F is not supposed to be (lax) symmetric monoidal, the image of an operad under F does not have to be an operad again.

3.3. Transfer of algebra structures over operads

In situations where one considers a (lax) symmetric monoidal functor, algebra structures over operads directly give operad structures on the image of the algebra. Parametrized endomorphism operads help to transfer operad structures on objects C in $\mathcal{C}$ to operad structures on $F(C)$ without restrictions on the monoidal properties of the functor F except the unit condition from (2.1) or (2.2).

Proposition 3.3.1. *If $C \in \mathcal{C}$ is an $\mathcal{O}$-algebra, then $F(C)$ has a natural structure of an algebra over* $\mathsf{End}^{\mathcal{O}}_F$.

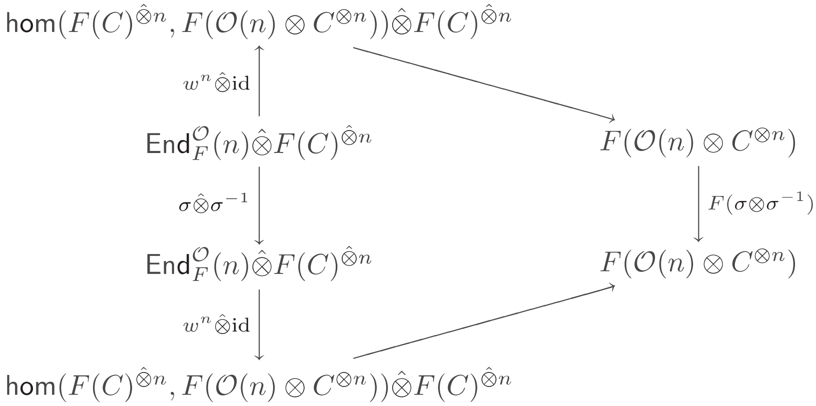
Proof. The structure map ξ of the operad action of $\mathsf{End}^{\mathcal{O}}_F$ on $F(C)$ is given by the composition

$$\begin{array}{ccc}\mathsf{End}^{\mathcal{O}}_F(n)\hat{\otimes}F(C)^{\hat{\otimes} n} & \xrightarrow{\psi} & F(\mathcal{O}(n) \otimes C^{\otimes n})\\ & \underset{\xi}{\searrow} & \downarrow F(\theta)\\ & & F(C).\end{array}$$

Here ψ is the composition of $w^n : \mathsf{End}_F^{\mathcal{O}}(n) \to \mathsf{hom}(F(C)^{\hat{\otimes} n}, F(\mathcal{O}(n) \otimes C^{\otimes n}))$ followed by the evaluation map from $\mathsf{hom}(F(C)^{\hat{\otimes} n}, F(\mathcal{O}(n) \otimes C^{\otimes n}))\hat{\otimes} F(C)^{\hat{\otimes} n}$ to $F(\mathcal{O}(n) \otimes C^{\otimes n})$, and θ is the action of $\mathcal{O}$ on C. As C is an $\mathcal{O}$-algebra there is an action map $\theta_0 : \mathcal{O}(0) \to C$. The map from $\mathsf{End}_F^{\mathcal{O}}(0) = F(\mathcal{O}(0))$ to $F(C)$ is therefore given by $F(\theta_0)$.

It is clear that the unit of the operad $\mathsf{End}_F^{\mathcal{O}}$ induces the unit action on C. The associativity of the action follows from some associativity properties of evaluation maps. We leave the details of this straightforward but tedious proof to the reader.

For the equivariance of the action, we have to show that the diagram



commutes.

For that, note that an action of a permutation $\sigma \in \Sigma_n$ on $\mathsf{End}_F^{\mathcal{O}}(n)$ results in an action of $\mathsf{hom}(\sigma, F(\sigma \otimes \sigma^{-1}))$ on the outcome of the binatural transformation w^n. Combined with the action of σ^{-1} on $F(C)^{\hat{\otimes} n}$ this leads to an action of $\mathsf{hom}(\mathrm{id}, F(\sigma \otimes \sigma^{-1}))$ on $\mathsf{hom}(F(C)^{\hat{\otimes} n}, F(\mathcal{O}(n) \otimes C^{\otimes n}))\hat{\otimes} F(C)^{\hat{\otimes} n}$, thus the diagram above commutes. The naturality of F and the fact that θ is an operad action on C yield $F(\theta) \circ F(\sigma \otimes \sigma^{-1}) = F(\theta \circ (\sigma \otimes \sigma^{-1})) = F(\theta)$. Consequently,

$$\begin{aligned} \xi \circ (\sigma\hat{\otimes}\sigma^{-1}) &= F(\theta) \circ \psi \circ (\sigma\hat{\otimes}\sigma^{-1}) = F(\theta) \circ F(\sigma \otimes \sigma^{-1}) \circ \psi \\ &= F(\theta \circ (\sigma \otimes \sigma^{-1})) \circ \psi = F(\theta) \circ \psi = \xi. \qquad \square \end{aligned}$$

Note that for any operad $\mathcal{O}$ the sequence $(F\mathcal{O}(n))_n$ is a graded algebra over $\mathsf{End}_F^{\mathcal{O}}$: The evaluation $\mathsf{End}_F^{\mathcal{O}}(n)\hat{\otimes} F\mathcal{O}(m_1)\hat{\otimes} \cdots \hat{\otimes} F\mathcal{O}(m_n) \to F(\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n))$ can be prolonged with F applied to the operad composition to yield an action map to $F\mathcal{O}(\sum_{i=1}^n m_i)$.

4. Quillen adjunctions

4.1. Adjoints to F on algebras over operads

In order to talk about adjunctions between categories of algebras over operads we will assume that the categories $\mathscr{C}$ and $\mathscr{D}$ posses sums and coequalizers. Recall from Section 2 that we assume that $\mathscr{C}$ and $\mathscr{D}$ are closed, therefore the functors $C\otimes$—respectively $D\hat{\otimes}$—preserve colimits for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$. We can associate a monad O to every operad $\mathscr{O}$ in $\mathscr{C}$ and every $\mathscr{O}$-algebra $C \in \mathscr{C}$ can be written as a coequalizer in the following way:

$$\mathsf{OO}(C) \rightrightarrows \mathsf{O}(C) \longrightarrow C.$$

Let us denote the monad associated to the operad $\mathsf{End}_F^{\mathscr{O}}$ by $\mathsf{E}_F^{\mathscr{O}}$. We also assume, that F possesses a left adjoint $G : \mathscr{D} \to \mathscr{C}$. The question is, whether we can construct a left adjoint $L_F^{\mathscr{O}}$ to F from the category of $\mathsf{End}_F^{\mathscr{O}}$-algebras to the category of $\mathscr{O}$-algebras

$$L_F^{\mathcal{O}} : \mathsf{End}_F^{\mathcal{O}}\text{-}\mathsf{alg} \rightleftarrows \mathcal{O}\text{-}\mathsf{alg} : F$$

There is a standard procedure to construct such a functor.

Proposition 4.1.1. *The functor F from $\mathscr{O}$-algebras in $\mathscr{C}$ to $\mathsf{End}_F^{\mathscr{O}}$-algebras in $\mathscr{D}$ has a left adjoint for every operad $\mathscr{O}$ in $\mathscr{C}$.*

Proof. In the following we will omit the forgetful functor from algebras over an operad to the underlying category

$$\begin{array}{ccc} \mathsf{End}_F^{\mathcal{O}}\text{-}\mathsf{alg} & \overset{L_F^{\mathcal{O}}}{\underset{F}{\rightleftarrows}} & \mathcal{O}\text{-}\mathsf{alg} \\ {\scriptstyle \mathsf{O}}\uparrow & & \uparrow{\scriptstyle \mathsf{E}_F^{\mathcal{O}}} \\ \mathcal{D} & \overset{G}{\underset{F}{\rightleftarrows}} & \mathcal{C}. \end{array}$$

It is clear that $L_F^{\mathscr{O}}$ applied to a free $\mathsf{End}_F^{\mathscr{O}}$-algebra $\mathsf{E}_F^{\mathscr{O}}(X)$ on an object $X \in \mathscr{D}$ has to be defined as $\mathsf{O}(G(X))$, because the adjunction property dictates

$$\begin{aligned} \mathrm{Hom}_{\mathsf{End}_F^{\mathscr{O}}\text{-}\mathsf{alg}}(\mathsf{E}_F^{\mathscr{O}}(X), F(B)) &\cong \mathrm{Hom}_{\mathscr{D}}(X, F(B)) \\ &\cong \mathrm{Hom}_{\mathscr{C}}(G(X), B) \cong \mathrm{Hom}_{\mathscr{O}\text{-}\mathsf{alg}}(\mathsf{O}(G(X)), B). \end{aligned}$$

As the functor $L_F^{\mathscr{O}}$ should become a left adjoint, it has to respect colimits. Thus, for an arbitrary $\mathsf{End}_F^{\mathscr{O}}$-algebra A we can define $L_F^{\mathscr{O}}(A)$ by the following coequalizer diagram:

$$L_F^{\mathscr{O}}(\mathsf{E}_F^{\mathscr{O}}(\mathsf{E}_F^{\mathscr{O}}(A))) = (\mathsf{O}G(\mathsf{E}_F^{\mathscr{O}}(A))) \rightrightarrows L_F^{\mathscr{O}}(\mathsf{E}_F^{\mathscr{O}}(A)) = \mathsf{O}G(A) \longrightarrow L_F^{\mathscr{O}}(A).$$

The maps in this diagram arise from the structure map of the $\mathsf{End}_F^{\mathscr{O}}$-algebra A from $\mathsf{E}_F^{\mathscr{O}}(A)$ to A and the second horizontal arrows on the left-hand side is given by the monad structure of $\mathsf{E}_F^{\mathscr{O}}$, namely $\mathsf{E}_F^{\mathscr{O}}(\mathsf{E}_F^{\mathscr{O}}(A)) \to \mathsf{E}_F^{\mathscr{O}}(A)$ via the composition $\mathsf{E}_F^{\mathscr{O}} \circ \mathsf{E}_F^{\mathscr{O}} \to \mathsf{E}_F^{\mathscr{O}}$ in the monad. The coequalizer diagram is split,

$$\mathsf{E}_F^{\mathcal{O}}(\mathsf{E}_F^{\mathcal{O}}(A)) \rightrightarrows \mathsf{E}_F^{\mathcal{O}}(A) \longrightarrow A$$

and therefore after applying $\mathrm{Hom}_{\mathsf{End}_F^{\mathscr{O}}\text{-}\mathsf{alg}}(-, B)$ the resulting diagram is a split equalizer. The two equalizers $\mathrm{Hom}_{\mathscr{O}\text{-}\mathsf{alg}}(L_F^{\mathscr{O}}(A), B)$ and $\mathrm{Hom}_{\mathsf{End}_F^{\mathscr{O}}\text{-}\mathsf{alg}}(A, F(B))$ have therefore to be isomorphic. □

We postpone discussions about (semi) model structures on operads and their algebras Section 8. Let us assume the following properties of $\mathscr{C}$ and $\mathscr{D}$

(1) The categories $\mathscr{C}$, $\mathscr{D}$ are cofibrantly generated model categories.
(2) The categories of operads in $\mathscr{C}$ and $\mathscr{D}$ posses (semi) model category structures as defined in 8.1.1 or 8.3.3.
(3) The categories of algebras over cofibrant operads in $\mathscr{C}$ and $\mathscr{D}$ posses a (semi) model structure. An alternative to (3) can be.
(3′) The categories of algebras over operads with underlying cofibrant symmetric sequence (compare Definition 8.0.3) posses a model structure.

In our situation, we consider the category of $\mathscr{O}$-algebras for some operad $\mathscr{O}$. We replace $\mathscr{O}$ by a weakly equivalent operad $Q\mathscr{O}$ which is cofibrant or whose underlying symmetric sequence is cofibrant in order to apply Theorem 8.1.3 or 8.1.2.

We know, that F maps $Q\mathscr{O}$-algebras to algebras over $\mathsf{End}_F^{Q\mathscr{O}}$. Depending on the situation we replace that operad again by $Q\mathsf{End}_F^{Q\mathscr{O}}$ where this replacement is either cofibrant or has at least an underlying cofibrant symmetric sequence. In order to ease notation we abbreviate $Q\mathscr{O}$ to $\mathscr{E}$ and $Q\mathsf{End}_F^{Q\mathscr{O}}$ to $\mathscr{E}'$. Note that every $\mathsf{End}_F^{Q\mathscr{O}}$-algebra is an $\mathscr{E}'$-algebra by means of the replacement map $\mathscr{E}' \to \mathsf{End}_D^{\mathscr{E}}$. The category of $\mathscr{E}'$-algebras is another semi-model category and the construction above gives an adjoint functor pair between these categories. The fibrations and acyclic fibrations are determined by the forgetful functors $U : \mathscr{E}\text{-alg} \to \mathscr{C}$ and $U' : \mathscr{E}'\text{-alg} \to \mathscr{D}$.

We will first discuss the general case of $\mathscr{O}$-algebras and show, that the functor F together with its left adjoint $L_F^{\mathscr{O}}$ gives rise to a Quillen adjoint pair, if the original adjunction (G, F) has been a Quillen pair already. Later in 5.7.1 and 6.5.1, we will deal with the examples off the functors involved in the Dold–Kan correspondence, i.e., the normalization adjunction (N, D) and the conormalization adjunction (D^*, N^*). In these examples the situation is in fact so nice, that *all* operads in $\mathscr{D}$, though not in $\mathscr{C}$, posses (genuine) model structures (cf. 8.3.4). In particular, we can use the corresponding parametrized endomorphism operads $\mathsf{End}_D^{\mathscr{O}}$ resp. $\mathsf{End}_{N^*}^{\mathscr{O}}$ for the Quillen adjunction instead of their cofibrant replacements.

4.2. *Quillen adjunction on the level of algebras over operads*

Assume that our adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is a Quillen pair, i.e., that F preserves fibrations and acyclic fibrations.

Our claim is that the functor F is part of a Quillen adjunction on the level of algebras for every operad $\mathscr{E}$ as above. Let E denote the associated monad to the operad $\mathscr{E}$ and similarly let E' be the monad corresponding to the operad $\mathscr{E}'$.

Theorem 4.2.1. *The adjunction* $(L_F^{\mathscr{E}}, F)$

$$L_F^{\mathcal{E}} : \mathcal{E}'\text{-alg} \rightleftarrows \mathcal{E}\text{-alg} : F$$

is a Quillen adjoint pair.

Proof. We will show that the functor F preserves fibrations and trivial fibrations. Let $f : A \to B$ be a fibration of $\mathscr{E}$-algebras. Then the map $U(f)$ on underlying objects is a fibration in the model category $\mathscr{C}$. But the functor F is part of the Quillen adjunction $G : \mathscr{C} \rightleftarrows \mathscr{D} : F$ and in this role as a right Quillen functor it perserves fibrations and acyclic fibrations in these model structures. So the only thing that is to check is, that $F(U(f))$ gives rise to a map of $\mathscr{E}'$-algebras.

We will check, that $F(U(f))$ is a map of algebras over the operad $\mathsf{End}_F^{\mathscr{E}}$; as $\mathscr{E}' \to \mathsf{End}_F^{\mathscr{E}}$ is a map of operads, the claim then follows. This procedure is legitimate, because we start with two $\mathsf{End}_F^{\mathscr{E}}$-algebras FA and FB, so the $\mathscr{E}'$-algebra structure on both of them is induced by the map $\mathscr{E}' \to \mathsf{End}_F^{\mathscr{E}}$.

The identity map on the operad $\mathsf{End}_F^{\mathscr{E}}$ tensorized with an n-fold tensor product of $F(f)$ yields a morphism

$$\mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n}(\mathscr{E}(n) \otimes -))\hat{\otimes}(F(A)^{\hat{\otimes} n}) \longrightarrow \mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n}(\mathscr{E}(n) \otimes -))\hat{\otimes}(F(B)^{\hat{\otimes} n}).$$

The operad action of the endomorphism operad gives maps on each term to $F(\mathscr{E}(n) \otimes A^{\otimes n})$ and $F(\mathscr{E}(n) \otimes B^{\otimes n})$ and by the very definition of this operad, these action maps are natural in A and B. On this level f induces a map

$$F(\mathrm{id} \otimes f \otimes \cdots \otimes f) : F(\mathscr{E}(n) \otimes A^{\otimes n}) \longrightarrow F(\mathscr{E}(n) \otimes B^{\otimes n}).$$

Thus the naturality of the action map makes the following diagram commute:

$$\begin{array}{ccc} \mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n}(\mathscr{E}(n) \otimes -))\hat{\otimes}(FA^{\hat{\otimes} n}) & \xrightarrow{\mathrm{id}\hat{\otimes} Ff^{\hat{\otimes} n}} & \mathsf{nat}(F^{\hat{\otimes} n}, F^{\otimes n}(\mathscr{E}(n) \otimes -))\hat{\otimes}(FB^{\hat{\otimes} n}) \\ \downarrow & \circlearrowleft & \downarrow \\ F(\mathscr{E}(n) \otimes A^{\otimes n}) & \xrightarrow[F(\mathrm{id} \otimes f^{\otimes n})]{} & F(\mathscr{E}(n) \otimes B^{\otimes n}) \end{array}$$

As f was assumed to be an $\mathscr{E}$-algebra map we can stack the above diagram on the following:

$$\begin{array}{ccc} F(\mathscr{E}(n) \otimes A^{\otimes n}) & \xrightarrow{F(\mathrm{id} \otimes f^{\otimes n})} & F(\mathscr{E}(n) \otimes B^{\otimes n}) \\ \downarrow & \circlearrowleft & \downarrow \\ FA & \xrightarrow[Ff]{} & FB. \end{array}$$

Then the underlying map of the morphism of $\mathscr{E}'$-algebras $U'F(f)$ is the same as $FU(f)$ and thus $F(f)$ is a fibration. That $F(-)$ preserves acyclic fibrations follows by the same sort of argument. □

4.3. A Quillen adjunction for homotopy algebras

We want to obtain a similar result as above for the functor F applied to a general homotopy $\mathcal{O}$-algebra, i.e., a $Q\mathcal{O}$-algebra such that $Q\mathcal{O}$ is a cofibrant replacement of $\mathcal{O}$ in the (semi) model category structure of operads in $\mathcal{C}$

$$* \rightarrowtail Q\mathcal{O} \xrightarrow{\sim} \mathcal{O}.$$

Similarly, in the category $\mathcal{D}$, we take a cofibrant replacement of the parametrized endomorphism operad $\mathsf{End}_F^{Q(\mathcal{O})}$ or a replacement with underlying cofibrant symmetric sequence

$$* \rightarrowtail Q(\mathsf{End}_F^{Q(\mathcal{O})}) \xrightarrow{\sim} \mathsf{End}_F^{Q(\mathcal{O})}.$$

Then it is straightforward to see, that the statement of Theorem 4.2.1 transfers to our situation and we obtain an adjunction on the level of homotopy categories.

Theorem 4.3.1. *The functor* $F : Q(\mathcal{O})$-algebras $\to Q(\mathsf{End}_F^{Q(\mathcal{O})})$-algebras *possesses a left adjoint*, $L_F^{\mathcal{O}}$, *and this adjoint pair is a Quillen adjunction*.

As the functor F is not lax symmetric monoidal, $F(\mathcal{O})$ is no operad in general and the operad $Q(\mathsf{End}_F^{Q(\mathcal{O})})$ will not be weakly equivalent to $F(\mathcal{O})$ for arbitrary functors F. We will later consider examples, however, where this is the case and where Theorem 4.3.1 above gives an actual statement about homotopy algebras.

4.4. Maps from the operad of associative monoids

So far, we did not assume that the functor $F : \mathcal{C} \to \mathcal{D}$ preserves the monoidal structures in $\mathcal{C}$ and $\mathcal{D}$. But if F is at least a lax monoidal functor, we can transfer more algebra structures to the images of algebras over operads than in the general case.

For every symmetric monoidal closed category $\mathcal{C}$, the adjunction for the internal homomorphism object $\mathsf{hom}_{\mathcal{C}}(-,-)$ gives a bijection

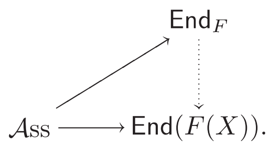
$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(C, C) &\cong \mathrm{Hom}_{\mathcal{C}}(C \otimes \mathbf{1}_{\mathcal{C}}, C)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(C, \mathsf{hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, C))\end{aligned}$$

and therefore the identity morphism on each object $C \in \mathcal{C}$ gives rise to a map from C to $\mathsf{hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, C)$. Using [2, 6.1.7] one sees that the composition with the forgetful functor $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, -)$ from $\mathcal{C}$ to sets sends $\mathsf{hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, C)$ to $\mathrm{Hom}(\mathbf{1}_{\mathcal{C}}, C)$, i.e., each object C gives rise to a natural morphism in $\mathcal{C}$ from the unit $\mathbf{1}_{\mathcal{C}}$ to C. We will denote this map by u_C.

Theorem 4.4.1. *Assume that the functor F is lax monoidal*.

(1) *The generalized endomorphism operad* End_F *possesses an operad map from the operad* $\mathcal{A}$ss *in* $\mathcal{D}$.
(2) *If an operad* $\mathcal{O}$ (*with* Σ-*action*) *has an operad map from the operad of associative monoids* $\mathcal{A}$ss in $\mathcal{C}$, *then there is a map of operads from* $\mathcal{A}$ss *in* $\mathcal{D}$ *to* $\mathsf{End}_F^{\mathcal{O}}$.

It is clear, that every image of a commutative monoid X under F is associative, so there is an action of the operad $\mathcal{A}\mathrm{ss}$ on $F(X)$; but we claim that this action factors over the operad End_F:



Proof of theorem 4.4.1. By assumption, F is lax monoidal, therefore there is a natural transformation $\Upsilon_2 : F(C)\hat{\otimes}F(D) \to F(C \otimes D)$ which obeys the associativity coherence conditions from [2, 6.27,6.28].

Note that by adjunction the morphisms from $\mathbf{1}_{\mathscr{D}}$ to $\mathsf{hom}(F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n),\ F(C_1 \otimes \cdots \otimes C_n))$ are in bijection with the morphisms in $\mathscr{D}$ from $F(C_1)\hat{\otimes}\cdots\hat{\otimes}\, F(C_n)$ to $F(C_1 \otimes \cdots \otimes C_n)$ for any n-tuple $(C_1, \ldots, C_n) \in \mathscr{C}^n$. The operad $\mathscr{A}\mathrm{ss}$ in the category $\mathscr{D}$ in degree n consists of the group ring $\mathbf{1}_{\mathscr{D}}[\Sigma_n] \cong \coprod_{\sigma\in\Sigma_n} \mathbf{1}_{\mathscr{D}}(\sigma)$ and we first specify the image of the component $\mathbf{1}_{\mathscr{D}}(\mathrm{id}_n)$ with $\mathrm{id}_n \in \Sigma_n$. Define the $(n-1)$-fold iteration

$$\Upsilon_n := \Upsilon_2 \circ (\Upsilon_2\hat{\otimes}\mathrm{id}) \circ \cdots \circ (\Upsilon_2\hat{\otimes}\mathrm{id}^{\hat{\otimes} n-2}).$$

Applied to $(C_1, \cdots, C_n)$ this gives a natural morphism in $\mathscr{D}$ from $F(C_1)\hat{\otimes}\cdots\hat{\otimes}\, F(C_n)$ to $F(C_1 \otimes \cdots \otimes C_n)$. We send the component $\mathbf{1}_{\mathscr{D}}(\sigma)$ of $\sigma \in \Sigma_n$ to the element in $\mathsf{hom}(F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n),\ F(C_1 \otimes \cdots \otimes C_n))$ which is uniquely determined by $\Upsilon_n.\sigma$. By the universal property of the end End_F, this gives maps $\mathscr{A}\mathrm{ss}(n) \longrightarrow \mathsf{End}_F(n)$ for all n which together yield a map of operads from $\mathscr{A}\mathrm{ss}$ to End_F.

If we start with an operad $\mathscr{O}$ which comes equipped with an operad map $\varphi : \mathscr{A}\mathrm{ss} \longrightarrow \mathscr{O}$, then we obtain a map $\phi : \mathscr{A}\mathrm{ss} \longrightarrow \mathsf{End}_F^{\mathscr{O}}$ in the following way. The map φ has as an nth component a map $\varphi(n) : \mathbf{1}_{\mathscr{C}}[\Sigma_n] \to \mathscr{O}(n)$ whose values are determined by $\varphi(n)$ applied to the component $\mathbf{1}_{\mathscr{C}}(\mathrm{id}_n) \in \mathbf{1}_{\mathscr{C}}[\Sigma_n]$ of the identity permutation in Σ_n. For any n-tuple $(C_1, \ldots, C_n) \in \mathscr{C}^n$ we choose as a morphism in $\mathscr{D}$ from $F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n)$ to $F(\mathscr{O}(n)\otimes C_1\otimes\cdots\otimes C_n)$ the composition $F(\varphi(n)|\mathbf{1}_{\mathscr{C}}(\mathrm{id}_n)\otimes\mathrm{id})\circ\Upsilon_n$ applied to $(C_1, \ldots, C_n)$. We have to show that this gives a well-defined map, if we send the copy $\mathbf{1}_{\mathscr{D}} \cong \mathbf{1}_{\mathscr{D}}(\sigma)$ for $\sigma \in \Sigma_n$ via ϕ to the morphism $(F(\varphi(n)|\mathbf{1}_{\mathscr{C}}(\mathrm{id}_n) \otimes \mathrm{id}) \circ \Upsilon_n).\sigma$ in $\mathscr{D}$. By the very definition of the Σ_n-action this morphism is

$$\mathrm{Hom}(\sigma, F(\sigma \otimes \sigma^{-1})) \circ F(\varphi(n)|\mathbf{1}_{\mathscr{C}}(\mathrm{id}_n) \otimes \mathrm{id}) \circ \Upsilon_n.$$

In terms of natural transformations, this maps $F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n)$ via σ to $F(C_{\sigma^{-1}(1)})\hat{\otimes}\cdots\hat{\otimes}F(C_{\sigma^{-1}(n)})$, applies then Υ_n which lands in $F(C_{\sigma^{-1}(1)} \otimes \cdots \otimes C_{\sigma^{-1}(n)})$. With $F(\varphi(n)|\mathbf{1}_{\mathscr{C}}(\mathrm{id}_n) \otimes \mathrm{id})$ this is transferred to $F(\mathscr{O}(n) \otimes C_{\sigma^{-1}(1)} \otimes \cdots \otimes C_{\sigma^{-1}(n)})$. Finally, the term $F(\sigma \otimes \sigma^{-1})$ brings the $C_{\sigma^{-1}(i)}$ in the old order and acts on the operad entry. As $(\varphi(n)|\mathbf{1}_{\mathscr{C}}(\mathrm{id}_n)).\sigma$ is precisely $\varphi(n)|\mathbf{1}_{\mathscr{C}}(\sigma)$, the claim follows. □

4.5. A homotopy Gerstenhaber structure for End_F

Gerstenhaber and Voronov describe in [7] a criterium which ensures that an operad $\mathscr{O}$ (in vector spaces) receives an operad map from the operad of associative monoids $\mathscr{A}\mathrm{ss}$:

the operad $\mathcal{O}$ has to have a multiplication $m \in \mathcal{O}(2)$. Let us denote the composition in the operad by γ. In order to qualify for a multiplication $m \in \mathcal{O}(2)$ must satisfy

$$m \circ m = 0 \quad \text{with} \quad m \circ m = \gamma(m; \mathrm{id}, m) - \gamma(m; m, \mathrm{id}),$$

i.e., the associator of m is trivial.

Theorem 4.5.1 (*Gerstenhaber and Voronov [7, Theorem 3.4]*). *A multiplication m on an operad $\mathcal{O}(n)$ in vector spaces defines the structure of a homotopy G-algebra on $\bigoplus_{n \geqslant 0} \mathcal{O}(n)$.*

The homotopy G-structure (see [7, Definition 2] for the precise definition) consists of a product, braces and a differential. These data are easily defined with the help of γ and m. For $w \in \mathcal{O}(n)$ let $|w|$ be n. Then the braces are defined as

$$w\{w_1, \ldots, w_n\} = \sum (-1)^{\varepsilon} \gamma(w; \mathrm{id}, \ldots, \mathrm{id}, w_1, \mathrm{id}, \ldots, \mathrm{id}, w_n, \mathrm{id}, \ldots, \mathrm{id}), \tag{4.1}$$

where the sum is taken over all possibilities to insert the w_i into the operad composition with the restriction that w_i appears before w_{i+1} and ε is an appropriate sign depending on the positions of the w_i.

The multiplication in $\bigoplus_n \mathcal{O}(n)$ is defined via m:

$$v \bullet w := (-1)^{|v|+1} \gamma(m; v, w) \quad \text{for all} \quad v, w \in \bigoplus_n \mathcal{O}(n) \tag{4.2}$$

and the differential of an element w is

$$d(w) = m \circ w - (-1)^{|w|} w \circ m. \tag{4.3}$$

Assuming that the functor $F : \mathcal{C} \to \mathcal{D}$ is lax monoidal, we obtain a canonical multiplication element in $\mathsf{End}_F(2)$ induced by the given natural transformation $\Upsilon_2 : F(-)\hat{\otimes} F(-) \longrightarrow F(- \otimes -)$. If $\mathcal{C}$ and $\mathcal{D}$ are abelian symmetric monoidal categories with coproducts, such that the coproducts are distributive with respect to the monoidal structure, then we can form the graded object associated to End_F,

$$\bigoplus_{n \geqslant 0} \mathsf{End}_F(n).$$

By a homotopy G-structure on that we understand that $\bigoplus \mathsf{End}_F(n)$ has braces, a multiplication and a differential as in (4.1)–(4.3).

Theorem 4.5.2. *With $\mathcal{C}$ and $\mathcal{D}$ as above and F being lax monoidal, the graded object $\bigoplus_n \mathsf{End}_F(n)$ has a structure of a homotopy G-algebra. If $\mathcal{O}$ is an operad in $\mathcal{C}$ with a map $\mathcal{A}ss \to \mathcal{O}$ then $\bigoplus_n \mathsf{End}_F^{\mathcal{O}}(n)$ is a homotopy G-algebra in $\mathcal{D}$.*

5. The inverse of the normalization

The classical Dold–Kan correspondence says that the normalization functor N from simplicial modules to differential graded modules is an equivalence of categories with inverse D

$$N : \mathsf{smod} \rightleftarrows \mathsf{dgmod} : D.$$

In particular the functor N is a left adjoint to D. The value of N on a simplicial k-module $X_\bullet$ in chain degree n is

$$N_n(X_\bullet) = \bigcap_{i=0}^{n-1} \ker(d_i : X_n \longrightarrow X_{n-1}),$$

where the d_i are the simplicial structure maps. The differential $d : N_n(X_\bullet) \to N_{n-1}(X_\bullet)$ is given by the remaining face map $(-1)^n d_n$.

For two arbitrary simplicial k-modules A and B let $A \hat{\otimes} B$ denote the degree-wise tensor product of A and B, i.e., $(A \hat{\otimes} B)_n = A_n \otimes_k B_n$. Here, the simplicial structure maps are applied in each component; in particular, the differential on $N_*(A \hat{\otimes} B)$ in degree n is $(-1)^n (d_n \otimes d_n)$.

On differential graded modules we take the usual monoidal structure with the tensor product of two chain complexes $(C^1_* \otimes C^2_*)_n = \bigoplus_{p+q=n} C^1_p \otimes C^1_q$ with differential $d(c^1 \otimes c^2) = d_{C^1}(c^1) \otimes c^2 + (-1)^{|c^1|} c^1 \otimes d_{C^2}(c^2)$.

Note that the functor D is compatible with the units in the monoidal structures on differential graded modules and simplicial modules in the sense of condition (2.1): it sends the chain complex $(k, 0)$ which has the ground ring k in dimension zero and is trivial in all other dimensions to the constant simplicial module $\underline{k}$ which is k in every simplicial degree with the identity on k as structure maps.

In [21] we proved that the functor D sends differential graded commutative algebras to algebras over an E_∞-operad. In fact, we showed that the endomorphism operad End_D of D is acyclic.

5.1. A left adjoint for D

Using 4.1.1 we know that the functor D has a left adjoint $L_D = L_D^{\mathscr{C}\mathrm{om}}$ from the category of End_D-algebras End_D-alg to the category of differential graded commutative algebras dgca:

$$L_D : \mathsf{End}_D\text{-}\mathsf{alg} \rightleftarrows \mathsf{dgca} : D$$

We denote the monad corresponding to the operad End_D by E_D. The functor which assigns the symmetric algebra on V to a differential graded module V is denoted by S. Then L_D applied to a free End_D-algebra $\mathsf{O}(X)$ on a simplicial module X has to be defined as $S(N(X))$ and for a general End_D-algebra A, $L_D(A)$ is given by the following coequalizer diagram:

$$L_D(\mathsf{O}(\mathsf{O}(A))) = (SN(\mathsf{O}(A))) \rightrightarrows L_D(\mathsf{O}(A)) = SN(A) \to L_D(A).$$

5.2. The generalized endomorphism operad of D

We briefly recall the explicit form of End_D: on an n-tuple of chain complexes $(C_1, \ldots, C_n)$ the functor $D^{\hat{\otimes} n}$ takes the external tensor product of the terms where D is applied to each single C_i, i.e., $D^{\hat{\otimes} n}(C_1, \ldots, C_n) = D(C_1)\hat{\otimes} \cdots \hat{\otimes}\, D(C_n)$. The functor $D^{\otimes n}$ is D applied to the internal tensor product of the differential graded modules, that is, $D^{\otimes n}(C_1, \ldots, C_n) = D(C_1 \otimes \cdots \otimes C_n)$. In this case, the generalized endomorphism operad End_D of D is explicitly given as follows: in simplicial degree ℓ the nth operad part consists of the natural transformations from $D^{\hat{\otimes} n}\hat{\otimes} k[\Delta_\ell]$ to $D^{\otimes n}$

$$\mathsf{End}_D(n)_\ell = \mathrm{Nat}(D^{\hat{\otimes} n}\hat{\otimes} k[\Delta_\ell], D^{\otimes n}).$$

We proved in [21, 4.1] that this operad (which was baptized $\mathcal{O}_D$ in [21]) has an augmentation to the operad which codifies commutative simplicial rings and this augmentation map is a weak equivalence.

5.3. *The parametrized versions of* End_D

We will consider a parametrized version of the generalized endomorphism operad End_D. Let $\mathcal{O}$ be an arbitrary operad in the category of differential graded modules. By results from the previous section we know:

Proposition 5.3.1. *If X is a non-negative chain complex, which is an $\mathcal{O}$-algebra then $D(X)$ is an algebra over the parametrized endomorphism operad $\mathsf{End}_D^{\mathcal{O}}$.*

In general, this result is a strict implication. For a typical algebra A over the operad $\mathsf{End}_D^{\mathcal{O}}$ the normalization $N(A)$ is in general no algebra over $\mathcal{O}$. For instance for every differential graded commutative algebra the image under D is an algebra over $\mathsf{End}_D \cong \mathsf{End}_D^{\mathcal{C}om}$, but for instance the normalization of a free End_D-algebra will not be strictly commutative.

5.4. E_∞*-structures are preserved by D*

As there are many different notions of E_∞ operads in the literature, let us specify what we mean by that. Let us assume, that $\mathcal{C}$ is a symmetric monoidal model category $\mathcal{C}$ which is cofibrantly generated (see [10, 2.1.3]). Then $\mathcal{C}$ has a canonical model category structure on its related category of symmetric sequences, i.e., on sequences $(C_0, C_1, \ldots)$ where each C_n has an action of the symmetric group Σ_n. The model structure is such that a map f of symmetric sequences is a weak equivalence resp. a fibration if each map $f_n : C_n \to C_n'$ is a weak equivalence resp. a fibration in $\mathcal{C}$ (see for instance [1, Section 3]). We discuss this in more detail in Section 8 in 8.0.3.

Definition 5.4.1. An operad $\mathcal{O}$ in $\mathcal{C}$ is called an E_∞ *operad*, if

(1) its zeroth term $\mathcal{O}(0)$ is isomorphic to $\mathbf{1}_{\mathcal{C}}$ and the augmentation

$$\varepsilon : \mathcal{O}(n) \cong \mathcal{O}(n) \otimes \mathcal{O}(0)^{\otimes n} \xrightarrow{\gamma} \mathcal{O}(0)$$

is a weak equivalence, and if

(2) its underlying symmetric sequence $(\mathcal{O}(n))_{n \geqslant 0}$ is cofibrant.

The fact which enables us to prove a comparison result is that the functor D is able to convert E_∞-algebras in the differential graded framework into E_∞-algebras in the category of simplicial modules. It remains to be shown that the parametrized version is again an E_∞-operad. For now, we drop the assumption that the underlying symmetric sequence is cofibrant, but we will force this condition later by using cofibrant replacements. So we have to prove:

Theorem 5.4.2. *For any E_∞-operad $\mathcal{O}$ the operad $\mathsf{End}_D^{\mathcal{O}}$ is weakly equivalent to the operad of commutative monoids via its augmentation map.*

Proof. In the proof of [21, Theorem 4.1] we identified the operad End_D with the total space of a simplicial–cosimplicial gadget. Similarly $\mathsf{End}_D^{\mathcal{O}}$ can be expressed this way as

$$\mathsf{End}_D^{\mathcal{O}}(n) \cong \mathsf{Tot}\ \underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n}(\mathcal{O}(n) \otimes -)).$$

Here $\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})_{(\ell,m)}$ are the natural transformations of functors from the n-fold product of the category of differential graded modules, dgmod^n, to the category of abelian groups, from $D^{\hat{\otimes} n}$ in degree m to $D^{\otimes n}$ in degree ℓ,

$$\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})_{(\ell,m)} = \mathrm{Nat}(D_m^{\hat{\otimes} n}, D_\ell^{\otimes n}).$$

We can use the Bousfield–Kan spectral sequence for the tower of fibrations from [3, IX, Section 4] (see also [8, VIII, Section 1]) belonging to the skeleton filtration of the total space to calculate the homotopy groups of our operad. The E_2-page looks as follows:

$$E_2^{p,q} = \pi^p \pi_q\ \underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n}(\mathcal{O}(n) \otimes -)).$$

In order to identify this E_2-term we use the Yoneda lemma for multilinear functors [21, Lemma 4.2]. The functor D is representable as $\mathrm{Hom}_{\mathsf{dgmod}}(N(k\Delta_\ell), X) = D(X)_\ell$ and therefore we can rewrite $\underset{\sim}{\mathsf{nat}}(D_\ell^{\hat{\otimes} n}, D^{\otimes n}(\mathcal{O}(n) \otimes -)_m)$ as

$$\underset{\sim}{\mathsf{nat}}(D_\ell^{\hat{\otimes} n}, D^{\otimes n}(\mathcal{O}(n) \otimes -)_m) \cong D(\mathcal{O}(n) \otimes N(k\Delta_\ell) \otimes \cdots \otimes N(k\Delta_\ell))_m. \tag{5.1}$$

We can write $\mathcal{O}(n)$ as $N(D(\mathcal{O}(n)))$ and calculate the homotopy groups in the E_2-tableau as

$$\pi_q(D(\mathcal{O}(n) \otimes N(k\Delta_\ell) \otimes \cdots \otimes N(k\Delta_\ell))_*) \cong \pi_q((D(\mathcal{O}(n)) \hat{\otimes} k\Delta_\ell \hat{\otimes} \cdots \hat{\otimes} k\Delta_\ell)_*).$$

The homotopy groups of $k\Delta_\ell \hat{\otimes} \cdots \hat{\otimes} k\Delta_\ell$ are trivial in all dimensions but zero. As $\mathcal{O}(n)$ is weakly equivalent to the chain complex $(k, 0)$, which is k in dimension zero and trivial in all other dimensions, and as D preserves weak equivalences, we can conclude that the homotopy groups are trivial in all dimensions but zero. The maps in cosimplicial direction come from maps which concern the index ℓ in the free k-module $k\Delta_\ell$ and they give trivial cohomotopy except in dimension zero. Thus the spectral sequence collapses and the operad $\mathsf{End}_D^{\mathscr{O}}$ is acyclic.

The given augmentation of the operad $\mathscr{O}$, $\varepsilon : \mathscr{O}(n) \to k$, composed with the augmentation $\tilde{\varepsilon} : \mathsf{End}_D(n) \to \mathsf{End}_D(0) \cong \underline{k}$ of the operad End_D gives the augmentation of the amalgamated operad $\mathsf{End}_D^{\mathscr{O}}$. The augmentation for End_D involves exactly the evaluation on tensor powers of $D(k) \cong D(k\Delta_0)$; therefore the weak equivalence to the operad of commutative monoids is given by the composition $\tilde{\varepsilon} \circ \varepsilon$. $\square$

Remark 5.4.3. (1) Note, that argument (5.1) proves as well, that our operad exists as an operad of simplicial k-modules, because the representability of D ensures that $\underset{\sim}{\mathsf{nat}}(D_\ell^{\hat{\otimes} n}, D^{\otimes n}(\mathscr{O}(n) \otimes -)_m)$ is a set and therefore the corresponding totalization is a simplicial set. The additional k-module structure is obvious.

(2) In addition, it is clear, that $\mathsf{End}_D^{\mathscr{O}}$ is not the empty set: the Alexander–Whitney map AW [16] gives rise to natural transformations

$$AW_n : D(C_1) \hat{\otimes} \cdots \hat{\otimes} D(C_n) \longrightarrow D(C_1 \otimes \cdots \otimes C_n).$$

In terms of the normalization, the Alexander–Whitney maps cares for a lax comonoidal structure on the images under the functor N. For any two simplicial k-modules A and B we have natural maps

$$\mathrm{AW} : N(A \hat{\otimes} B) \longrightarrow N(A) \otimes N(B).$$

As AW is given in terms of evaluation of front and back side as

$$\mathrm{AW}(a_n \otimes b_n) = \sum_{i=0}^{n} \tilde{d}^{n-i}(a_n) \otimes d_0^i(b_n)$$

with $a_n \in A_n$, $b_n \in B_n$ and $\tilde{d}^{n-i} = d_{i+1} \circ \cdots \circ d_n$, it is *not* lax *symmetric* monoidal. Nevertheless, it suffices to obtain

$$\begin{array}{ccc} AW_2 : D(C_1)\hat{\otimes} D(C_2) \cong DN(D(C_1)\hat{\otimes} D(C_2)) & \xrightarrow{D(AW)} & D(ND(C_1) \otimes ND(C_2)) \\ & & \downarrow \cong \\ & & D(C_1 \otimes C_2). \end{array}$$

Choosing fixed elements in $\mathscr{O}(n)$ gives then for instance non-trivial elements in $\mathsf{End}_D^{\mathscr{O}}(n)$. But note, that one cannot cobble these choices together to obtain a map of operads $\mathscr{O} \to \mathsf{End}_D^{\mathscr{O}}$.

(3) Using Theorem 4.4.1 part (1) the operad $\mathsf{End}_D = \mathsf{End}_D^{\mathscr{C}om}$ possesses a map from the operad of associative monoids $\mathscr{A}ss$: One can send the identity map in $\mathscr{A}ss(n)$ to the n-fold iteration of the Alexander–Whitney transformation and extend this map Σ_n-equivariantly.

(4) Again by Theorem 4.4.1 part (3) we see that $\mathsf{End}_D^{\mathcal{O}}$ gives associative E_∞-structures if $\mathcal{O}$ has been an E_∞-operad in chain complexes with a map from $\mathcal{A}$ss.

5.5. The functor D and general homotopy algebras

In the following we will always assume that our operads are *reduced*, i.e., $\mathcal{O}(0) \cong \mathbf{1}_{\mathcal{C}}$. A map between reduced operads is always assumed to be the identity on the zeroth term.

So far we considered operads with an action of the symmetric groups. Recall that a non-Σ-operad $\mathcal{O}'$ in some symmetric monoidal category is a sequence of objects $(\mathcal{O}(n)')_{n \geqslant 0}$ which obeys the axioms of an operad with the sole difference that we do not require any action of symmetric groups on $\mathcal{O}'$.

Example 5.5.1. The non-Σ-version $\mathcal{A}$ss$'$ of the operad of associative algebras in the category of k-modules consists of the ground ring k in every operad degree. As algebras over the operad $\mathcal{A}$ss$'$ do not have to satisfy any equivariance condition, they are just unital associative algebras in the ordinary sense with the multiplication given by the unit of k in $\mathcal{A}$ss$'(2)$.

An arbitrary non-Σ-operad $\mathcal{P}$ gives of course rise to an operad $\mathsf{End}_D^{\mathcal{P}}$ as well. We have to view this operad as a non-Σ operad, because we cannot define any reasonable Σ-action on this parametrized operad. In particular, if we start with an A_∞-operad $\mathcal{P}$, i.e., a non-Σ-operad for which the augmentation $\mathcal{P}(n) \to \mathcal{P}(0)$ is a weak equivalence, we get an operad which codifies homotopy associativity again:

Corollary 5.5.2. *For any A_∞-operad $\mathcal{P}$, the parametrized endomorphism operad $\mathsf{End}_D^{\mathcal{P}}$ is again an A_∞-operad. Therefore the image of an arbitrary differential graded A_∞-algebra A_* under D is a simplicial A_∞-algebra.*

To every non-Σ operad $\mathcal{O}'$ in the category of k-modules one can associate an ordinary operad $\mathcal{O}$ with an action of symmetric groups by inducing up with the regular representation, i.e.,

$$\mathcal{O}(n) := k[\Sigma_n] \otimes_k \mathcal{O}'(n).$$

As the category of k-modules is a full subcategory in the categories dgmod and smod we obtain the following non-Σ analog of Theorem 4.4.1.

Corollary 5.5.3. *Every non-Σ-operad $\mathcal{O}'$ in the category of k-modules gives rise to a map of non-Σ-operads from $\mathcal{O}'$ to $\mathsf{End}_D^{\mathcal{O}'}$.*

Inducing the actions of the symmetric groups up, we obtain maps of (genuine) operads from $\mathcal{O}$ to $\mathsf{End}_D^{\mathcal{O}'}$. Here the symmetric group action on $\mathcal{O}'$ in $\mathsf{End}_D^{\mathcal{O}'}$ is defined to be trivial.

In Example 5.5.1 we get a map of non-Σ-operads from $\mathcal{A}$ss$'$ to End_D or if we prefer a map from $\mathcal{A}$ss to End_D which is the same as the one in Theorem 4.4.1.

Note, that the arguments in 5.5.2 work, because we consider operads, which are weakly equivalent to the unit of the category of differential graded modules. If $\tilde{\mathcal{O}}$ is an operad in differential graded modules which comes with an operad map to a reduced operad $\mathcal{O}$ then the augmentation map from $\mathsf{End}_D^{\tilde{\mathcal{O}}}$ to $D\mathcal{O}$ is the composition of the operad map $\tilde{\mathcal{O}} \to \mathcal{O}$ followed by the augmentation of the operad $\mathsf{End}_D^{\mathcal{O}}$

$$\begin{array}{ccc} \mathsf{End}_D^{\mathcal{O}}(n) \cong \mathsf{End}_D^{\mathcal{O}}(n) \hat{\otimes} D(k)^{\hat{\otimes} n} & \xrightarrow{w^n} & \mathsf{hom}(D(k)^{\hat{\otimes} n}, D(\mathcal{O}(n) \otimes k^{\otimes n})) \hat{\otimes} D(k)^{\hat{\otimes} n} \\ & & \downarrow \mathrm{ev} \\ & & D(\mathcal{O}(n) \otimes k^{\otimes n}) \cong D(\mathcal{O}(n)). \end{array}$$

For any operad $\mathcal{O}$ in the category dgmod which is concentrated in degree zero, $D(\mathcal{O}(n))$ is the constant simplicial object which has $\mathcal{O}(n)$ in every degree and identity maps as simplicial structure maps. Therefore, in this cases the sequence $(D(\mathcal{O}(n)))_n$ defines an operad in the category of simplicial k-modules and the map above is easily seen to be an operad map.

Definition 5.5.4. For an operad in the category of k-modules, $\mathcal{O}$, and for any operad $\tilde{\mathcal{O}}$ in dgmod which is weakly equivalent to $\mathcal{O}$ via a map of operads, we call an $\tilde{\mathcal{O}}$-algebra a *weak homotopy-$\mathcal{O}$-algebra*.

Usually, one calls a cofibrant operad together with a map of operads down to $\mathcal{O}$ which is a weak equivalence a *homotopy $\mathcal{O}$-operad* and algebras over such operads would be called *homotopy $\mathcal{O}$-algebras*. Let us summarize the observations which we made above as follows:

Theorem 5.5.5. *For any reduced operad $\tilde{\mathcal{O}}$ in* dgmod *which is weakly equivalent to an operad $\mathcal{O}$ in the category of modules via a map of operads, the operad* $\mathsf{End}_D^{\tilde{\mathcal{O}}}$ *has an operad map to $D(\mathcal{O})$, which is a weak equivalence, therefore*

(1) *every $\mathcal{O}$-algebra X in* dgmod *gives rise to a weak homotopy-$D(\mathcal{O})$-algebra $D(X)$ in* smod *and*
(2) *the functor D maps every weak homotopy-$\mathcal{O}$-algebra X in* dgmod *to a weak homotopy-$D(\mathcal{O})$-algebra $D(X)$ in* smod.

5.6. Another way to pass differential graded homotopy algebras to spectra

In [21] we suggested a very straightforward way to pass from differential graded commutative algebras to spectra. The inverse of the normalization D maps commutative algebras to simplicial E_∞-algebras. As there are lax symmetric monoidal models for a functor H which associates a generalized Eilenberg–MacLane spectrum to a simplicial abelian group [22], we proposed to take H of the operad End_D as an E_∞-operad in spectra which gives $H(D(A_*))$ an E_∞-structure for any differential commutative algebra A_*. However, any symmetric monoidal category of spectra which models the stable homotopy category and fulfills some other reasonable properties has to have deficiencies [14] (see [1, 4.6.4]): as a consequence, either it has a cofibrant unit or a symmetric monoidal fibrant replacement functor, but not both. Therefore the operads in that symmetric monoidal category will have no model structure [1, 3.1]. But all known models are known to be enriched in simplicial sets or topological spaces; operads therein have a nice model structure and algebras over cofibrant operads obtain a model structure as well.

For defineteness, let us work in the category of simplicial symmetric spectra *à la* [12] where we take the standard model for Eilenberg–MacLane spectra. Given a simplicial abelian group $B_\bullet$, the nth term in the spectrum $H(B_\bullet)$ is $B_\bullet \hat{\otimes} \bar{\mathbb{Z}}[\mathbb{S}^n]$, where we take the 1-sphere $\mathbb{S}^1$ as the quotient $\mathbb{S}^1 = \Delta^1/\partial\Delta^1$ of the 1-simplex divided by its boundary and the higher spheres $\mathbb{S}^n$ as iterated smash powers of $\mathbb{S}^1$ [12].

Theorem 5.6.1. (1) *There is an operad in the category of simplicial sets which turns the spectrum $H(D(A_*))$ into an E_∞-monoid in the category of symmetric simplicial spectra.*

(2) *If A_* is a homotopy $\mathcal{O}$-algebra with $\mathcal{O}$ a reduced operad in modules, then there is an operad in simplicial sets which gives $H(D(A_*))$ a weak homotopy $H(D(\mathcal{O}))$-structure.*

Proof. In both cases we take the parametrized generalized endomorphism operad, which is an object in simplicial k-modules, as the corresponding operad in simplicial sets. We have to define the action map. As the second claim includes the first claim, we will prove it.

Let K and L be two simplicial abelian groups with basepoint the zero element. There is a natural map from the smash product to the tensor product of simplicial abelian groups

$$\phi : K \wedge L \longrightarrow K \hat{\otimes} L$$

which is induced by the natural map from the product to the tensor product. A map from a tensor product of symmetric spectra $X \wedge Y$ to a third spectrum Z is determined by a family of $\Sigma_p \times \Sigma_q$-equivariant maps $X_p \wedge Y_q \to Z_{p+q}$ (see [12, Section 2]) which commute with the S-module structure.

On the mth term of our spectrum $(\mathsf{End}_D^{\mathcal{O}}(n) \wedge H(D(A_*))^{\wedge n})$ we therefore have to specify equivariant maps

$$\mathsf{End}_D^{\mathcal{O}}(n) \wedge H(D(A_*))_{r_1} \wedge \cdots \wedge H(D(A_*))_{r_n} \longrightarrow H(D(A_*))_{\sum r_i}.$$

To obtain these maps, we use the map ϕ to send

$$\mathsf{End}_D^{\mathcal{O}}(n)_m \wedge \big(H(D(A_*))_{r_1} \wedge \cdots \wedge H(D(A_*))_{r_n}\big)_m$$

to

$$\mathsf{End}_D^{\mathcal{O}}(n)_m \otimes H(D(A_*)_{r_1} \hat{\otimes} \cdots \hat{\otimes} D(A_*)_{r_n})_m$$

which is equal to

$$\mathsf{End}_D^{\mathcal{O}}(n)_m \otimes (D(A_*) \hat{\otimes} \bar{\mathbb{Z}}[\mathbb{S}^{r_1}] \hat{\otimes} \cdots \hat{\otimes} D(A_*) \hat{\otimes} \bar{\mathbb{Z}}[\mathbb{S}^{r_n}])_m.$$

Shuffling the factors $D(A_*)$ to the left and using the smashing map on spheres we finally arrive at

$$\mathsf{End}_D^{\mathcal{O}}(n)_m \otimes D(A_*)_m \otimes \cdots \otimes D(A_*)_m \otimes (\bar{\mathbb{Z}}[\mathbb{S}^{r_1+\cdots+r_n}])_m.$$

Taking simplicial degrees together we can apply our action map of $\mathsf{End}_F^{\mathcal{O}}(n)$ on the n copies of $D(A_*)$ to arrive at $H(D(A_*))$, such that all transformations involved are preserve the monoidal structure. As the S-module structure only affects the $\bar{\mathbb{Z}}[\mathbb{S}^{r_i}]$-factors, the constructed map yields a well-defined action of the operad $\mathsf{End}_D^{\mathcal{O}}$ on $H(D(A_*))$. □

Remark 5.6.2. Note, that the E_∞ structure we get on $H(D(A_*))$ for a differential graded commutative algebra A_* preserves the *strict* associativity of A_*, because the operad map from the operad of associative simplicial k-modules $\mathscr{A}$ss to End_D still induces an action of $\mathscr{A}$ss on $HD(A_*)$.

5.7. Model structures on differential graded modules and simplicial modules

We will consider the so-called *projective* model category structure on differential graded modules, alias non-negatively graded chain complexes. Here all modules are taken over some commutative ring k with unit. Already Quillen made this structure explicit in [19, II.4]; more recent accounts are [23, Section 4] and [10, 2.3]. The projective model structure is cofibrantly generated by

- the set of generating cofibrations $I = \{\mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n, n \geqslant 1\}$ and
- the set of generating acyclic cofibrations $J = \{0 \longrightarrow \mathbb{D}^n, n \geqslant 1\}$.

The disk chain complex $\mathbb{D}^n$ has $(\mathbb{D}^n)_p = k$ for $k = n, n-1$ and is trivial in all other degrees. Its only non-trivial differential is the identity on k. The sphere chain complex $\mathbb{S}^n$ is concentrated in degree n where it is the ground ring k. Chain maps from the n-sphere to a chain complex correspond to the n-cycles in that chain complex, whereas chain maps from the n-disk correspond to the degree-n part of the chain complex.

Fibrations are maps of chain complexes which are surjective in positive degrees, and weak equivalences are maps which induce isomorphisms on homology groups. Cofibrations are then determined by having the left lifting property with respect to acyclic fibrations.

This model structure is inherited from the model structure on simplicial modules. Inherited means, that a map of simplicial modules is a fibration, cofibration or weak equivalence if and only if the normalization of this map is a fibration, cofibration or weak equivalence in differential graded modules.

The model category of simplicial modules is easily seen to be left and right proper, i.e., pushouts along cofibrations and pullbacks along fibrations preserve weak equivalences. The Quillen equivalence (N, D) between these model categories therefore yields a proper model structure on differential graded modules.

We can apply the Berger–Moerdijk criterium (8.2.2, [1, Proposition 4.1]) for the existence of model structures on the category of algebras over operads in simplicial modules and differential graded modules.

Theorem 5.7.1. *Let $\mathscr{O}$ be a reduced operad in* dgmod. *The adjunction $(L_D^{\mathscr{O}}, D)$ passes to a Quillen adjunction between the model categories of $Q(\mathscr{O})$-algebras and* $\mathsf{End}_D^{Q\mathscr{O}}$*-algebras, where $Q\mathscr{O}$ is a cofibrant replacement of the operad $\mathscr{O}$.*

Remark 5.7.2. Of course, we would like to clarify whether this Quillen adjunction is in fact a Quillen equivalence. It is straightforward to see that the unit of the adjunction $\mathrm{id} \to D \circ L_D^{\mathscr{O}}$ is a weak equivalence on free algebras, but so far we have not been able to extend this result to arbitrary cofibrant objects.

Remark 5.7.3. Schwede and Shipley provide in [23, Theorem 1.1 (3)] a Quillen equivalence between the category of simplicial k-algebras and differential graded k-algebras. However, they use the normalization as one of the Quillen functors and the Quillen equivalence does not involve the functor D.

6. The conormalization functor

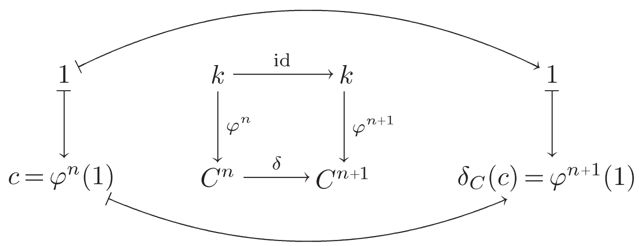
6.1. Cosimplicial modules and cochain complexes

On unbounded differential graded k-modules there is still a model structure (see [10, 2.3]) but here

- the fibrations are maps of chain complexes which are surjective in *every* degree,
- the weak equivalences are again maps which induce isomorphisms on homology groups, and
- the cofibrations are determined by the left lifting property with respect to acyclic fibrations.

We can still use the generating cofibrations and acyclic cofibrations, but now we need sphere to disk inclusions $\{\mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n\}$ for all integers n, and inclusions from the trivial module to disks $\{0 \longrightarrow \mathbb{D}^n\}$ in all degrees as well.

Taking the model structure on unbounded chain complexes, it is clear that the category of unbounded cochain complexes has a cofibrantly generated model structure as well. We can view unbounded cochain complexes (C^*, δ_C) as unbounded chain complexes $(\tilde{C}_*, d_{\tilde{C}})$ with $\tilde{C}_* = C^{-*}$. They inherit generating cofibrations and acyclic cofibrations from the category of chain complexes, namely *codisks* $(\mathbb{D}_n)_{n\in\mathbb{Z}}$ with $\tilde{\mathbb{D}}_n = \mathbb{D}^{-n}$ and *cospheres* $(\mathbb{S}_n)_{n\in\mathbb{Z}}$ with $\tilde{\mathbb{S}}_n = \mathbb{S}^{-n}$. A homomorphisms φ of cochain complexes of k-modules from $\mathbb{D}_n$ to some cochain complex (C^*, δ_C) therefore picks an element $c \in C^n$ and its coboundary $\delta_C(c) \in C^{n+1}$:



Therefore we obtain the following useful representation of the nth degree part of a cochain complex

$$C^n \cong \mathrm{Hom}_{\delta\mathsf{mod}}(\mathbb{D}_n, C^*).$$

Under this identification the cochain complexes which are concentrated in non-negative degrees correspond to chain complexes concentrated in degrees $\leqslant 0$. Fibrations in that inherited model structure are given by surjective maps. That fibrations have to be surjective can be easily seen, because we have to lift the acyclic cofibrations $0 \longrightarrow \mathbb{D}_n$ for all $n \geqslant 0$. It is straightforward to check, that this model structure is right and left proper.

Again, we can transfer this model structure on cochain complexes in non-negative degrees to a model structure on cosimplicial modules via the Dold–Kan equivalence (D^*, N^*). As the original model structure was proper, the transferred one is proper as well.

The Dold–Kan correspondence between the category of cosimplicial k-modules cmod and the category of cochain complexes of k-modules concentrated in non-negative degrees $\delta\mathsf{mod}$ is of the following form: the conormalization functor (compare [3, X.7.1] or [8, VIII.1]) on a cosimplicial module $A^\bullet$ is given as

$$N^n(A^\bullet) = \bigcap_{i=0}^{n-1} \ker(\sigma^i : A^n \longrightarrow A^{n-1}),$$

where the σ^i are the cosimplicial structure maps. The differential is then given by the alternating sum $\delta = \sum_{i=0}^{n}(-1)^i \delta^i$. Equivalently, the conormalization can be expressed as a quotient, namely

$$N^n(A) \cong A^n / \sum \delta^i(A^{n-1}). \tag{6.1}$$

6.2. *Alexander–Whitney and shuffle transformations*

Dual to the case of chain complexes and simplicial modules, the Alexander–Whitney map will give rise to the monoidal structure on normalized cochains whereas shuffle maps constitute a lax symmetric comonoidal transformation.

The Alexander–Whitney map: The Alexander–Whitney map on normalized cochains is a transformation

$$\mathsf{aw} : \bigoplus_{p+q=n} N^p(A^\bullet) \otimes N^q(B^\bullet) \longrightarrow N^n(A^\bullet) \otimes N^n(B^\bullet). \tag{6.2}$$

We define its (p, q)-component $\mathsf{aw}_{p,q}$ from $A^p \otimes B^q$ to $A^n \otimes B^n$ as

$$\mathsf{aw}_{p,q} : A^p \otimes B^q \ni a \otimes b \mapsto \tilde{\delta}^q(a) \otimes (\delta^0)^p(b),$$

where $\tilde{\delta}^q$ is the composition $\delta^{n-1} \circ \cdots \circ \delta^p$ of the 'last' coface maps. Dualizing the proof for chain complexes and simplicial modules yields, that $\mathsf{aw} = \bigoplus_{p,q} \mathsf{aw}_{p,q}$ gives rise to a map of cochain complexes, i.e., on summands we get

$$\delta \circ \mathsf{aw}_{p,q} = \mathsf{aw}_{p+1,q} \circ \delta \otimes \mathrm{id} + (-1)^p \mathsf{aw}_{p,q+1} \circ \mathrm{id} \otimes \delta.$$

With the help of the reformulation in (6.1) it is straightforward to check that aw gives a well-defined transformation on the associated conormalized cochain complexes.

The shuffle-transformations: For the comonoidal structure we will consider the shuffle-transformation. We start with an element $a \otimes b$ in $N^n(A^\bullet \hat{\otimes} B^\bullet)$ which is a submodule of $A^n \otimes B^n$. In order to reduce the degrees of a and b use the structure maps $\sigma^i : A^n \to A^{n-1}$ (resp. $B^n \to B^{n-1}$)

$$\mathsf{sh} : a \otimes b \mapsto \sum_{\xi \in SH(p,q)} \operatorname{sign}(\xi) \sigma^{s_1} \circ \cdots \circ \sigma^{s_q}(a) \otimes \sigma^{t_1} \circ \cdots \circ \sigma^{t_p}(b),$$

where ξ is a (p, q)-shuffle permutation, which is determined by its sequences of values $t_1 < \cdots < t_p$ and $s_1 < \cdots < s_q$. Note that the order of the structure maps σ^{s_j} and σ^{t_j} increases from left to right. The map sh gives a transformation of cochain complexes and passes to the conormalization.

Note, that dual to the case of chain complexes, the composition $\mathsf{sh} \circ \mathsf{aw} = \mathrm{id}$ whereas the composition $\mathsf{aw} \circ \mathsf{sh}$ is only homotopic to the identity.

The conormalization has an inverse $D^* : \delta\mathsf{mod} \longrightarrow \mathsf{cmod}$. Therefore the value of N^* on any cosimplicial module $A^\bullet$ in cochain degree n is given as

$$N^n(A^\bullet) = \mathrm{Hom}_{\delta\mathsf{mod}}(\mathbb{D}_n, N^*(A^\bullet)) \cong \mathrm{Hom}_{\mathsf{cmod}}(D^*(\mathbb{D}_n), A^\bullet).$$

6.3. *The generalized endomorphism operad for N^**

Let us first introduce the internal hom-object in the category of cochain complexes.

Definition 6.3.1. Let C^* and D^* be two cochain complexes of k-modules. The *cochain complexes of homomorphisms* $\mathsf{hom}'(C^*, D^*)$ in cochain degree n is

$$\mathsf{hom}'(C^*, D^*)^n := \prod_{\ell \geqslant 0} \mathrm{Hom}_{k-\mathrm{mod}}(C^\ell, D^{\ell+n}).$$

The coboundary map δ evaluated on such a sequence of morphisms $\psi = (\psi^\ell)_{\ell \geqslant 0}$ is $(\delta(\psi))^\ell = (\delta_{\mathsf{hom}}(\psi))^\ell = \psi^{\ell+1} \circ \delta_C + (-1)^{n+1} \delta_D \circ \psi^\ell$.

Remark 6.3.2. The above cochain complex is in general not bounded. In the following we will use the truncated variant with

$$\mathsf{hom}(C^*, D^*)^n := \begin{cases} \prod_{\ell \geqslant 0} \mathrm{Hom}_{k-\mathrm{mod}}(C^\ell, D^{\ell+n}) & \text{for} \quad n > 0, \\ \text{cocycles in } \prod_{\ell \geqslant 0} \mathrm{Hom}_{k-\mathrm{mod}}(C^\ell, D^\ell) & \text{for} \quad n = 0, \\ 0 & \text{for} \quad n < 0. \end{cases}$$

This cochain complex has the same cohomology as $\mathsf{hom}'(C^*, D^*)$ in degrees greater or equal to zero. We will establish a spectral sequence which converges weakly to the cohomology groups of $\mathsf{hom}(C^*, D^*)$ for any two cochain complexes C^* and D^*. We interpret our cochain complex as the total complex of the following homological second quadrant bicomplex $X_{*,*}$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & \mathrm{Hom}_{k-\mathrm{mod}}(C^2, D^2) & \longleftarrow & 0 & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & \mathrm{Hom}_{k-\mathrm{mod}}(C^1, D^2) & \longleftarrow & \mathrm{Hom}_{k-\mathrm{mod}}(C^1, D^1) & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longleftarrow & \mathrm{Hom}_{k-\mathrm{mod}}(C^0, D^2) & \longleftarrow & \mathrm{Hom}_{k-\mathrm{mod}}(C^0, D^1) & \longleftarrow & \mathrm{Hom}_{k-\mathrm{mod}}(C^0, D^0).
\end{array}$$

Thus, $X_{p,q} = \mathrm{Hom}_{k-\mathrm{mod}}(C^q, D^{-p})$ and the total complex is given by

$$\begin{aligned}
\mathrm{Tot}(X_{*,*})_{-\ell} &= \prod_{p+q=-\ell} X_{p,q} = \prod_{p+q=-\ell} \mathrm{Hom}_{k-\mathrm{mod}}(C^q, D^{-p}) \\
&= \prod_q \mathrm{Hom}_{k-\mathrm{mod}}(C^q, D^{\ell+q}).
\end{aligned}$$

Filtering the bicomplex $X_{*,*}$ by columns gives a standard spectral sequence with E_1-term

$$E_1^{p,q} = H_q^{\mathrm{vert}}(X_{p,*})$$

and E_2-term

$$E_2^{p,q} = H_p^{\mathrm{hor}} H_q^{\mathrm{vert}}(X_{*,*})$$

with horizontal homology H_*^{hor} and vertical homology H_*^{vert}. As our bicomplex is concentrated in the second quadrant the filtration by columns is complete and exhaustive; therefore the associated spectral sequence weakly converges to the homology of the associated product total complex [26, p. 142].

6.4. *Preservation of homotopy structures*

Hinich and Schechtman proved in [9] that the conormalization functor maps commutative cosimplicial rings to algebras over an acyclic operad; in particular every conormalization of such a ring can be viewed as an E_∞-algebra in the category of cochain complexes. However, if one wants to generalize their approach in order to deal with homotopy algebras, one has to modify the construction. They consider the *endomorphism operad* of the functor N^* and in their context the nth part of the Hinich–Schechtman operad $\mathscr{HS}(n)$ consists of the natural transformations from the n-fold tensor power of N^* to N^*

$$\mathscr{HS}(n) = \mathsf{nat}(N^{*\otimes n}, N^*).$$

As the target consists of one single copy of the functor N^* there is no space for implementing operad actions. But it will turn out that parametrized generalized endomorphism operads can handle this problem.

We will construct the generalized endomorphism operad End_{N^*} for the conormalization functor $N^* : \mathsf{cmod} \longrightarrow \delta\mathsf{mod}$ and its parametrized version $\mathsf{End}^{\mathcal{O}}_{N^*}$ for an arbitrary operad $\mathcal{O}$ in the category of cosimplicial modules. To this end we use the internal hom-object defined before.

Let $(N^*)^{\otimes n}$ be the functor from cmod^n to $\delta\mathsf{mod}$ which sends any n-tuple $(A^\bullet_1, \ldots, A^\bullet_n)$ of cosimplicial k-modules to the tensor product of cochain complexes $N^*(A^\bullet_1) \otimes \cdots \otimes N^*(A^\bullet_n)$ and let $(N^*)^{\hat{\otimes} n} : \mathsf{cmod}^n \to \delta\mathsf{mod}$ be the functor which applies N^* to the tensor product of the $A^\bullet_i$:

$$(N^*)^{\hat{\otimes} n}(A^\bullet_1, \ldots, A^\bullet_n) := N^*(A^\bullet_1 \hat{\otimes} \cdots \hat{\otimes} A^\bullet_n).$$

Proposition 6.4.1. *The generalized endomorphism operad* End_{N^*} *for the functor* N^* *is given by*

$$\mathsf{End}_{N^*}(n) := \mathsf{nat}((N^*)^{\otimes n}, (N^*)^{\hat{\otimes} n}),$$

i.e., $\mathsf{End}_{N^*}(n)$ *in cochain degree m consists of the natural transformations from* $(N^*)^{\otimes n}$ *to* $(N^*)^{\hat{\otimes} n}$ *which raise degree by* $m \geqslant 0$:

$$\mathsf{End}_{N^*}(n)^m = \begin{cases} \prod_{\ell \geqslant 0} \mathsf{nat}\left(((N^*)^{\otimes n})^\ell, ((N^*)^{\hat{\otimes} n})^{\ell+m}\right) & \text{for } m > 0, \\ \text{cocycles in } \prod_{\ell \geqslant 0} \mathsf{nat}\left(((N^*)^{\otimes n})^\ell, ((N^*)^{\hat{\otimes} n})^{\ell}\right) & \text{for } m = 0. \end{cases}$$

As N^* *satisfies the unit condition 2.1, we define* $\mathsf{End}_{N^*}(0)$ *to be the cochain complex which consists of the ground ring k concentrated in degree zero.*

Let us comment on the existence of this object: we saw that the conormalization functor is representable as $N^m(A^\bullet) = \mathrm{Hom}_{\mathsf{cmod}}(D^*(\mathbb{D}_m), A^\bullet)$. Applying the multilinear Yoneda lemma again we see that we can write the natural transformations from the functor $(N^*)^{\otimes n}$ in cosimplicial degree s to the functor $(N^t)^{\hat{\otimes} n}$ in cosimplicial degree t as

$$\mathsf{nat}\left((N^{*\otimes n})^s, (N^t)^{\hat{\otimes} n}\right) = \prod_{r_1+\cdots+r_n=s} N^t(D^*(\mathbb{D}_{r_1}) \hat{\otimes} \cdots \hat{\otimes} D^*(\mathbb{D}_{r_n}))$$

and this is clearly a set. Taking the total complex of this gives our operad.

Before we analyze the cohomology of the operad End_{N^*}, let us investigate how the representability via the cochain complexes $\mathbb{D}_m$ work in detail: Any cochain complex C^* in degree m is isomorphic to the cochain homomorphisms from $\mathbb{D}_m$ to C^*. Such a morphism φ picks an element $\varphi^m(1) = c \in C^m$. The coboundary of c is then given by $\varphi^{m+1}(1)$. Interpreted as a map from $\mathbb{D}_{m+1}$ to $\mathbb{D}_m$ the coboundary corresponds to the map which sends the generator in degree $m+2$ to zero and the generator in degree $m+1$ to the generator in degree $m+1$ in $\mathbb{D}_m$:

$$\begin{array}{lccc} & \vdots & & \vdots \\ m+2: & k & \rightarrow & 0 \\ & \mathrm{id}\uparrow & & \uparrow \\ m+1: & k & \stackrel{\mathrm{id}}{\rightarrow} & k \\ & \uparrow & & \mathrm{id}\uparrow \\ m: & 0 & \rightarrow & k \end{array}$$

In this way we get a complex of cochain complexes

$$\mathbb{D}_* := (\cdots \longrightarrow \mathbb{D}_{m+1} \longrightarrow \mathbb{D}_m \longrightarrow \mathbb{D}_{m-1} \longrightarrow \cdots).$$

Lemma 6.4.2. *Any n-fold tensor product of the complex of codisk cochain complexes* ($\mathbb{D}_*^{\otimes n}, n \geqslant 1$) *is acyclic, i.e., it is quasiisomorphic to the complex* $(k, 0)$ *which is the ground ring k concentrated in degree zero.*

Proof. It is obvious from the definition of $\mathbb{D}_*$ that it is acyclic, because its defining sequence is exact. Its homology in degree zero is given by $\mathbb{D}_0$ divided by the boundaries coming from $\mathbb{D}_1$ and thus only $\mathbb{D}_0^0 \cong k$ remains in degree zero. The claim then follows from the Künneth theorem because all modules involved are free and $(\mathbb{D}_0^0)^{\otimes n} \cong (k, 0)$. □

Theorem 6.4.3. *The operad* End_{N^*} *is acyclic.*

Proof. In order to calculate the cohomology groups of our operad End_{N^*}, we apply the spectral sequence constructed in 6.3 to the unbounded variant of our cochain complex of natural transformations

$$\mathsf{End}'_{N^*}(n)^m = \prod_{\ell \geqslant 0} \mathsf{nat}\left(((N^*)^{\otimes n})^\ell, ((N^*)^{\hat{\otimes} n})^{\ell+m}\right) \quad \text{for all} \quad m \in \mathbb{Z}.$$

In our case, the E_1-term calculates the vertical homology of the complex $X_{p,q} = \mathsf{nat}$ $(((N^*)^{\otimes n})^p, (N^{\hat{\otimes} n})^{-q})$, but each of these groups $X_{p,q}$ is isomorphic to

$$N^{-q}\left(\bigoplus_{r_1+\cdots+r_n=p} D^*(\mathbb{D}_{r_1})\hat{\otimes}\cdots\hat{\otimes}D^*(\mathbb{D}_{r_n})\right).$$

Thus, homology in vertical direction is the homology of

$$N^{-q}\left(\bigoplus_{r_1+\cdots+r_n=*} D^*(\mathbb{D}_{r_1})\hat{\otimes}\cdots\hat{\otimes}D^*(\mathbb{D}_{r_n})\right).$$

As the functor D^* is part of an equivalence of categories which gives rise to a Quillen equivalence of the corresponding model categories and as the functor D^* preserves the monoidal structure up to weak equivalence, these homology groups are isomorphic to the homology of

$$N^{-q}\left(D^*\left(\bigoplus_{r_1+\cdots+r_n=*} \mathbb{D}_{r_1}\otimes\cdots\otimes\mathbb{D}_{r_n}\right)\right)\cong N^{-q}D^*(\mathbb{D}_*^{\otimes n})$$

and we saw in the lemma above that the complex $\mathbb{D}_*^{\otimes n}$ is exact. Therefore, on the E_2-term we are left with the horizontal homology in direction of the conormalization applied to the constant cosimplicial object which consists of k in every degree. Therefore the cohomology of the cochain complex $\mathsf{End}'_{N^*}(n)$ is isomorphic to k concentrated in degree zero.

As the truncated cochain complex $\mathsf{End}_{N^*}(n)$ has the same cohomology as $\mathsf{End}'_{N^*}(n)$ in non-negative degrees we obtain that $H^*\mathsf{End}_{N^*}(n)\cong(k,0)$. That this isomorphism is given by the augmentation map corresponds to the fact that the evaluation map is precisely given by the evaluation on the n-fold $\hat{\otimes}$-product of the constant cosimplicial object which is k in every degree; this object is isomorphic to

$$D^*(H_*\mathbb{D}_*)\hat{\otimes}\cdots\hat{\otimes}D^*(H_*\mathbb{D}_*)=D^*(k,0)\hat{\otimes}\cdots\hat{\otimes}D^*(k,0)$$

and this isomorphism causes the spectral sequence to collapse. □

Given an arbitrary operad $\mathscr{O}$ in the category of cosimplicial modules one can prove in a similar manner that the parametrized generalized endomorphism operad with parameter $\mathscr{O}$, $\mathsf{End}^{\mathscr{O}}_{N^*}$, is weakly equivalent to the cochain complex $N^*(\mathscr{O})$ which is however no operad in general.

Proposition 6.4.4. *The operad* $\mathsf{End}^{\mathscr{O}}_{N^*}$ *in cochain complexes is defined as*

$$\mathsf{End}^{\mathscr{O}}_{N^*}(n):=\mathsf{nat}((N^*)^{\otimes n},N^{*\hat{\otimes}n}(\mathscr{O}(n)\hat{\otimes}-))$$

with $\mathsf{End}^{\mathscr{O}}_{N^*}(0)$ *being* $N^*(\mathscr{O}(0))$.

If an operad $\mathscr{P}$ in cosimplicial modules has an augmentation to a reduced operad $\mathscr{O}$ which is constant in the cosimplicial direction then $\mathsf{End}^{\mathscr{P}}_{N^*}$ has a natural augmentation to the operad $N^*(\mathscr{O})$.

Corollary 6.4.5. *The operad* $\mathsf{End}^{\mathscr{O}}_{N^*}$ *is weakly equivalent as a cochain complex to* $N^*(\mathscr{O})$. *For operads* $\mathscr{O}$ *concentrated in degree zero, the functor* N^* *maps* $\mathscr{O}$*-algebras to weak homotopy* $\mathscr{O}$*-algebras and (weak) homotopy* $\mathscr{O}$*-algebras to weak homotopy* $\mathscr{O}$*-algebras.*

6.5. Quillen adjunctions for the conormalization

Castiglioni and Cortiñas [4] show that the Dold–Kan correspondence passes to an equivalence between the homotopy category of cosimplicial rings and the homotopy category of cochain rings. We will provide a Quillen adjunction $(L^{\mathscr{O}}_{N^*},N^*)$ for every operad in cosimplicial modules.

Using the Berger–Moerdijk model structure from (8.2.2, [1, Proposition 4.1]) again we get the following result:

Theorem 6.5.1. *The adjunction (D^*, N^*) gives rise to an adjoint pair of functors $(L^{\mathcal{O}}_{N^*}, N^*)$ between the category of $\mathcal{O}$-algebras and the category of $\mathsf{End}^{\mathcal{O}}_{N^*}$-algebras. If $\mathcal{O}$ is reduced and cofibrant then this adjunction is a Quillen pair on the corresponding model structures.*

7. Coalgebra structures

We saw that parametrized endomorphism operads transfer algebra structures over operads. Dually one can ask which constructions would help to save some aspects of coalgebra structures.

If X is a cocommutative comonoid in the category $\mathscr{C}$, i.e., there is a comultiplication $\psi : X \to X \otimes X$ which commutes with the natural symmetry operator in the symmetric monoidal structure $\mathscr{C}$, then the image of X under F is a coalgebra over the operad Coend_F which we define in degree n as the end of the bifunctor which maps a tuple $((C_1, \ldots, C_n), (C'_1, \ldots, C'_n))$ to

$$\mathrm{Hom}(F(C_1 \otimes \cdots \otimes C_n), F(C_1)\hat{\otimes} \cdots \hat{\otimes} F(C_n)).$$

However, in our examples, we do not know how to control the homotopy type of the above operads. This is why we will have to find a way around these constructions.

7.1. (Co)algebra structures via (co)actions of (co)operads

If one consider operads and cooperads, there are several possibilities how an action or coaction can arise. In the following we set $K = k_1 + \cdots + k_n$. If $\mathcal{O}$ is an operad with composition maps γ (in some symmetric monoidal category $\mathscr{C}$),

(1a) it can act on an algebra X, i.e., there are action maps $\vartheta_n : \mathcal{O}(n) \otimes X^{\otimes n} \longrightarrow X$ which are compatible with the operad composition (see [13, I, 2.1]).

(1b) It can coact on an algebra X, thus we have coaction maps $\psi_n : X^{\otimes n} \longrightarrow \mathcal{O}(n) \otimes X$ such that the following diagram commutes

$$\begin{array}{ccc}
X^{\otimes k_1} \otimes \ldots \otimes X^{\otimes k_n} = X^{\otimes K} & \xrightarrow{\psi_K} & \mathcal{O}(K) \otimes X \\
\downarrow{\scriptstyle \bigotimes_{i=1}^n \psi_{k_i}} & & \\
\mathcal{O}(k_1) \otimes X \otimes \ldots \otimes \mathcal{O}(k_n) \otimes X & & \uparrow{\scriptstyle \gamma\circ(\mathrm{shuffle}\otimes\mathrm{id})} \\
\downarrow{\scriptstyle \mathrm{shuffle}} & & \\
(\bigotimes_{i=1}^n \mathcal{O}(k_i)) \otimes X^{\otimes n} & \xrightarrow{\mathrm{id}\otimes\psi_n} & (\bigotimes_{i=1}^n \mathcal{O}(k_i)) \otimes \mathcal{O}(n) \otimes X
\end{array}$$

(1c) An operad can also parametrize a coalgebra structure via an action, i.e., there are maps $\vartheta^n : \mathcal{O}(n) \otimes X \longrightarrow X^{\otimes n}$ such that

$$\begin{array}{ccc}
\mathcal{O}(n) \otimes (\bigotimes_{i=1}^{n} \mathcal{O}(k_i)) \otimes X & \xrightarrow{\gamma \otimes \mathrm{id}} & \mathcal{O}(K) \otimes X \\
\downarrow{\scriptstyle \mathrm{id} \otimes \vartheta^n \circ \mathrm{shuffle}} & & \\
(\bigotimes_{i=1}^{n} \mathcal{O}(k_i)) \otimes X^{\otimes n} & & \downarrow{\scriptstyle \vartheta^K} \\
\downarrow{\scriptstyle \mathrm{shuffle}} & & \\
\bigotimes_{i=1}^{n} (\mathcal{O}(k_i) \otimes X) & \xrightarrow{\bigotimes_{i=1}^{n} \vartheta^{k_i}} & X^{\otimes K}
\end{array}$$

commutes.

(1d) And finally there can be a coaction of $\mathcal{O}$ turning something into a coalgebra over $\mathcal{O}$, namely structure maps $\psi^n : X \longrightarrow \mathcal{O}(n) \otimes X^{\otimes n}$ such that

$$\begin{array}{ccc}
X & \longrightarrow & \mathcal{O}(K) \otimes X^{\otimes K} \\
\downarrow{\scriptstyle \psi^n} & & \uparrow{\scriptstyle (\gamma \otimes \mathrm{id}) \circ \mathrm{shuffle}} \\
\mathcal{O}(n) \otimes X^{\otimes n} & \xrightarrow{\mathrm{id} \otimes \bigotimes_{i=1}^{n} \psi^{k_i}} & \mathcal{O}(n) \otimes \bigotimes_{i=1}^{n} (\mathcal{O}(k_i) \otimes X^{\otimes k_i})
\end{array}$$

commutes.

A lax symmetric monoidal functor such as N or D^* preserves algebra structures as in (1a), but it can destroy the other (co)algebra structures.

Dually, a cooperad, i.e., a sequence of objects $(\mathcal{O}(n))_n$ with decomposition maps

$$\ell = \ell_{n,k_1,\ldots,k_n} : \mathcal{O}\left(\sum_{i=1}^{n} k_i\right) = \mathcal{O}(K) \longrightarrow \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n)$$

gives rise to dual actions and coactions.

(2a) There can be an action of the cooperad $\mathcal{O}$ on an algebra $u_n : \mathcal{O}(n) \otimes X^{\otimes n} \longrightarrow X$ with a coherence condition dual to the one in (1d).

(2b) Dually, $\mathcal{O}$ can coact on an algebra with structure maps $v_n : X^{\otimes n} \longrightarrow \mathcal{O}(n) \otimes X$. Here the coherence condition is dual to (1c).

(2c) An action of $\mathcal{O}$ can be given by action maps

$$u^n : \mathcal{O}(n) \otimes X \longrightarrow X^{\otimes n}$$

such that the coherence property dual to (1b) is fulfilled.

(2d) Last but not least, $\mathcal{O}$ can coact with $v^n : X \longrightarrow \mathcal{O}(n) \otimes X^{\otimes n}$ to give a coalgebra structure on X such that the coaction map is coassociative in the sense of the dual of (1a).

7.2. Endomorphism operads parametrized by cooperads

Let $\mathcal{O}$ be a cooperad in the category $\mathcal{D}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a functor between symmetric monoidal categories, then the following categorical end (if it exists) defines an operad in $\mathcal{D}$:

Definition 7.2.1. We denote by $\mathsf{End}^F_{\mathcal{O}}(n)$ the end

$$\mathsf{nat}(\mathcal{O}(n)\hat{\otimes}F^{\hat{\otimes}n}, F^{\otimes n}) = \int_{\mathcal{C}^n} \mathsf{hom}(\mathcal{O}(n)\hat{\otimes}F(C_1)\hat{\otimes}\cdots\hat{\otimes}F(C_n), F(C_1\otimes\cdots\otimes C_n))$$

in $\mathcal{D}$.

As the proof of the operad property is similar to the one for $\mathsf{End}^{\mathcal{O}}_F$ we will omit it and will just specify the operadic composition map

$$\gamma : \mathsf{End}^F_{\mathcal{O}}(n)\hat{\otimes}\mathsf{End}^F_{\mathcal{O}}(k_1)\hat{\otimes}\cdots\hat{\otimes}\mathsf{End}^F_{\mathcal{O}}(k_n) \longrightarrow \mathsf{End}^F_{\mathcal{O}}(K)$$

with $K = k_1 + \cdots + k_n$.

In every closed symmetric monoidal category $\mathcal{D}$ one has partial composition maps

$$\mathsf{hom}(A\hat{\otimes}B, C)\hat{\otimes}\mathsf{hom}(D, B) \longrightarrow \mathsf{hom}(A\hat{\otimes}D, C).$$

We use the maps w,

$$w^n : \mathsf{End}^F_{\mathcal{O}}(n) \longrightarrow \mathsf{hom}\left(\mathcal{O}(n)\hat{\otimes}\hat{\bigotimes_{j=1}^{n}} F\left(\bigotimes_{i=1}^{k_j} C_i^j\right), F(C_1^1\otimes\cdots\otimes C_{k_n}^n)\right)$$

and

$$w^{k_i} : \mathsf{End}^F_{\mathcal{O}}(k_i) \longrightarrow \mathsf{hom}(\mathcal{O}(k_i)\hat{\otimes}F(C_1^i)\hat{\otimes}\cdots\hat{\otimes}F(C_{k_i}^i), F(C_1^i\otimes\cdots\otimes C_{k_i}^i))$$

for arbitrary objects $C_i^j \in \mathcal{C}$. With the help of the partial composition maps we can send the $\hat{\otimes}$-product of these internal homomorphism objects to

$$\mathsf{hom}\left(\mathcal{O}(n)\hat{\otimes}\hat{\bigotimes_{i=1}^{n}}\mathcal{O}(k_i)\hat{\otimes}F(C_1^i)\hat{\otimes}\cdots\hat{\otimes}F(C_{k_i}^i), F(C_1^1\otimes\cdots\otimes C_{k_n}^n)\right).$$

A shuffle map followed by the decomposition map ℓ of our cooperad $\mathcal{O}$ in the contravariant part then yields a map to $\mathsf{hom}(\mathcal{O}(K)\hat{\otimes}\hat{\bigotimes}_{i,j}F(C_j^i), F(\bigotimes_{i,j}C_j^i))$. The universal property of ends gives a composition

$$\gamma : \mathsf{End}^F_{\mathcal{O}}(n)\hat{\otimes}\mathsf{End}^F_{\mathcal{O}}(k_1)\hat{\otimes}\cdots\hat{\otimes}\mathsf{End}^F_{\mathcal{O}}(k_n) \longrightarrow \mathsf{End}^F_{\mathcal{O}}(K).$$

7.3. (Co)algebra structures and the Dold–Kan correspondence

The functor N : smod $\to$ dgmod maps cocommutative coalgebras to E_∞-coalgebras [21]: the functor N applied to the operad $N\mathsf{End}_D$ is an operad again and for every cocommutative coalgebra $A_\bullet$ in simplicial modules the action map from $N\mathsf{End}_D(n) \otimes NA\bullet$ to $(N(A_\bullet))^{\otimes n}$ is

$$\begin{array}{ccc}
N\mathsf{End}_D(n) \otimes N(A_\bullet) & \longrightarrow & N\mathsf{End}_D(n) \otimes N(A_\bullet^{\hat{\otimes} n}) \\
 & & \Big\downarrow {\scriptstyle \mathrm{id} \otimes N(\eta^{\hat{\otimes} n})} \\
N(\mathsf{End}_D(n) \hat{\otimes} DNA_\bullet^{\hat{\otimes} n}) & \longleftarrow & N\mathsf{End}_D(n) \otimes N(DNA_\bullet^{\hat{\otimes} n}) \\
\Big\downarrow & & \\
N(D((NA_\bullet)^{\otimes n})) & \xrightarrow[\epsilon]{} & NA_\bullet^{\otimes n}
\end{array}$$

However, it is *not* clear that this transfers to operadic coalgebra structures in general. If a simplicial k-module $A_\bullet$ has a coalgebra structure with respect to an action by an operad $\mathcal{O}$, then there is a map

$$N\mathcal{O}(n) \otimes N\mathsf{End}_D(n) \otimes N(A_\bullet) \longrightarrow NA_\bullet^{\otimes n}$$

defined in a similar manner as above, but as the actions of $N\mathcal{O}$ and $N\mathsf{End}_D$ do not necessarily commute, this need not give rise to an $N\mathcal{O} \otimes N\mathsf{End}_D$-structure on $N(A_\bullet)$. A similar warning concerns the functor D^*.

However, we can impose combined actions and coactions on images under D and N^*. Note that D and N^* are lax cosymmetric comonoidal, hence they preserve cooperads and coalgebra structures.

Theorem 7.3.1. *If $X \in \mathsf{dgmod}$ is an algebra over a cooperad $\mathcal{O}$ with respect to a coaction as in* 7.1 (2b) *then there are combined action and coaction maps*

$$\mathsf{End}_D(n) \hat{\otimes} D(X)^{\hat{\otimes} n} \longrightarrow D(X^{\otimes n}) \to D(\mathcal{O}(n) \otimes X) \to D(\mathcal{O}(n)) \hat{\otimes} D(X)$$

of the operad End_D and the cooperad $D(\mathcal{O})$.

Similarly, an algebra $A^\bullet \in \mathsf{cmod}$ over a cooperad $\mathcal{P}$ leads to an End_{N^*}-$N^*(\mathcal{P})$-structure on $N^*(A^\bullet)$.

In the following, we will just discuss the properties of the functor D. We leave it to the reader to draw the analogous conclusions for N^*.

Theorem 7.3.2. *If A has a coaction map $A \to \mathcal{O}(n) \hat{\otimes} A^{\hat{\otimes} n}$ with respect to a cooperad $\mathcal{O}$ in simplicial modules then the normalization of A is an $N\mathsf{End}^D_{\mathcal{O}}$-algebra. If $\mathcal{O}$ is in addition degreewise projective, then $N\mathsf{End}^D_{\mathcal{O}}(n)$ is acyclic if each $\mathcal{O}(n)$ is. In particular, if A is an E_∞-coalgebra, then $N(A)$ is one as well.*

Proof. The structure map for the action first uses the coaction map on A

$$\mathrm{id} \otimes N(\nu^n) : N(\mathsf{End}^D_{\mathcal{O}}(n)) \otimes N(A) \longrightarrow N(\mathsf{End}^D_{\mathcal{O}}(n)) \otimes N(\mathcal{O}(n) \hat{\otimes} A^{\hat{\otimes} n}).$$

The n-fold $\hat{\otimes}$-product of the unit map $\eta : A \to DNA$ then sends this to $N(\mathsf{End}^D_{\mathcal{O}}(n)\hat{\otimes}\mathcal{O}(n)\hat{\otimes}DNA^{\hat{\otimes}n})$ and from there the action map of $\mathsf{End}^D_{\mathcal{O}}$ followed by the counit ε sends the outcome to $NA^{\otimes n}$.

If $\mathcal{O}(n)$ is degreewise projective, we can use the adjunction

$$\begin{aligned}&\mathsf{hom}(\mathcal{O}\hat{\otimes}D(X_1)\hat{\otimes}\cdots\hat{\otimes}D(X_n), D(Y_1\otimes\cdots\otimes Y_n))\\ &\quad\cong\mathsf{hom}(\mathcal{O}(n), \mathsf{hom}(D(X_1)\hat{\otimes}\ldots\hat{\otimes}D(X_n), D(Y_1\otimes\cdots\otimes Y_n))\end{aligned}$$

to control the homotopy groups of $\mathsf{End}^D_{\mathcal{O}}$. The above isomorphism gives an induced isomorphism of ends between $\mathsf{hom}(\mathcal{O}(n), \mathsf{End}_D(n))$ and our operad terms $\mathsf{End}^D_{\mathcal{O}}(n)$. So the homotopy groups of the operad part $\mathsf{End}^D_{\mathcal{O}}(n)$ are isomorphic to the homotopy of $\mathsf{hom}(\mathcal{O}(n), \mathsf{End}_D(n))$ and following [3, 3.3] we can write this as $\pi_*\mathsf{Tot}\ \underset{\sim}{\mathsf{hom}}(\mathcal{O}(n), \mathsf{End}_D(n))$ where

$$\underset{\sim}{\mathsf{hom}}(\mathcal{O}(n), \mathsf{End}_D(n))_{\ell,m} = \mathsf{hom}_{k\text{-mod}}(\mathcal{O}(n)_\ell, \mathsf{End}_D(n)_m).$$

Therefore the Bousfield–Kan spectral sequence has as $E_2^{p,q}$-term

$$E_2^{p,q} = \pi^p\pi_q\ \underset{\sim}{\mathsf{hom}}(\mathcal{O}(n), \mathsf{End}_D(n)).$$

As $\mathcal{O}(n)$ was assumed to be degreewise projective and as we know that $\mathsf{End}_D(n)$ is acyclic, the homotopy groups in simplicial direction q give $\underset{\sim}{\mathsf{hom}}(\mathcal{O}(n), k)$ and the homotopy groups of $\mathsf{End}^D_{\mathcal{O}}$ are trivial. □

8. (Semi) Model categories of algebras over operads

For an arbitrary operad it is unlikely that there will be a ‘well-behaved’ model category structure on the category of algebras over this operad. For instance in an arbitrary symmetric monoidal model category $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ the operad of commutative monoids given by $\mathcal{C}\mathrm{om}(n) = \mathbf{1}_{\mathcal{C}}$ will not have a model category structure, in which the fibrations and weak equivalences are determined by the forgetful functor from the category of commutative monoids in $\mathcal{C}$ to the underlying category $\mathcal{C}$. Even for ‘nice’ operads such as E_∞-operads there was no known model category structure for the algebras over such operads for quite a long time. Mandell provided such a structure in cases where one can rely on operads which are built out of the linear isometries operad [17].

For a general operads one cannot expect to obtain a full model category structure on the algebras over the operad. We will briefly discuss two approaches to that problem: first we will introduce the Spitzweck’s concept of semi-model structures in order to deal with the most general situations. In the second part we will focus on the work by Berger and Moerdijk, they establish genuine model structures in more specific cases. We will make explicit how this approach can be used in our examples of the Dold–Kan correspondence where we have (co)chain complexes and (co)simplicial modules as underlying model categories. Let us briefly recall the definition of symmetric sequences and their model structure.

Definition 8.0.3. Let $\mathscr{C}$ be a category. The category of symmetric sequences has as objects sequences of objects of $\mathscr{C}$, $(C_0, C_1, C_2, \ldots)$ such that the nth object C_n has a right action of the symmetric groups on n letters, Σ_n. Morphisms are sequences of equivariant morphisms. Equivalently, the category of symmetric sequences is the category of functors from Σ to $\mathscr{C}$, where Σ is the small category with objects $\underline{n} = \{1, \ldots, n\}$, $\underline{0} = \varnothing$ and morphisms being bijections of finite sets.

If $\mathscr{C}$ is a model category Berger and Moerdijk [1, Section 3] take the model structure on symmetric sequences (called 'collections', *loc. cit.*) which defines fibrations and weak equivalences by the forgetful functor down to sequences of objects. If $\mathscr{C}$ is cofibrantly generated with generators I for the cofibrations and J for the trivial cofibrations, there is a cofibrantly generated model structure on symmetric sequences with generators $(I[\Sigma_n])_n$, respectively $(J[\Sigma_n])_n$ (see [24] for details); it is easy to see that both structures agree in this situation.

8.1. Semi-model category structures

Hovey introduced the notion of semi-model structures and applied the concept to the case of algebras over a commutative monoid in [11, Theorem 3.3]. In [24], Spitzweck defined 'semi-model category structures' and constructed them on the category of $\mathscr{O}$-algebras, if $\mathscr{O}$ is either a 'cofibrant' operad or at least if the underlying sequence $(\mathscr{O}(n))_{n \geqslant 0}$ of objects $\mathscr{O}(n)$ with Σ_n-actions is cofibrant in the model category of symmetric sequences. To obtain statements about the homotopy category it suffices to have a semi-model category at hand.

Definition 8.1.1 (*Spitzweck [24, p. 5]*). Let $(\mathscr{C}, \otimes, \mathbf{1}_{\mathscr{C}})$ be a closed symmetric monoidal model category, let $\mathscr{D}$ be any category, and let $F : \mathscr{C} \to \mathscr{D}$ be a left adjoint. The category $\mathscr{D}$ with specified classes of weak equivalences, fibrations and cofibrations is a *semi-model category* with these classes if the following conditions are satisfied:

- The right adjoint preserves fibrations and trivial fibrations.
- The category $\mathscr{D}$ is bicomplete, satisfies the two-out-of-three axiom for the weak equivalences and the retract axiom for all three classes.
- Cofibrations in $\mathscr{D}$ have the left lifting property with respect to acyclic fibration and acyclic cofibrations whose domain is cofibrant have the left lifting property with respect to fibrations.
- Every map in $\mathscr{D}$ can be functorially factored into a cofibration followed by an acyclic fibration. If the domain of the map is cofibrant then it has a functorial factorization into an acyclic cofibration and a fibration.
- Cofibrations in $\mathscr{D}$ whose domain is cofibrant become cofibrations in $\mathscr{C}$ and the initial object in $\mathscr{D}$ is mapped to a cofibrant object.
- Fibrations and acyclic fibrations are closed under pullbacks in $\mathscr{D}$.

Let $\mathscr{C}$ be in addition a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathscr{O}$ be an arbitrary operad in $\mathscr{C}$ with an underlying cofibrant symmetric sequence, e.g. an E_∞-operad, and let O be the associated monad. There is a forgetful functor U from the category of $\mathcal{O}$-algebras to the category $\mathcal{C}$ and a left adjoint, which maps an object $C \in \mathcal{C}$ to the free $\mathcal{O}$-algebra $\mathsf{O}(C)$ generated by C.

Theorem 8.1.2 (*Spitzweck [24, Theorems 4.7, 2.9]*). *The category of $\mathcal{O}$-algebras is a semi-model category with weak equivalences and fibrations defined via the forgetful functor and generating cofibrations $\mathsf{O}(I)$ and generating acyclic cofibrations $\mathsf{O}(J)$.*

Remark 8.1.3. A cofibrant operad gives rise to a similar semi-model category structure [24, Theorem 4.3].

The model categories in our examples are the categories of non-negative chain complexes, of simplicial modules, of non-negative cochain complexes and of cosimplicial modules. They all satisfy the assumption of Theorem 8.1.2, but we will see, that we can actually get (genuine) model structures on the target of the functors D and N^*.

8.2. The Berger–Moerdijk model structures

We will briefly recall the results of [1] on model category structures on operads, respectively, on algebras over operads.

Assume $\mathcal{C}$ is a monoidal model category. Let $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ denote the monoidal structure. There is always a factorization $\mathbf{1}_{\mathcal{C}} \coprod \mathbf{1}_{\mathcal{C}} \to H \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}}$. If there is such a factorization such that H is a Hopf-object, i.e., has a multiplication and comultiplication which fit together in the canonical way [1, Section 1], then $\mathcal{C}$ is said to admit a Hopf interval.

Berger and Moerdijk proved the following criterium:

Theorem 8.2.1 (*Berger and Moerdijk [1, Theorem 3.1]*). *Let $\mathcal{C}$ be as above such that the model structure on $\mathcal{C}$ is cofibrantly generated, the model category of objects over the unit $\mathbf{1}_{\mathcal{C}}$ has a monoidal fibrant replacement functor and $\mathcal{C}$ admits a commutative Hopf-interval. Then the category of reduced operads has a cofibrantly generated model category structure, where a map $f : \mathcal{O} \to \mathcal{P}$ is a fibration or weak equivalence if $f(n) : \mathcal{O}(n) \to \mathcal{P}(n)$ is a fibration or weak equivalence in $\mathcal{C}$ for all n.*

Under some mild extra assumptions, the category of algebras over operads possesses a model structure as well.

Proposition 8.2.2 (*Berger and Moerdijk [1, Propostion 4.1]*). *Let $\mathcal{C}$ be a cofibrantly generated monoidal model category such that the unit $\mathbf{1}_{\mathcal{C}}$ is cofibrant and $\mathcal{C}$ has a monoidal fibrant replacement functor for the objects over $\mathbf{1}_{\mathcal{C}}$. Then*

(a) *If there exists an operad map $j : \mathcal{P} \to \mathcal{P} \otimes \mathcal{O}$ and an interval in $\mathcal{C}$ which is a $\mathcal{O}$-coalgebra, then the category of $\mathcal{P}$-algebras has a model structure.*
(b) *If there exists an interval in $\mathcal{C}$ with a coassociative comultiplication, then Σ-split operads posses a model structure for their algebras.*
(c) *If there is an interval in $\mathcal{C}$ which is coassociative and cocommutative, then for all operads there is a model structure for their algebras.*

In all these cases, the fibrations and weak equivalences in the model structures are determined by the forgetful functor.

An operad $\mathscr{O}$ is Σ-split, if is it a retract of $\mathscr{O} \otimes \mathscr{A}ss$.

Their result implies for instance that cofibrant operads in unbounded chain complexes allow a model structure for their algebras. In the category of simplicial sets or simplicial modules groups algebras over an arbitrary operad posses a model structure (see [1, Remark 4.2]).

8.3. Operads and algebras in cochain complexes

In this part we use notation that was introduced at the beginning of Section 6. For cochain complexes which are concentrated in non-negative degrees we have to provide a factorization of the folding map $\mathbb{S}_0 \oplus \mathbb{S}_0 \longrightarrow \mathbb{S}_0$ into a cofibration followed by a weak equivalence,

$$\mathbb{S}_0 \oplus \mathbb{S}_0 \rightarrowtail H \xrightarrow{\sim} \mathbb{S}_0$$

such that H is a commutative Hopf-object in the category of cochain complexes.

But cofibrations in the category of cochain complex do not have to satisfy any condition in their zeroth degree, so we can take $H = \mathbb{S}_0$ which is a canonical Hopf-object with commutative multiplication $k \otimes_k k \cong k$ and cocommutative comultiplication $k \cong k \otimes_k k$. So we have proved the following result:

Proposition 8.3.1. *The category of cochain complexes which are concentrated in non-negative degrees possesses a commutative Hopf-interval.*

Furthermore, it is easy to see that this model category has a monoidal fibrant replacement functor over the unit.

Proposition 8.3.2. *For every cochain complex $X^* \in \delta\mathsf{mod}$ over $\mathbb{S}_0$ there is a functorial factorization*

$$X^* \overset{\sim}{\rightarrowtail} R(X^*) \twoheadrightarrow \mathbb{S}_0.$$

Proof. Adding a zero codisk $\mathbb{D}_0$ does not change the cohomology and is a cofibration. There is a well-defined projection down to $\mathbb{S}_0$ sending the generator in degree zero of $\mathbb{D}_0$ to the generator in $\mathbb{S}_0^0$ and sending $\mathbb{D}_0^1$ to zero. Therefore we set $R(X^*) := X^* \oplus \mathbb{D}_0$. This is clearly functorial and monoidal using the following map as monoidal transformation:

$$\begin{array}{ccc} (X^* \oplus \mathbb{D}_0) \otimes (Y^* \oplus \mathbb{D}_0) & \xrightarrow{\cong} & X^* \otimes Y^* \oplus X^* \otimes \mathbb{D}_0 \oplus \mathbb{D}_0 \otimes Y^* \oplus \mathbb{D}_0 \otimes \mathbb{D}_0 \\ & & \downarrow \mathrm{id} \oplus 0 \oplus 0 \oplus 0 \\ & & (X^* \otimes Y^*) \oplus \mathbb{D}_0. \quad \square \end{array}$$

For the category of cochain complexes, the requirements of [1, Theorem 3.1] are therefore fulfilled and we obtain the following result:

Theorem 8.3.3. *Reduced operads in the category of cochain complexes posses a cofibrantly generated model category structure, such that a map $f : \mathcal{O} \to \mathcal{P}$ is a fibration or weak equivalence, if for all $n \geqslant 1$ the map $f(n) : \mathcal{O}(n) \to \mathcal{P}(n)$ is a fibration or weak equivalence of cochain complexes.*

As the interval $H = \mathbb{S}_0$ is naturally coassociative and cocommutative we get model category structures on algebras as well because the criterium [1, Proposition 4.1.(c)] applies.

Proposition 8.3.4. *The category of algebras over an arbitrary operad in non-negative cochain complexes is a model category with fibrations and weak equivalences given by the forgetful functor. In particular the category of* End_{N^*}*-algebras is a model category.*

Remark 8.3.5. It is tempting to try to transfer this result directly to the category of cosimplicial modules with the help of the functor D^*, but the unit $\eta : \mathrm{id} \to D^*N^*$ is *not* monoidal. Take $A^\bullet = D^*(\mathbb{S}_1)$ and consider cosimplicial degree one. We know that $D^1(\mathbb{S}_1) = k$, but $D^1(N^*D^*(\mathbb{S}_1)^{\otimes 2}) \cong D^1(\mathbb{S}_2) \cong 0$. Thus in the diagram

$$\begin{array}{ccccc} D^1(\mathbb{S}_1)\hat{\otimes}D^1(\mathbb{S}_1) & \xrightarrow{\eta\hat{\otimes}\eta} & D^1N^*(D^*(\mathbb{S}_1)) \otimes D^1N^*(D^*(\mathbb{S}_1)) & \xrightarrow{\psi} & D^1(N^*D^*(\mathbb{S}_1)^{\otimes 2}) \\ & \searrow^{\eta} & & \swarrow^{\mathsf{AW}} & \\ & & D^1N^*(D^*(\mathbb{S}_1)\hat{\otimes}D^*(\mathbb{S}_1)) & & \end{array}$$

the upper composition factors over zero whereas the lower unit $\eta_{D^*(\mathbb{S}_1)\hat{\otimes}D^*(\mathbb{S}_1)}$ is non-trivial, because it fits into the following commutative diagram:

$$\begin{array}{ccc} D^*(\mathbb{S}_1 \otimes \mathbb{S}_1) & \xrightarrow{\sim} & D^*(\mathbb{S}_1)\hat{\otimes}D^*(\mathbb{S}_1) \\ \downarrow \cong & & \downarrow \eta_{D^*(\mathbb{S}_1)\hat{\otimes}D^*(\mathbb{S}_1)} \\ D^*N^*D^*(\mathbb{S}_1 \otimes \mathbb{S}_1) & \xrightarrow{\sim} & D^*N^*(D^*(\mathbb{S}_1)\hat{\otimes}D^*(\mathbb{S}_1)) \end{array}$$

and the non-trivial homotopy group is concentrated in degree one. Therefore the composition

$$A^\bullet \xrightarrow{\eta} D^*N^*A^\bullet \longrightarrow D^*(N^*A^\bullet \oplus \mathbb{D}_0) \longrightarrow D^*(\mathbb{S}_0) \cong \bar{k}$$

will not yield a monoidal fibrant replacement of $A^\bullet$ in general.

8.4. Remarks on cofibrancy

In Theorems 5.4.2 and 6.4.3 we constructed operads End_D and End_{N^*} which have operad maps to the operad of commutative monoids, such that this augmentation is a weak equivalence. This might seem alarming for somebody who is used to work with actual homotopy invariant information.

Sometimes one runs into difficulties if the operad is not cofibrant in a stronger sense. If $\mathcal{O}$ is an operad then a homotopy $\mathcal{O}$-algebra usually is an algebra over a cofibrant resolution of $\mathcal{O}$ in an appropriate model category of operads. So a homotopy $\mathcal{C}$om-algebra in that strong sense is an algebra over $Q(\mathscr{C}om)$, where $Q(\mathscr{C}om)$ is a cofibrant replacement of the operad of commutative monoids.

In our examples, a theorem of Berger and Moerdijk ensures that it makes no difference whether we work with (old-fashioned) E_∞-operads or with operads of the form $Q(\mathscr{C}om)$.

Theorem 8.4.1 (*Berger and Moerdijk [1, 4.5]*). *If cofibrant operads have a model structure for their algebras via the forgetful functor in the sense of [1, 2.5], if $\mathscr{C}$ is left proper and the unit $\mathbf{1}_{\mathscr{C}}$ is cofibrant. Let $\mathscr{O}$ be an arbitrary operad in $\mathscr{C}$. Then for every operad $\tilde{\mathscr{O}} \xrightarrow{\sim} \mathscr{O}$ which has a cofibrant symmetric sequence and has a model structure for its algebras à la [1, 2.5] there is a Quillen equivalence between the category of $\tilde{\mathscr{O}}$-algebras and the category of $Q(\mathscr{O})$-algebras.*

The model category of simplicial modules is (left) proper, hence every operad has such a model structure for its algebras. Consequently, the model category of a Σ-cofibrant replacement of $\mathsf{End}_D^{\mathscr{O}}$ is Quillen equivalent to the model category of $Q(\mathsf{End}_D^{\mathscr{O}})$-algebras. For non-negative cochain complexes the situation is similar and the model category of algebras over a Σ-cofibrant replacement of $\mathsf{End}_{N^*}^{\mathscr{O}}$ is Quillen equivalent to the model category of $Q(\mathsf{End}_{N^*}^{\mathscr{O}})$-algebras. In particular, the homotopy categories agree in both cases.

Note, that the underlying symmetric sequence of a cofibrant operad is again cofibrant. A proof for this fact is a direct transfer of an argument in [1, 4.3] to our setting. So there are no problems to induce an action of an E_∞-operad in either sense on $D(X)$ and $N^*(A^\bullet)$ for E_∞-algebras X and $A^\bullet$.

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