Algebraic models for topological spaces

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So the above models help to decide whether $X_{\mathbb{Q}} \sim Y_{\mathbb{Q}}$.
What can we expect, if we want to control the actual (weak) homotopy type?

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Jim McClure, Jeff Smith: Multivariable cochain operations and little *n*-cubes. J. Amer. Math. Soc. 16 (2003).
They construct an E_∞-operad out of such cochain operations and their generalizations. This operad acts on cochains of a space.

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Formally: We have chain complexes $E(n)$ that are contractible and free as Σ_n-chain complexes, together with actions

$$E(n) \otimes_{\Sigma_n} S^*(X)^{\otimes n} \to S^*(X),$$

satisfying a long list of coherence conditions...

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Theorem [R-Shipley 2017]
There is a zigzag of Quillen equivalences between the category of differential graded E_∞-algebras and the category of commutative $\mathcal{I}$-chain-algebras.

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- The functors $X \mapsto \mathrm{hocolim}_{\mathcal{I}} A^{\mathcal{I}}(X; k)$ and $X \mapsto S^*(X; k)$ from simplicial sets to E_∞-algebras are naturally quasi-isomorphic.

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What about the other models? So what about differential graded cocommutative coalgebras and Lie-algebras?

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The Day convolution product gives $\mathsf{Ch}^{\mathcal{I}}$ a symmetric monoidal structure. Explicitly, for two $\mathcal{I}$-chain complexes X_*, Y_*

$$(X_* \boxtimes Y_*)(\mathsf{n}) = \mathrm{colim}_{\mathcal{I}(\mathsf{p} \sqcup \mathsf{q}, \mathsf{n})} X_*(\mathsf{p}) \otimes Y_*(\mathsf{q}).$$

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Definition: Commutative $\mathcal{I}$-chain algebras are commutative monoids in $\mathsf{Ch}^{\mathcal{I}}$.

Free things

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For any $\mathcal{I}$-chain complex X_*, the free commutative $\mathcal{I}$-chain algebra on X_* is

$$\mathrm{S}^{\mathcal{I}}(X_*) = \bigoplus_{n \geq 0} X_*^{\boxtimes n}/\Sigma_n.$$

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$$\bigoplus_{[f_q|\ldots|f_1] \in N_q\mathcal{I}} X_p(\mathrm{source}(f_1))$$

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If C_* is a cocommutative comonoid in $\mathrm{Ch}^{\mathcal{I}}$, what can we say about $\mathrm{hocolim}_{\mathcal{I}} C_*$?

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In the category of symmetric sequences of chain complexes, Ch^{Σ}, the norm map *is* an iso on reduced objects [Stover, Fresse].

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The co-free cocommutative coalgebra on a chain complex C_*, rationally, is described via Σ-invariants:

$$S'_*(C_*) = \bigoplus_{n \geq 0} (C_*^{\otimes n})^{\Sigma_n}.$$

Then one uses that rationally the norm map

$$N_n \colon C_*^{\otimes n}/\Sigma_n \to (C_*^{\otimes n})^{\Sigma_n}$$

is an equivalence in order to determine the homotopy type.

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Theorem There are reduced $X_* \in \mathrm{Ch}^{\mathcal{I}}$ (i.e., $X_*(0) = 0$) such that

$$N_n \colon X_*^{\boxtimes n}/\Sigma_n \to (X_*^{\boxtimes n})^{\Sigma_n}$$

is *not* an isomorphism.

A counterexample

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So this element is invariant under the Σ_2-action, but it is not in the image of the norm map, unless 2 is invertible in k.

For any chain complex C_*, for every m and for every $p \geq 1$ the norm $N_n = \sum_{\sigma \in \Sigma_n} \sigma \in \mathbb{Z}[\Sigma_n]$ induces an isomorphism of chain complexes

$$N_n \colon (F_p^{\mathcal{I}}(C_*)^{\boxtimes n}/\Sigma_n)(\mathrm{m}) \to ((F_p^{\mathcal{I}}(C_*)^{\boxtimes n})^{\Sigma_n}(\mathrm{m}).$$

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This follows from the fact that $(F_p^{\mathcal{I}}(C_*))^{\boxtimes n} \cong F_{pn}^{\mathcal{I}}(C_*^{\otimes n})$ and that the Σ_n-action is free on $\mathcal{I}(\mathrm{pn}, \mathrm{m})$ as long as $p \geq 1$.

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Note that this implies that the free commutative monoid on $F_p^{\mathcal{I}}(C_*)$ is isomorphic to the free divided power algebra and the cofree cocommutative coalgebra generated on $F_p^{\mathcal{I}}(C_*)$.

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Proposition Even if we work over a field, the Künneth map is in general *not* an isomorphism.
There is a concrete counterexample.

Consider a chain complex C_* over a field with a chosen zero cycle c_0 and let $\mathrm{Sym}^{\mathcal{I}}(C_*) \in \mathrm{Ch}^{\mathcal{I}}$ be defined as

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This results in $D^1 \oplus_{S^0} D^1$ which has nontrivial H_1.

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Thank you!