A model for the stable homotopy category II

Birgit Richter, January 10, 2022

1. Brown representability and the Bousfield-Friedlander structure

Theorem [Br] [Edgar Henry Brown, December 27, 1926–December 22, 2021)] Every reduced cohomology theory $E^*(-)$ on the category of pointed CW complexes and pointed maps has the form $E^n(X) \cong [X, E_n]$ for some Ω-spectrum $(E_n)_n$.

- Singular cohomology is represented by Eilenberg-Mac Lane spaces: For an abelian group A ([Ma, Chapter 22]):

$$\tilde{H}^n(X; A) \cong [X, K(A, n)].$$

- Bott periodicity tells us that $BU \simeq \Omega SU$ which extends to $\mathbb{Z} \times BU \simeq \Omega U$. The equivalence $U \simeq \Omega BU$ loops to $\Omega U \simeq \Omega^2 BU$. The composite is the equivalence $\mathbb{Z} \times BU \simeq \Omega^2(BU) \simeq \Omega^2(\mathbb{Z} \times BU)$. So

$$\widetilde{KU}^0(X) = [X, \mathbb{Z} \times BU]$$

  and the odd degree groups are given by homotopy classes of maps from X to U.

In the following, when I say based spaces, I mean pointed, compactly generated, weak Hausdorff topological spaces.

The Bousfield-Friedlander category of sequential spectra has sequences of based spaces $E_n, n \in \mathbb{N}_0$ with structure maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$ as objects and morphisms $f \colon E \to F$ of spectra are families of pointed continuous maps $f_n \colon E_n \to F_n$ such that

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\sigma_n} & E_{n+1} \\ {\scriptstyle \Sigma f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_n & \xrightarrow{\sigma_n} & F_{n+1} \end{array}$$

commutes for all n.

Aldridge Knight Bousfield (April 5, 1941–October 4, 2020; known as Pete Bousfield)

The kth stable homotopy group of a sequential spectrum E is

$$\pi_k^s E = \mathrm{colim}_n \pi_{n+k} E_n.$$

A morphism $f \colon E \to F$ of sequential spectra is a

- *strict weak equivalence*, if each f_n is a weak equivalence of topological spaces (ie, induces an iso on ordinary homotopy groups).
- *strict fibration*, if each f_n is a Serre fibration (ie, has the RLP wrt $\mathbb{D}^k \hookrightarrow \mathbb{D}^k \times [0, 1]$ for all k).
- *strict cofibration* if it has the LLP wrt acyclic fibrations.

One then gets the Bousfield-Friedlander stable model category structure on sequential spectra by a Bousfield localization. This yields the stable model structure on sequential spectra:

A morphism $f\colon E \to F$ of sequential spectra is a *stable homotopy equivalence*, if

$$\pi_k^s(f)\colon \pi_k^s(E) = \mathrm{colim}_n \pi_{n+k} E_n \to \mathrm{colim}_n \pi_{n+k} F_n = \pi_k^s(F)$$

is an isomorphism for all k.

It is a *stable cofibration* if it is a strict cofibration and the stable fibrations are the maps with the RLP wrt stable cofibrations that are stable homotopy equivalences.

Why are we not happy with that structure? We want to do algebra in the category of spectra. To that end we need an analogue of a tensor product, that is called a smash product. Sequential spectra have a smash product, but this isn't good enough.

2. Symmetric and orthogonal spectra

The idea of symmetric spectra and orthogonal spectra is, to require the extra data of suitable group actions on the spaces in a spectrum that are compatible with the structure maps. The resulting smash products give rise to a symmetric monoidal category structure.

We start with symmetric spectra [HSS, Sch].

Definition A *symmetric spectrum* E is a family E_n of based topological spaces together with based maps

$$\sigma\colon S^1 \wedge E_n \to E_{n+1}$$

for each $n \geq 0$ such that there is a base-point preserving left action of Σ_n on E_n such that the composition

$$\sigma^p := \sigma \circ (S^1 \wedge \sigma) \circ \ldots \circ (S^{p-1} \wedge \sigma)\colon S^p \wedge E_n \to E_{p+n}$$

is $\Sigma_p \times \Sigma_n$-equivariant for all $p \geq 1$ and $n \geq 0$.

A map $f\colon E \to F$ of symmetric spectra is a family of Σ_n-equivariant based maps $f_n\colon E_n \to F_n$ such that the diagram

$$\begin{array}{ccc} S^1 \wedge E_n & \xrightarrow{\sigma} & E_{n+1} \\ {\scriptstyle S^1 \wedge f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge F_n & \xrightarrow{\sigma} & F_{n+1} \end{array}$$

commutes for all $n \geq 0$.

We denote by Sp^Σ the category of symmetric spectra.

For orthogonal spectra you just replace the symmetric groups by the family of orthogonal groups:

Definition An *orthogonal spectrum* E is a family E_n of based topological spaces together with based maps

$$\sigma\colon S^1 \wedge E_n \to E_{n+1}$$

for each $n \geq 0$ such that there is a base-point preserving left action of $O(n)$ on E_n such that the composition

$$\sigma^p := \sigma \circ (S^1 \wedge \sigma) \circ \ldots \circ (S^{p-1} \wedge \sigma)\colon S^p \wedge E_n \to E_{p+n}$$

is $O(p) \times O(n)$-equivariant for all $p \geq 1$ and $n \geq 0$.

A map $f\colon E \to F$ of orthogonal spectra is a family of $O(n)$-equivariant based maps $f_n\colon E_n \to F_n$ such that the diagram

$$\begin{array}{ccc} S^1 \wedge E_n & \xrightarrow{\sigma} & E_{n+1} \\ {\scriptstyle S^1\wedge f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge F_n & \xrightarrow{\sigma} & F_{n+1} \end{array}$$

commutes for all $n \geq 0$.

We denote by Sp^O the category of symmetric spectra.

For symmetric and orthogonal spectra you use the canonical inclusions $\Sigma_p \times \Sigma_n \subset \Sigma_{p+n}$ and $O(p) \times O(n) \subset O(p+n)$. Note that the monomorphism $\Sigma_n \to O(n)$ that sends a permutation to the corresponding permutation matrix ensures that every orthogonal spectrum is a symmetric spectrum. If you forget the symmetric group action, you get a spectrum in the sense of Bousfield-Friedlander.

For both types of spectra it is now relatively straightforward to define a smash product:

Definition

- Let E and F be symmetric spectra. Their *smash product,* $E \wedge F$ is a symmetric spectrum whose nth space $(E \wedge F)_n$ is the coequalizer of the maps

$$\bigvee_{p+1+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} E_p \wedge S^1 \wedge F_q \overset{\beta_n}{\underset{\alpha_n}{\rightrightarrows}} \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} E_p \wedge F_q.$$

  Here, α_n maps the $(p,1,q)$-summand $(\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} E_p \wedge S^1 \wedge F_q$ to $(\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} E_{p+1} \wedge F_q$ by using the permutation $E_p \wedge S^1 \wedge F_q \to S^1 \wedge E_p \wedge F_q$ followed by the structure map $\sigma \wedge F_q\colon S^1 \wedge E_p \wedge F_q \to E_{p+1} \wedge F_q$ and β_n is induced by $E_p \wedge \sigma\colon E_p \wedge S^1 \wedge F_q \to E_p \wedge F_{q+1}$.

  The structure map $\sigma\colon S^1 \wedge (E \wedge F)_n \to (E \wedge F)_{n+1}$ of the smash product is induced by the structure map on E: On the summand $E_p \wedge F_q$ one has $\sigma \wedge F_q\colon S^1 \wedge E_p \wedge F_q \to E_{p+1} \wedge F_q$.

  Note that we could have also switched the source to $E_p \wedge S^1 \wedge F_q$ in order to apply the structure map of F. As we defined the smash product as a coequalizer, these two maps agree.
- For orthogonal spectra, you replace the symmetric groups by the orthogonal groups.

Before we explain, why this gives rise to symmetric monoidal structures, we first introduce some standard examples. From now on we focus on symmetric spectra.

(1) We use the explicit model of S^n iteratively as $S^1 \wedge S^{n-1}$. Then Σ_n permutes the coordinates and $\sigma\colon S^1 \wedge S^n \to S^{n+1}$ is the identity. Hence this defines a symmetric spectrum, S, which forgets to model of the sphere spectrum in the Bousfield-Friedlander category.

(2) Similarly, the model of suspension spectra that we considered last time can be lifted to a model in symmetric spectra. Given a based space X we set $(\Sigma^\infty X)_n := S^n \wedge X$ and declare that the Σ_n-action only takes place on the S^n-factor.

(3) Let A be an abelian group. The symmetric version of the Eilenberg-Mac Lane spectrum of A is best defined via simplicial sets. So we take the simplicial model of the 1-sphere, $S^1 := \Delta_1/\partial\Delta_1$ and set $S^n := (S^1)^{\wedge n}$. Then
$$(HA)_n := A \otimes \tilde{\mathbb{Z}}[S^n]$$
where the right hand factor is the reduced free simplicial abelian group generated by the simplicial n-sphere. If you want a model in spaces, then just take the geometric realization.

The sphere spectrum is then the unit in $(Sp^\Sigma, \wedge)$: One can describe the smash product of symmetric spectra as
$$E \wedge F = E \otimes_S F.$$
Here, $E \otimes F$ is the smash product of symmetric sequences, so
$$(E \otimes F)_n = \bigvee (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} E_p \wedge F_q.$$
Symmetric spectra are then nothing but S-modules for this structure and thus $S \otimes_S F \cong F$ and $E \otimes_S S \cong E$. Associativity of the smash product is clear. Symmetry is interesting:

For $E \otimes F \cong F \otimes E$ one sends a representative $(\tau; e, f) \in (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} E_p \wedge F_q$. to $(\tau \circ \chi; f, e)$ where χ is the (q,p)-shuffle in Σ_n, that sends the first q numbers to $p+1, \ldots, p+q$ and the last p numbers to $1, \ldots, p$.

3. Model category structures

The first guess would be to define the weak equivalences of symmetric spectra to be the maps that induce an isomorphism on stable homotopy groups. That does *not* work.

We start with a level structure:

Definition A morphism $f\colon E \to F$ of symmetric spectra is a *level equivalence* if each $f_n\colon E_n \to F_n$ is a weak equivalence of topological spaces. It is a *level fibration*, if each f_n is a Serre fibration. *Level cofibrations* are the maps with the LLP wrt acyclic fibrations.

This is indeed a model structure [Sch, Theorem 1.13].

One can show the following: If f is a map of symmetric spectra such that $\pi_* f$ is an isomorphism, then f is a stable equivalence. But there are more:

There are free functors from spaces to symmetric spectra: For $m \in \mathbb{N}_0$ and a based space X we define the symmetric spectrum $F_m X$ as
$$(F_m X)_n := (\Sigma_n)_+ \wedge_{\Sigma_{n-m} \times \{id_m\}} S^{n-m} \wedge X.$$
If $m > n$, then this is set to be a point. The structure map $\sigma\colon S^1 \wedge (F_m X)_n \to (F_m X)_{n+1}$ merges the S^1 with the S^{n-m}.

The functor F_m is left adjoint to the forgetful functor that sends a symmetric spectrum to its mth space.

Note that $F_0 X$ is nothing but the suspension spectrum on X, so in particular $\pi_0 F_0 S^0 = \pi_0^s = \mathbb{Z}$ whereas $\pi_0 F_1 S^1$ has countably many copies of the integers, because $(F_S^1)_n = (\Sigma_n)_+ \wedge_{\Sigma_{n-1}} S^1 \wedge S^{n-1} \simeq \bigvee_{i=1}^n S^n$ and the structure maps for the colimit that calculate π_0 are inclusions of direct summands. Therefore the map $F_1 S^1 \to F_0 S^0$ (that is left adjoint to the identity map $S^1 \to (F_0 S^0)_1 = S^1$) is not an isomorphism on stable homotopy groups.

But stably it shouldn't matter if you take the free spectrum starting in level 0 of S^0 or the free spectrum starting in level 1 of S^1, so this map should be a stable equivalence of symmetric spectra.

References

[BF] Aldridge Bousfield, Eric Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, Springer Lecture Notes in Math., Vol. 658, Springer, Berlin, 1978, pp. 80–130.

[Br] Edgar H. Brown, Cohomology theories, Annals of Mathematics, Second Series, 75, 1962, 467–484.

[HSS] Mark Hovey, Brooke Shipley and Jeff Smith, Symmetric spectra, J. Amer. Math. Soc. 13 (2000), 149–208.

[MMSS] Michael A. Mandell, J. Peter May, Stefan Schwede, Brooke Shipley, Model categories of diagram spectra, Proc. London Math. Soc. 82 (2001), 441–512.

[Ma] J. Peter May, A concise course in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. x+243 pp.

[Sch] Stefan Schwede, An untitled book project about symmetric spectra, available on `https://www.math.uni-bonn.de/people/schwede/SymSpec.pdf`