LODAY CONSTRUCTIONS OF TAMBARA FUNCTORS

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Abstract. Building on work of Hill, Hoyer and Mazur we propose an equivariant version of a Loday construction for G-Tambara functors where G is an arbitrary finite group. For any finite simplicial G-set and any G-Tambara functor, our Loday construction is a simplicial G-Tambara functor. We study its properties and examples. For a circle with rotation action by a finite cyclic group our construction agrees with the twisted cyclic nerve of Blumberg, Gerhardt, Hill, and Lawson. We also show how the Loday construction for genuine commutative G-ring spectra relates to our algebraic one via the $\underline{\pi}_0$-functor. We describe Real topological Hochschild homology as such a Loday construction.

1. Introduction

In this paper we generalize the Loday construction for commutative rings [Pir00] to the equivariant context where the groups involved are finite. For every genuine commutative G-ring spectrum R, Brun showed [Bru07] that $\underline{\pi}_0(R)$ is a G-Tambara functor – this is roughly a Mackey functor with compatible commutative ring structures and multiplicative norms; see [Tam93, §2] for the definition. We will define a Loday construction for G-Tambara functors for any finite group G and investigate some of its properties.

Our work builds on the Hill-Hopkins notion of a G-symmetric monoidal category [HH]. In equivariant algebra there is an important difference between commutative monoids in the category of G-Mackey functors – these are the commutative G-Green functors – and G-commutative monoids – these are the G-Tambara functors. Mazur, Hill-Mazur and Hoyer [Maz13, HM19, Hoy14] prove that for any finite group and any G-Tambara functor $\underline{R}$ there is a compatible definition of the tensor product of a finite G-set X with $\underline{R}$.

Their construction leads directly to our definition of a Loday construction in Section 2 because their naturality statements ensure that one can extend this tensor product to a tensor product of a G-Tambara functor with a finite simplicial G-set. We study basic properties like the behaviour of the Loday construction on free Tambara functors and on norms in Section 3 and we confirm that restricting a G-Loday construction to H for a subgroup $H < G$ gives the H-Loday construction of the H-restricted Tambara functor. We prove in Section 4 that the Loday construction is homotopy invariant with respect to G-homotopy equivalences. Even for constant Tambara functors the Loday construction detects interesting features of the G-simplicial set and embedding commutative rings into the equivariant setting by taking their constant Tambara functors doesn't produce any undesired properties. We discuss these facts in Section 5. We show how the $\underline{\pi}_0$-functor relates Loday constructions for genuine G-equivariant commutative ring spectra to the Loday construction of the corresponding Tambara functors in Section 6.

Section 7 is the heart of the paper where we present several explicit examples of Loday constructions. We relate our Loday construction for a circle with rotation action to the twisted cyclic nerve of [BGHL19] and show that at the free level we just obtain a subdivision of the ordinary Loday construction. We also identify Loday constructions for unreduced suspensions where we either fix the suspension apices or a C_2-action flips them.

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We show that Real topological Hochschild homology can be expressed as an equivariant Loday construction for flat and well-pointed genuine commutative C_2-ring spectra. In this example the finite simplicial C_2-set is a circle with flip action.

For the classical Loday construction working relative to a base-ring is often crucial for performing calculations. We propose a relative version of the Loday construction for Tambara functors in Section 8.

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2. Definition and basic properties

Let G be a finite group, and let $\underline{R}$ be a G-Tambara functor, as in [Tam93, §2]. There is a tensor product of $\underline{R}$ with finite G-sets and our definition of Loday constructions for G-Tambara functors is based on that. In the context of cyclic p-groups for a prime p, this was developed in Mazur's thesis [Maz13, Theorem 2.3.1] and published in joint work of Hill and Mazur [HM19, Theorem 5.2]. For general finite groups Hoyer provided a construction in [Hoy14, §2.4].

In the following let G be a finite group and we denote by Mack_G the category of G-Mackey functors. This is a symmetric monoidal category with respect to the $\square$-product of Mackey functors. The starting point is a G-symmetric monoidal structure in the sense of [HH, Definition 3.3.]. Let $\mathsf{Sets^f}_G$ denote the category of finite G-sets and we consider the wide subcategory where we restrict the morphisms to isomorphisms of finite G-sets. There is a functor

$$(-) \otimes (-) \colon (\mathsf{Sets^f}_G, G\text{−isoms}) \times \mathsf{Mack}_G \to \mathsf{Mack}_G \tag{2.1}$$

which satisfies the following properties:

(1) For all X and Y in $\mathsf{Sets^f}_G$ and $\underline{M}$, $\underline{N}$ in Mack_G, there are natural isomorphisms

$$(X \amalg Y) \otimes \underline{M} \cong (X \otimes \underline{M}) \square (Y \otimes \underline{M}) \tag{2.2}$$

and

$$X \otimes (\underline{M} \square \underline{N}) \cong (X \otimes \underline{M}) \square (X \otimes \underline{N}). \tag{2.3}$$

(2) There is a natural isomorphism

$$X \otimes (Y \otimes \underline{M}) \cong (X \times Y) \otimes \underline{M}. \tag{2.4}$$

(3) On the category with objects finite sets with trivial G-action and morphisms consisting only of isomorphisms, the functor restricts to exponentiation $X \otimes \underline{M} = \square_{x \in X} \underline{M}$.

If we now turn to the category of G-Tambara functors, Tamb_G, Definition 5.3 and Proposition 5.4 in [HM19] for cyclic p-groups and [Hoy14, Theorem 2.7.4] in the general case allow us to extend the definition in (2.1) to the category $\mathsf{Sets^f}_G$ where morphisms are all G-maps,

$$(-) \otimes (-) \colon \mathsf{Sets^f}_G \times \mathsf{Tamb}_G \to \mathsf{Tamb}_G. \tag{2.5}$$

The tensor product from (2.5) has an explicit description. By [HM19, Theorem 5.2] for the case of G a cyclic p-group and more generally [Hoy14, Theorem 2.7.4] we get that for every subgroup $H < G$ tensoring with G/H can be identified as

$$G/H \otimes \underline{R} \cong N_H^G i_H^* \underline{R}, \tag{2.6}$$

where i_H^* is the restriction functor $i_H^* \colon \mathsf{Tamb}_G \to \mathsf{Tamb}_H$ that takes a G-Tambara functor $\underline{R}$ to the H-Tambara functor

$$i_H^* \underline{R}(H/K) = \underline{R}(G \times_H H/K) \cong \underline{R}(G/K)$$

for any subgroup $K \leqslant H$; N_H^G is the norm, as in [HM19, Definition 3.13] and [Hoy14, Theorem 2.1.1]. Using (2.2), we immediately get that if $X = \amalg_{\alpha \in A} G/H_\alpha$ for subgroups $H_\alpha \leqslant G$, then

$$X \otimes \underline{R} \cong \square_{\alpha \in A} N_{H_\alpha}^G i_{H_\alpha}^* \underline{R}. \tag{2.7}$$

Remark 2.1. Since Tambara functors are formally just a diagram category with entries that are commutative rings, we could use direct limits of commutative rings to give us direct limits of Tambara functors, which still have the required structure for being Tambara functors. This allows us to extend the functor from (2.5) to G-sets that are not necessarily finite:

$$(-) \otimes (-) \colon \mathsf{Sets}_G \times \mathsf{Tamb}_G \to \mathsf{Tamb}_G. \tag{2.8}$$

This generalizes a construction in the nonequivariant setting, that is crucial for the Loday construction. For any finite set X and any commutative ring R, the assignment

$$(X, R) \mapsto X \otimes R = \bigotimes_{x \in X} R$$

is functorial in X and R. For an arbitrary (possibly infinite set) X this can be defined as the colimit over finite subsets of X of this construction. Maps $f \colon X \to Y$ between finite sets are sent to

$$f_* \colon \bigotimes_{x \in X} R \to \bigotimes_{y \in Y} R, \qquad f_*(\bigotimes_{x \in X} r_x) = \bigotimes_{y \in Y} b_y \quad \text{with} \quad b_y = \prod_{x \in f^{-1}(y)} r_x.$$

This is well-defined because R is assumed to be commutative.

An analogous definition works for simplicial rings and for commutative ring spectra.

This construction is the basis of the (nonequivariant) Loday construction, which takes a simplicial set $X.$ and a commutative ring R and sends them to the simplicial commutative ring

$$\mathcal{L}_{X.}(R) = \{[n] \mapsto X_n \otimes R = \bigotimes_{x \in X_n} R\} \tag{2.9}$$

such that the simplicial structure maps are induced by the corresponding maps in X. If $R.$ is a simplicial commutative ring, then we take the diagonal of the bisimplicial result of the above definition. That results in a simplicial commutative ring. If R is a commutative ring spectrum, we use the smash product instead of the tensor product and take the realization of the resulting simplicial commutative ring spectrum to obtain a commutative ring spectrum.

Definition 2.2. Let G be a group, $X.$ be a simplicial G-set, and $\underline{R}$ be a G-Tambara functor. *The equivariant Loday construction of $\underline{R}$ with respect to $X.$* is defined to be the simplicial G-Tambara functor

$$\mathcal{L}_{X.}^G(\underline{R}) = \{[n] \mapsto X_n \otimes \underline{R}\},$$

where $X_n \otimes \underline{R}$ is defined by the functor in (2.8). The simplicial structure maps are induced by the corresponding maps in X.

Remark 2.3. By construction the equivariant Loday construction is natural in the simplicial G-set $X.$ and in the G-Tambara functor $\underline{R}$. We will mostly consider finite simplicial G-sets $X.$ These are simplicial objects in finite G-sets.

Proposition 2.4. *The equivariant Loday construction satisfies the following properties:*

(1) *For all finite simplicial G-sets $X.$ and $Y.$ and G-Tambara functors $\underline{R}$ and $\underline{T}$, there are natural isomorphisms of simplicial Tambara functors*

$$\mathcal{L}_{X.\amalg Y.}^G(\underline{R}) \cong \mathcal{L}_{X.}^G(\underline{R}) \square \mathcal{L}_{Y.}^G(\underline{R})$$

and

$$\mathcal{L}_{X.}^G(\underline{R} \square \underline{T}) \cong \mathcal{L}_{X.}^G(\underline{R}) \square \mathcal{L}_{X.}^G(\underline{T}).$$

(2) *For all finite simplicial G-sets $X.$ and $Y.$ and any G-Tambara functor $\underline{R}$, there is a natural isomorphism between the diagonal of the bisimplicial Tambara functor $\mathcal{L}^G_{X.}(\mathcal{L}^G_{Y.}(\underline{R}))$ and the simplicial Tambara functor $\mathcal{L}^G_{X.\times Y.}(\underline{R})$.*

(3) *If $X.$ and $Y.$ are finite simplicial G-sets containing a common simplicial G-subset $Z.$ and $\underline{R}$ is G-Tambara functor, then*

$$\mathcal{L}^G_{X.\sqcup_{Z.}Y.}(\underline{R}) \cong \mathcal{L}^G_{X.}(\underline{R})\Box_{\mathcal{L}^G_{Z.}(\underline{R})}\mathcal{L}^G_{Y.}(\underline{R}).$$

Proof. These are all proved levelwise, using the isomorphisms of (2.2), (2.3), and (2.4), and the fact that for all $n \geqslant 0$,

$$(X_n \sqcup_{Z_n} Y_n) \otimes \underline{R} \cong (X_n \otimes \underline{R})\Box_{Z_n \otimes \underline{R}}(Y_n \otimes \underline{R})$$

also using (2.2). Note that $Z.$ has to contain entire orbits to be a simplicial G-subset. □

Remark 2.5. There is an explicit description of the action of the Weyl group of H in G, $W_G(H)$, on terms of the form $N^G_H i^*_H \underline{R}$ for instance in [HM19, Proof of Proposition 5.10] in the case of cyclic p-groups. This combines a cyclic permutation action and a coordinatewise action. It is also observed in [HM19, Theorem 5.11] that $N^G_H i^*_H \underline{R}$ with this Weyl action is isomorphic to $N^G_H i^*_H \underline{R}$ with the Weyl action from [HM19, Fact 5.8]. So for the Loday construction there is a choice to make and we choose to work with the first Weyl action.

3. Basic constructions

3.1. **Free Tambara functors.** For every G-Tambara functor $\underline{R}$ the commutative ring $\underline{R}(G/e)$ carries a G-action that is compatible with the ring structure. We call the category of such rings together with equivariant ring maps the category of commutative G-rings and we denote it by $\mathsf{cGrings}$.

Lemma 3.1. *([Bru05, §2]) There exists a left adjoint $\underline{F}(-)\colon \mathsf{cGrings} \to \mathsf{Tamb}_G$ to the functor*

$$\mathsf{Tamb}_G \to \mathsf{cGrings}, \quad \underline{R} \mapsto \underline{R}(G/e).$$

Proof. Brun first shows that the category of commutative G-rings is equivalent to the category of fG-Tambara functors [Bru05, Lemma 6 (i)]. Here, an fG-Tambara functor just accepts free orbits as input. He then constructs the free G-Tambara functor $\underline{F}(R)$ as the left Kan extension of the fG-Tambara functor corresponding to R along the inclusion $U_{fG} \hookrightarrow U_G$ [Bru05, p. 241]. Here, U_{fG} (U_G) denotes the category of bispans based on finite free G-sets (all finite G-sets). □

A concrete formula of $\underline{F}(-)$ in the case of $G = C_p$ can be found in [Maz13, Example 1.4.8].

Remark 3.2. Note that the right adjoint functor U that sends a G-Tambara functor $\underline{R}$ to $\underline{R}(G/e)$ is not faithful: for simplicity assume that $G = C_2$ and let $\underline{R}$ be a C_2-Tambara functor. Let M be a free $\underline{R}(C_2/C_2)$-module of rank 1 with generator m. Then we can define a new C_2-Tambara functor $\underline{R} \rtimes M$ as $\underline{R} \rtimes M(C_2/e) = \underline{R}(C_2/e)$ and $\underline{R} \rtimes M(C_2/C_2) := \underline{R}(C_2/C_2) \oplus M$ with the square-zero multiplication. We keep the norm and transfer maps from $\underline{R}$ on $\underline{R} \rtimes M$ and define the restriction of m to be zero. Then

$$\begin{aligned} &\mathsf{cGrings}(U(\underline{R} \rtimes M), U(\underline{R} \rtimes M)) = \mathsf{cGrings}(\underline{R} \rtimes M(C_2/e), \underline{R} \rtimes M(C_2/e)) \\ =&\mathsf{cGrings}(\underline{R}(C_2/e), \underline{R}(C_2/e)) = \mathsf{cGrings}(U(\underline{R}), U(\underline{R})) \end{aligned}$$

whereas we can define at least two different self-maps of the C_2-Tambara functor $\underline{R} \rtimes M$ by taking the identity on the free level and one morphism that sends $(r, m) \in \underline{R} \rtimes M(C_2/C_2)$ to $(r, 0)$ and a second one that sends (r, m) to (r, m).

A consequence is that the counit of the adjunction $\varepsilon\colon \underline{F} \circ U \to \mathrm{Id}$ is *not* an epimorphism (see e.g. [Ric20, Proposition 2.4.11]).

In the following we denote by res^G_H the forgetful functor from G-sets to H-sets for a subgroup $H \leqslant G$. We recall the following standard isomorphism. Let Z be a finite G-set. Then

$$G \times_H \mathsf{res}^G_H Z \cong G/H \times Z \tag{3.1}$$

where the isomorphism is given by $[g, z] \mapsto (gH, gz)$ with inverse $(gH, z) \mapsto [g, g^{-1}z]$. This isomorphism transforms the G-action on the left factor of $G \times_H \mathsf{res}^G_H Z$ to the diagonal G-action on $G/H \times Z$.

Lemma 3.3. *Let G be a finite group, and let Y be a finite G-set. Then the functor*

$$Y \otimes (-)\colon \mathsf{Tamb}_G \to \mathsf{Tamb}_G$$

is a left adjoint. Its right adjoint sends $\underline{T}$ to $\underline{T}^Y$ which maps a finite G-set Z to

$$\underline{T}^Y(Z) := \underline{T}(Y \times Z).$$

Proof. Again, it suffices to prove the claim for orbits. By definition $G/H \otimes \underline{R} = N^G_H i^*_H \underline{R}$ and $N^G_H(-)$ is left adjoint to restriction. Therefore for every G-Tambara functor $\underline{T}$

$$\mathsf{Tamb}_G(N^G_H i^*_H \underline{R}, \underline{T}) \cong \mathsf{Tamb}_H(i^*_H \underline{R}, i^*_H \underline{T}).$$

But the restriction functor i^*_H is itself a left adjoint and Strickland calls its right adjoint coinduction [Str, Prop 18.3]. For an H-Tambara functor $\underline{S}$ and a finite G-set X the latter is defined as $\mathsf{coind}^G_H \underline{S}(X) := \underline{S}(\mathsf{res}^G_H(X))$. Therefore

$$\mathsf{Tamb}_H(i^*_H \underline{R}, i^*_H \underline{T}) \cong \mathsf{Tamb}_G(\underline{R}, \mathsf{coind}^G_H i^*_H \underline{T}).$$

We know that $i^*_H \underline{T}(-) = \underline{T}(G \times_H (-))$ and $\mathsf{coind}^G_H \underline{T}(-) = \underline{T}(\mathsf{res}^G_H(-))$. The isomorphism from (3.1) then finishes the proof. $\square$

On the level of Mackey functors, coinduction is actually the left and right adjoint of restriction i^*_H (see [TW95, p. 1871], where coinduction is called induction), but on the level of Tambara functors $N^G_H(-)$ is the left adjoint and coind^G_H is the right adjoint.

If $T.$ is a simplicial commutative G ring, we define $\underline{F}(T.)$ as $\underline{F}(T.)_n := \underline{F}(T_n)$.

Note that for any commutative G-ring R and any finite set X, $X \otimes R = \bigotimes_{x \in X} R$ is a commutative G-ring as well, with G acting on all copies of R simultaneously. Similarly, for any finite simplicial set $X.$, the usual Loday construction $\mathcal{L}_{X.}(R) = X. \otimes R$ with

$$(X. \otimes R)_n := \bigotimes_{x \in X_n} R$$

with simplicial structure maps as in (2.9) and G acting on all copies of R simultaneously is a simplicial commutative G-ring.

Remark 3.4. For any finite set X and commutative G-ring R, the coproduct of copies of R indexed by X in the category of commutative rings $X \otimes R$ is also the corresponding coproduct in the category of commutative G-rings. The crucial point is that the inclusion of any copy of R indexed by $x \in X$ into $X \otimes R$ is G-equivariant by the choice of action on $X \otimes R$, and because of that compatibility factoring G-equivariant maps from X copies of R through $X \otimes R$ will automatically give a G-equivariant map from $X \otimes R$.

We identify the equivariant Loday construction of free Tambara functors as follows:

Proposition 3.5. *For every finite group G, for every commutative G-ring R and every finite simplicial G-set $X.$ we have*

$$\mathcal{L}^G_{X.} \underline{F}(R) \cong \underline{F}(\mathcal{L}_{X.}(R)),$$

where the Loday construction $\mathcal{L}_{X.}(R)$ is defined using the underlying simplicial set $X.$ where we forget the G-action.

Proof. It suffices to check the claim degreewise and because of (2.2) it suffices to check it on orbits G/H. We write $|G/H|$ for the set G/H after forgetting the G-action. For every G-Tambara functor $\underline{T}$ we have

$$\begin{aligned}
\mathsf{Tamb}_G(G/H \otimes \underline{F}(R), \underline{T}) &\cong \mathsf{Tamb}_G(\underline{F}(R), \underline{T}(G \times_H \mathsf{res}^G_H(-))) && \text{by Lemma 3.3}\\
&\cong \mathsf{cGrings}(R, \underline{T}(G \times_H \mathsf{res}^G_H G/e)) && \text{by Lemma 3.1}\\
&\cong \mathsf{cGrings}(R, \underline{T}(G/H \times G/e))\\
&\cong \mathsf{cGrings}(R, \underline{T}(|G/H| \times G/e))\\
&\cong \mathsf{cGrings}(R, \prod_{|G/H|} \underline{T}(G/e))\\
&\cong \mathsf{cGrings}(|G/H| \otimes R, \underline{T}(G/e)) && \text{by Remark 3.4}\\
&\cong \mathsf{Tamb}_G(\underline{F}(|G/H| \otimes R), \underline{T}) && \text{by Lemma 3.1.}
\end{aligned}$$

For the first and second unlabelled isomorphism we use the isomorphism of G-sets from (3.1) $G \times_H \mathsf{res}^G_H G/e \cong G/H \times G/e$. We then compose this isomorphism with the automorphism that sends $(gH, \tilde{g}e)$ to $(\tilde{g}^{-1}gH, \tilde{g}e)$. This reduces the diagonal G-action to the G-action on the right factor G/e. Thus in the last three rows of the equation, we regard G/H as a set rather than as a G-set.

For the third unlabelled isomorphism, we use that $\underline{T}$ is a Tambara functor, so it turns disjoint unions of finite G-sets into products. □

3.2. **Turning a commutative ring into a Tambara functor via the norm.** Let R be a commutative ring. We can view R as an e-Tambara functor, where e denotes the group with one element. For all finite groups G, $N^G_e R$ is then a G-Tambara functor.

An immediate consequence of Proposition 2.4 (2) is the following fact:

Proposition 3.6. *Let R be a commutative ring and let $\underline{R}$ be any G-Tambara functor with $i^*_e\underline{R} = \underline{R}(G/e) = R$. Then for all finite simplicial G-sets X.*

$$\mathcal{L}^G_{X.}(N^G_e R) \cong \mathcal{L}^G_{X.}(N^G_e i^*_e \underline{R}) \cong \mathcal{L}^G_{X.}(G/e \otimes \underline{R}) \cong \mathcal{L}^G_{X. \times G}(\underline{R})$$

where we view G as the constant simplicial set on the right.

We also apply N^G_e to a simplicial commutative ring degreewise and obtain a simplicial G-Tambara functor. If we apply N^G_e to the non-equivariant Loday construction of a commutative ring, we get the following relationship:

Proposition 3.7. *Fix a commutative ring R. Let $\underline{R}$ be any G-Tambara functor with $\underline{R}(G/e) = R$ and let $X.$ be any finite simplicial set. Then*

$$N^G_e \mathcal{L}_{X.}(R) \cong \mathcal{L}^G_{G \times X.}(\underline{R}).$$

Proof. The claim follows directly from Hoyer's naturality requirement in [Hoy14, Definition 2.7.2] that he proves in [Hoy14, Theorem 2.7.4]: For any finite group G, every pair of subgroups $H \leqslant K \leqslant G$ and every finite H-set S there is an isomorphism of K-Tambara functors

$$(K \times_H S) \otimes (i^*_K \underline{R}) \cong N^K_H (S \otimes (i^*_H \underline{R})) \tag{3.2}$$

which is natural in the H-set S. This implies that on the level of Loday constructions we obtain for every finite simplicial H-set X.

$$\mathcal{L}^K_{K \times_H X.}(i^*_K \underline{R}) \cong N^K_H (\mathcal{L}^H_{X.}(i^*_H \underline{R})).$$

We apply his result in the situation where $H = e$ and $K = G$. In this case, we identify $\mathcal{L}^G_{G \times_e X.}(i^*_G \underline{R}) = \mathcal{L}^G_{G \times X.}(\underline{R})$ with $N^G_e(\mathcal{L}_{X.}(i^*_e \underline{R}))$. As $i^*_e \underline{R}(e/e) = \underline{R}(G/e) = R$, this proves the claim. □

Remark 3.8. Note that the norm N_e^G of a commutative ring R does *not* agree with the free G-Tambara-functor, if we start with a commutative G-ring R viewed as a commutative G-ring with trivial G-action: By adjunction $\mathsf{Tamb}_G(N_e^G R, \underline{T}) \cong \mathsf{Tamb}_e(R, \underline{T}(G/e))$ where we view $\underline{T}(G/e)$ just as a commutative ring. In contrast, $\mathsf{Tamb}_G(\underline{F}(R), \underline{T}) \cong \mathsf{cGrings}(R, \underline{T}(G/e))$; so here, we *do* remember the G-action on the commutative ring $\underline{T}(G/e)$. As R carries the trivial action this yields ring morphisms from R into the G-fixed points of $\underline{T}(G/e)$, $\underline{T}(G/e)^G$.

3.3. **Change of groups.** Hoyer's naturality result from (3.2) immediately gives the following isomorphism:

Proposition 3.9. *For every finite group G with a subgroup $H < G$, and every finite simplicial H-set X.*

$$\mathcal{L}^G_{G\times_H X.}(\underline{R}) \cong N_H^G\left(\mathcal{L}^H_{X.}(i_H^*\underline{R})\right)$$

as simplicial G-Tambara functors.

We also have an isomorphism on Loday constructions with respect to restrictions. This follows from the next Theorem 3.11.

Proposition 3.10. *For every finite group G with a subgroup $H < G$, and every finite simplicial G-set X.*

$$i_H^*\mathcal{L}^G_{X.}(\underline{R}) \cong \mathcal{L}^H_{i_H^* X.}(i_H^*\underline{R})$$

as simplicial H-Tambara functors.

Theorem 3.11. *For every finite G-set S and G-Tambara functor $\underline{R}$ there is isomorphism*

$$i_H^*(S \otimes \underline{R}) \cong i_H^*(S) \otimes i_H^*(\underline{R}) \tag{3.3}$$

natural in S and $\underline{R}$.

Proof. In Hoyer's work [Hoy14, Theorem 2.5.1], it is shown that there is an isomorphism

$$i_H^*(G/K \otimes \underline{R}) \cong \square_{\gamma\in H\backslash G/K} N^H_{H\cap^\gamma K}(i^*_{H\cap^\gamma K}\underline{R})$$

that is natural in $\underline{R}$. The naturality in the finite G-set is not stated because it only makes sense when $\underline{R}$ is a G-commutative monoid and Hoyer proved the isomorphism in greater generality for G-Mackey functors. However, the arguments in his proof also show the functoriality in finite G-sets for G-Tambara functors:

We first recall some definitions from [Hoy14]. Let A_G be the Burnside category of isomorphism classes of spans of finite G-sets. Then G-Mackey functors are product preserving functors $A_G \to \mathrm{Set}$ such that the image is levelwise grouplike. For $f\colon X \to Y$ a map of finite G-sets, there are certain maps $\mathsf{res}_f \in A_G(Y, X)$ and $\mathsf{tr}_f \in A_G(X, Y)$, which give the restriction and transfer maps for Mackey functors. A span $X \xleftarrow{f} Z \xrightarrow{g} Y$ is equivalent to the composite $\mathsf{tr}_g\mathsf{res}_f$.

The norm functor for Mackey functors $\mathsf{Mack}_K \longrightarrow \mathsf{Mack}_G$ is the left Kan extension along $\mathrm{Map}_K(G, -)\colon A_K \longrightarrow A_G$. Using the explicit formula for the left Kan extension via coends, we obtain that for $\underline{M} \in \mathsf{Mack}_K$ and for a finite G-set Z, $N_K^G\underline{M}(Z)$ can be expressed as

$$N_K^G\underline{M}(Z) \cong \int^{X\in A_K} A_G(\mathrm{Map}_K(G, X), Z) \times \underline{M}(X).$$

We can therefore represent elements of $N_K^G\underline{M}(Z)$ as

$$\left(\mathrm{Map}_K(G, X) \xleftarrow{f} W \longrightarrow Z, x \in \underline{M}(X)\right) \tag{3.4}$$

up to coend identification, where W is a finite G-set and X is a finite K-set. Note that via the adjunction $(i_K, \mathrm{Map}_K(G, -))$, the G-map f factors as

$$W \xrightarrow{\eta} \mathrm{Map}_K(G, i_K^* W) \xrightarrow{\mathrm{Map}_K(G,\bar{f})} \mathrm{Map}_K(G, X),$$

where in the second map $\bar{f}\colon i_K^*W \longrightarrow X$ is the adjoint of f. Note that for a finite G-set W, the restriction i_K^*W just views W as a K-set. So in the coend, using the morphism $\mathsf{res}_{\bar{f}} \in A_K$, the element (3.4) is identified with the following element in "standard form"

$$\Big(\mathrm{Map}_K(G, i_K^*W) \xleftarrow{\eta} W \longrightarrow Z, w = \mathsf{res}_{\bar{f}}(x) \in \underline{M}(i_K^*W)\Big).$$

Note that the left map corresponding to the restriction map in the standard form is always the counit η.

To obtain elements in $i_H^*(G/K \otimes \underline{R}) = i_H^* N_K^G i_K^* \underline{R}$, we use that the restriction functor $i_H^*\colon \mathsf{Mack}_G \longrightarrow \mathsf{Mack}_H$ is precomposition with $G \times_H -$. Then, the standard form of an element in $i_H^* N_K^G i_K^* \underline{R}(Y)$ for an H-set Y is

$$\Big(\mathrm{Map}_K(G, i_K^*W) \longleftarrow W \longrightarrow G \times_H Y, a \in \underline{R}(G \times_K i_K^*W)\Big). \tag{3.5}$$

Up to isomorphism, $W \cong G \times_H B$ for some H-set B, and (3.5) can be rewritten as

$$\Big(\mathrm{Map}_K(G, i_K^*(G \times_H B)) \longleftarrow G \times_H B \xrightarrow{G \times_H f} G \times_H Y, a \in \underline{R}(G \times_K i_K^*(G \times_H B))\Big). \tag{3.6}$$

To map this element to $i_H^*(G/K) \otimes i_H^*\underline{R} \cong \square_{\gamma \in H\backslash G/K} N_{H\cap^\gamma K}^H i_{H\cap^\gamma K}^* \underline{R}$, Hoyer shows that there is an isomorphism of K-sets

$$\hat{\theta}\colon \coprod_\gamma \gamma^{-1} \cdot {}^\gamma K \times_{H\cap^\gamma K} i_{H\cap^\gamma K}^* B \cong i_K^*(G \times_H B).$$

This produces an isomorphism of G-sets

$$\bar{\theta} = G \times_K \hat{\theta}\colon \coprod_\gamma G \times_{H\cap^\gamma K} i_{H\cap^\gamma K}^* B \cong G \times_K i_K^*(G \times_H B). \tag{3.7}$$

As the left map in (3.6) does not carry any extra information, such elements are in bijection with

$$\Big(\prod \mathrm{Map}_{H\cap^\gamma K}(H, i_{H\cap^\gamma K}^* B) \longleftarrow B \xrightarrow{f} Y, \mathsf{res}_{\bar{\theta}}(a) \in \prod \underline{R}(G \times_{H\cap^\gamma K} i_{H\cap^\gamma K}^* B)\Big). \tag{3.8}$$

This is the standard form of elements in $\square_\gamma N_{H\cap^\gamma K}^H i_{H\cap^\gamma K}^* \underline{R}(Y)$. After checking that this is well-defined for choices of standard forms, Hoyer proved the isomorphism (3.3).

To prove the functoriality, we first consider the case where $K < L$ are subgroups of G and $G/K \to G/L$ is the quotient map. The isomorphism $\bar{\theta}$ is compatible with restrictions in the sense that the following diagram commutes, whose vertical maps are quotients:

$$\begin{tikzcd}
\coprod_{\gamma \in H\backslash G/K} G \times_{H\cap^\gamma K} i_{H\cap^\gamma K}^* B \arrow[r, "\bar{\theta}"] \arrow[d] & G \times_K i_K^*(G \times_H B) \arrow[d] \\
\coprod_{\gamma' \in H\backslash G/L} G \times_{H\cap^{\gamma'} L} i_{H\cap^{\gamma'} L}^* B \arrow[r, "\bar{\theta}"'] & G \times_L i_L^*(G \times_H B).
\end{tikzcd}$$

This proves that for the projection $G/K \to G/L$, the induced map

$$i_H^*(G/K) \otimes i_H^*(\underline{R}) \longrightarrow i_H^*(G/L) \otimes i_H^*(\underline{R})$$

via the isomorphism (3.3) is induced by the projection

$$\coprod_{\gamma \in H\backslash G/K} G \times_{H\cap^\gamma K} i_{H\cap^\gamma K}^* B \longrightarrow \coprod_{\gamma' \in H\backslash G/L} G \times_{H\cap^{\gamma'} L} i_{H\cap^{\gamma'} L}^* B.$$

In other words, it is induced by tensoring $i_H^*(\underline{R})$ with the restriction i_H^* of $G/K \longrightarrow G/L$. The functoriality with respect to conjugation $G/K \longrightarrow G/{}^\gamma K$, the initial morphisms $\varnothing \longrightarrow G/K$ and the fold maps $\nabla\colon G/K \coprod G/K \longrightarrow G/K$ follows similarly. $\square$

4. Homotopy invariance

For a cofibrant commutative ring spectrum A, the non-equivariant Loday construction $\mathcal{L}_{X.}(A)$ is, by [EKMM97, Chapter VII.3], a homotopy invariant of $|X.|$, and therefore, so is $\pi_*(\mathcal{L}_{X.}(A))$. We prove the G-homotopy invariance for Loday constructions of G-spectra later in Proposition 6.1.

It follows from [Pir00, Theorem 2.4] that for two finite simplicial sets $X., Y.$ whose homology groups are isomorphic as graded cocommutative k-coalgebras, the algebraic Loday constructions $\mathcal{L}^k_{X.}(A)$ and $\mathcal{L}^k_{Y.}(A)$ have isomorphic homotopy groups if A is a commutative k-algebra and k is a field. We expect an analogous G-homotopy invariance result for the equivariant version $\mathcal{L}^G_{X.}(\underline{R})$ where $X.$ is a finite simplicial G-set and $\underline{R}$ a G-Tambara functor. For now, we only prove a weaker result:

Theorem 4.1. *Let $X.$ and $Y.$ be two finite simplicial G-sets, equipped with simplicial G-maps $f\colon X. \to Y.$ and $g\colon Y. \to X.$ and simplicial G-homotopies $f \circ g \simeq \mathrm{id}_{X.}$ and $g \circ f \simeq \mathrm{id}_{Y.}$, where G acts trivially on $\Delta^1 = \Delta(-,[1])$. Then for any G-Tambara functor $\underline{R}$, there is a homotopy equivalence*

$$\mathcal{L}^G_{X.}(\underline{R}) \simeq \mathcal{L}^G_{Y.}(\underline{R}).$$

This is, in the usual way, a corollary of the following result.

Proposition 4.2. *Let $f, g\colon X. \to Y.$ be two simplicial G-maps between two finite simplicial G-sets, and assume that there is a simplicial G-homotopy $\mathcal{H}\colon X. \times \Delta^1 \to Y.$ between them, where G acts trivially on Δ^1, with $\mathcal{H}(x, s_0^n(0)) = f(x)$ and $\mathcal{H}(x, s_0^n(1)) = g(x)$ for all $n \geqslant 0$, $x \in X_n$. Then there is a homotopy between $f_*, g_*\colon \mathcal{L}^G_{X.}(\underline{R}) \to \mathcal{L}^G_{Y.}(\underline{R})$.*

A proof in the non-equivariant context can for instance be found in [And71, p. 3].

Proof. We view $\mathcal{L}^G_{(-)}(\underline{R})$ as a functor from finite simplicial G-sets to simplicial G-Tambara functors. We use an assembly map. For $t_n \in \Delta([n],[1])$ we get a map $X_n \to X_n \times \{t_n\}$ sending x_n to (x_n, t_n) and hence

$$\mathcal{L}^G_{X_n}(\underline{R}) \to \mathcal{L}^G_{X_n \times \{t_n\}}(\underline{R}).$$

This assembles into a G-equivariant simplicial map

$$\mathcal{A}\colon \mathcal{L}^G_{X.}(\underline{R}) \times \Delta(-,[1]) \to \mathcal{L}^G_{X. \times \Delta^1}(\underline{R}).$$

Here, G acts trivially on $\Delta(-,[1])$. Finally, we compose

$$\mathcal{L}^G_{\mathcal{H}}(\underline{R}) \circ \mathcal{A}\colon \mathcal{L}^G_{X.}(\underline{R}) \times \Delta^1 \to \mathcal{L}^G_{Y.}(\underline{R})$$

to get the desired homotopy. □

Remark 4.3. Note, that there is no simplicial G-Tambara structure on $\mathcal{L}^G_{X.}(\underline{R}) \times \Delta(-,[1])$, so the homotopy as a map $\mathcal{L}^G_{X.}(\underline{R}) \times \Delta^1 \to \mathcal{L}^G_{Y.}(\underline{R})$ cannot be a morphism in this category. However, we started with maps f, g of simplicial G-sets, so they induce morphisms of simplicial G-Tambara functors. In the situation of Theorem 4.1 they then induce an isomorphism on homotopy groups.

Example 4.4. If $X.$ is any simplicial G-set and $\underline{R}$ is a G-Tambara functor, then there is a homotopy equivalence $\mathcal{L}^G_{CX.}(\underline{R}) \simeq \underline{R}$ where $CX.$ is the cone on $X.$ with G-action induced from that on $X.$ (fixing the cone point) and $\underline{R}$ is interpreted as the constant simplicial object at $\underline{R}$. This holds because $CX.$ is G-simplicially homotopy equivalent to a point.

5. Calculations with constant Tambara functors

Definition 5.1. Let G be a finite group. For any commutative ring R we denote by $\underline{R}^c$ the constant G-Tambara functor with $\underline{R}^c(G/H) = R$ for any subgroup $H \leqslant G$, with $\mathsf{norm}\colon \underline{R}^c(G/H) \to \underline{R}^c(G/K)$ given by $\mathsf{norm}(a) = a^{[K:H]}$ and $\mathsf{tr}\colon \underline{R}^c(G/H) \to \underline{R}^c(G/K)$ given by $\mathsf{tr}(a) = [K:H] \cdot a$ for all $H \leqslant K$ and all $a \in R$, and all restriction maps equal to the identity. The action of all Weyl groups in $\underline{R}^c$ is trivial. We can similarly define the constant G-Tambara functor on a simplicial commutative ring $R.$ as $(\underline{R.})^c_n := \underline{R^c_n}$.

The following result is a sanity check about importing non-equivariant objects into the equivariant setting:

Proposition 5.2. *Let $X.$ be a simplicial set with trivial G-action and let R be any commutative ring. Then*

$$\mathcal{L}^G_{X.}(\underline{R}^c) \cong \underline{\mathcal{L}_{X.}(R)}^c,$$

where $\mathcal{L}_{X.}(R)$ is the nonequivariant Loday construction from (2.9).

Proof. By [LRZ24, Lemma 5.1] we know that the box product of two constant Tambara functors corresponding to commutative rings is the constant Tambara functor corresponding to the tensor product of these rings. So in every simplicial degree n of $\mathcal{L}^G_{X.}(\underline{R}^c)$, we have $\Box_{x\in X_n}\underline{R}^c \cong \underline{(\otimes_{x\in X_n} R)}^c$. □

We now restrict our attention to cyclic groups $G = C_p$ for p a prime and consider the Burnside Tambara functor which is the initial object in the category Tamb_{C_p}. The Burnside Tambara functor of the trivial group $\{e\}$ is just $\underline{\mathbb{Z}}^c$ and this in turn can be identified with the commutative ring $\mathbb{Z}$. As $\mathbb{Z}$ is the initial object in the category of commutative rings and as the norm functor is a left adjoint, it sends initial objects to initial objects, thus

$$N_e^{C_p} i_e^*(\underline{\mathbb{Z}}^c) \cong N_e^{C_p}\mathbb{Z} \cong \underline{A}, \tag{5.1}$$

where $\underline{A}$ denotes the C_p-Burnside Tambara functor. Note that $\underline{A}$ is the unit with respect to the box product of C_p-Mackey functors, and that i_e^* of the constant C_p-Tambara functor $\underline{\mathbb{Z}}^c$ is the constant $\{e\}$-Tambara functor $\underline{\mathbb{Z}}^c$ which is just the commutative ring $\mathbb{Z}$.

In the following example, the geometric properties of a finite simplicial C_p-space determine the behaviour of the Loday construction. The existence of fixed points decides about the Loday construction:

Proposition 5.3. *If $X.$ is a finite simplicial C_p-set, then*

$$\mathcal{L}^{C_p}_{X.}(\underline{\mathbb{Z}}^c) \cong \begin{cases} \underline{\mathbb{Z}}^c, & \text{if } X.^{C_p} \neq \emptyset, \\ \underline{A}, & \text{if } X.^{C_p} = \emptyset, \end{cases}$$

where in both cases $\underline{\mathbb{Z}}^c$ and $\underline{A}$ denote simplicial Tambara functors that are constant as simplicial objects.

Proof. If $X.^{C_p} = \emptyset$, then at every simplicial level X_n is a finite disjoint union of free orbits, $X_n = \bigsqcup_{e\in E_n} C_p/e$, and therefore

$$X_n \otimes \underline{\mathbb{Z}}^c \cong \Box_{x\in E_n} N_e^{C_p} i_e^*\underline{\mathbb{Z}}^c.$$

By (5.1) each factor is the Burnside Tambara functor for C_p, and as this is the unit for the box product, we obtain

$$X_n \otimes \underline{\mathbb{Z}}^c \cong \underline{A}.$$

As the simplicial structure maps induce multiplication and insertion of units, we obtain that in this case the equivariant Loday construction is isomorphic to the constant simplicial Tambara functor with value $\underline{A}$.

If $X.^{C_p} \neq \emptyset$, there is an $n \geqslant 0$ and some $x \in X_n$ which is fixed under C_p. Applying iterations of d_0 and s_0 then yields a fixed point in every simplicial degree. Therefore for all $n \geqslant 0$,

$$X_n \cong \bigsqcup_{x\in E_1} C_p/C_p \sqcup \bigsqcup_{x\in E_2} C_p/e$$

with $|E_1| \geqslant 1$ and we obtain

$$X_n \otimes \underline{\mathbb{Z}}^c \cong (\square_{x\in E_1} \underline{\mathbb{Z}}^c)\square(\square_{x\in E_2}\underline{A}).$$

As $\underline{A}$ is the unit for the box product and as $\underline{\mathbb{Z}}^c\square\underline{\mathbb{Z}}^c \cong \underline{\mathbb{Z}}^c$ by [LRZ24, Lemma 5.1] we obtain

$$X_n \otimes \underline{\mathbb{Z}}^c \cong \underline{\mathbb{Z}}^c,$$

because there is at least one trivial orbit. Again, the simplicial structure maps induce the identity under this isomorphism.

□

Note that $\underline{\mathbb{Z}}^c$ is *not* the unit for the box product of Mackey functors. Despite this fact, we obtain the following result:

Corollary 5.4. *For any commutative ring R and any simplicial C_p-set $X.$ for which $X.^{C_p} \neq \emptyset$,*

$$\mathcal{L}^{C_p}_{X.}(\underline{R}^c) \cong \underline{\mathbb{Z}}^c\square\mathcal{L}^{C_p}_{X.}(\underline{R}^c).$$

Proof. Since $R \cong \mathbb{Z}\otimes R$, by [LRZ24, Lemma 5.1] we get that $\underline{R}^c \cong \underline{\mathbb{Z}}^c\square\underline{R}^c$. The claim follows by applying part (1) of Proposition 2.4 and Proposition 5.3. □

6. The linearization map

Several authors have observed, that one can form Loday constructions for G-equivariant commutative ring spectra. One approach uses the fact that the category of G-spectra can be turned into a G-symmetric monoidal structure using the Hill-Hopkins-Ravenel norm construction [Maz13, Example 2.1.1] such that G-equivariant commutative ring spectra are the G-commutative monoids [Hil20, Corollary 17.4.35]. A different approach [BDS] uses the fact that for orthogonal spectra the category of objects with G-action is equivalent to genuine G-equivariant spectra. This makes it possible to use the classical Loday construction for commutative ring spectra and to endow it with G-actions.

If we work with the first approach, then we can form a Loday construction $\mathcal{L}^G_X(R)$, of a G-equivariant commutative ring spectrum R with respect to a finite simplicial G-set X. This is based on tensoring such ring spectra with finite G-sets, such that on orbits G/H we obtain

$$G/H \otimes R \cong N^G_H i^*_H R.$$

Here, i^*_H denotes the spectral version of the restriction functor and N^G_H is the HHR-norm [HHR16, Definition A.52].

We first proof the G-homotopy invariance for Loday constructions of G-spectra:

Proposition 6.1. *Let $f\colon X. \to Y.$ be a morphism of simplicial G-sets, such that the realization $|f|\colon |X.| \to |Y.|$ is a homotopy equivalence, and let R be a G-equivariant commutative ring spectrum. Then $f_*\colon \mathcal{L}^G_{X.}(R) \to \mathcal{L}^G_{Y.}(R)$ induces an equivalence on geometric realizations.*

Proof. Following [HHR16, Sec 2.3.1], we use Comm^G to denote the category of G-commutative ring spectra with G-commutative maps, and in contrast use Comm_G to denote the category with the same objects but with non-equivariant multiplicative maps. Then Comm^G is enriched in spaces and Comm_G is enriched in G-spaces. For two arbitrary G-equivariant commutative ring spectra R and T and for any finite G-set X there is a homeomorphism [HHR16, §2.3.1, p. 25]

$$\mathrm{Comm}^G(X\otimes R, T) \cong G\mathrm{Top}(X, \mathrm{Comm}_G(R,T)). \tag{6.1}$$

Furthermore, when Z is a G-space with trivial action, there is a chain of G-equivariant homeomorphisms

$$\underline{\mathrm{Top}_G}(Z, \mathrm{Comm}_G(R,T)) \cong \mathrm{Comm}_G(Z\otimes R, T) \cong \mathrm{Comm}_G(R, T^Z). \tag{6.2}$$

Here, $\underline{\mathrm{Top}_G}$ means the G-space of non-equivariant maps.

With these preparations in place we prove the claim. Let $\mathbf{S}_\bullet : \mathrm{Comm}_G \to s\mathrm{Comm}_G$ be the total singular complex functor that sends T to $(\mathbf{S}_\bullet T)_n = T^{\Delta^n}$. By adjunction we get the following chain of homeomorphisms

$$\begin{aligned}
\mathrm{Comm}^G(|\mathcal{L}^G_{X.}(R)|, T) &\cong s\mathrm{Comm}^G(\mathcal{L}^G_{X.}(R), \mathbf{S}_\bullet T) \\
&\cong sG\mathrm{Top}(X., \mathrm{Comm}_G(R, \mathbf{S}_\bullet T)) && \text{by (6.1)} \\
&\cong sG\mathrm{Top}(X., \mathbf{S}_\bullet \mathrm{Comm}_G(R, T)) && \text{by (6.2)} \\
&\cong G\mathrm{Top}(|X.|, \mathrm{Comm}_G(R, T))
\end{aligned}$$

The Yoneda lemma then implies the claim. □

We can compare $\underline{\pi}_0$ of the spectral Loday construction for R to the Loday construction on the G-Tambara functor $\underline{\pi}_0(R)$:

Proposition 6.2. *Let X be a finite simplicial G-set and let R be a connective G-equivariant commutative ring spectrum. Then there is an isomorphism of simplicial G-Tambara functors:*

$$\underline{\pi}_0(\mathcal{L}^G_X(R)) \cong \mathcal{L}^G_X(\underline{\pi}_0(R)).$$

Proof. We consider the nth simplicial degree $\underline{\pi}_0(\mathcal{L}^G_X(R))_n = \underline{\pi}_0(X_n \otimes R)$ and we decompose X_n into G-orbits: $X_n \cong G/H_1 \sqcup \ldots \sqcup G/H_k$. Then, as R is a G-commutative monoid,

$$X_n \otimes R \cong (G/H_1 \otimes R) \wedge \ldots \wedge (G/H_k \otimes R) \cong (N^G_{H_1} i^*_{H_1}(R)) \wedge \ldots \wedge (N^G_{H_k} i^*_{H_k}(R)).$$

Hoyer [Hoy14, §2.3.2] proves that there are natural isomorphisms $\underline{\pi}_0 N^G_H H(\underline{M}) \cong N^G_H \underline{M}$ for any G-Mackey functor $\underline{M}$, where the latter is the norm functor that he constructed for Mackey functors and H denotes the equivariant Eilenberg-MacLane spectrum.

If E is a connective G-equivariant spectrum, then by [Ull, Lemma 5.11] the map $E \to H\underline{\pi}_0(E)$ induces an isomorphism

$$\underline{\pi}_0(N^G_H E) \cong \underline{\pi}_0(N^G_H H(\underline{\pi}_0 E)).$$

As we assume R to be connective and as the smash factors $N^G_{H_j} i^*_{H_j}(R)$ are connective as well, we obtain

$$\begin{aligned}
\underline{\pi}_0((N^G_{H_1} i^*_{H_1}(R)) \wedge \ldots \wedge (N^G_{H_k} i^*_{H_k}(R))) &\cong \underline{\pi}_0(N^G_{H_1} i^*_{H_1}(R)) \Box \ldots \Box \underline{\pi}_0(N^G_{H_k} i^*_{H_k}(R)) \\
&\cong N^G_{H_1} i^*_{H_1} \underline{\pi}_0(R) \Box \ldots \Box N^G_{H_k} i^*_{H_k} \underline{\pi}_0(R).
\end{aligned}$$

Therefore, in every simplicial degree,

$$\underline{\pi}_0(\mathcal{L}^G_X(R))_n \cong (\mathcal{L}^G_X(\underline{\pi}_0(R)))_n.$$

The simplicial structure maps induce morphisms that come from the norm-restriction adjunction or that are induced by the multiplicative structure on R and $\underline{\pi}_0(R)$. As $\underline{\pi}_0$ is strong symmetric monoidal for connective spectra and as the norm and restriction functors are also strong symmetric monoidal, we get an isomorphism of simplicial G-Tambara functors. □

Ullman showed that there is no lax symmetric Eilenberg-MacLane spectrum functor from the category of G-Mackey functors to the category of connective G-spectra, if G is a non-trivial finite group [Ull, Theorem 6.2]. However, for a G-Tambara functor $\underline{R}$ the multiplication on $H\underline{R}$ is still induced by the multiplication map $\underline{R} \Box \underline{R} \to \underline{R}$: Ullman shows [Ull, Theorem 5.2] that for a connective commutative G-ring spectrum E and a G-Tambara functor $\underline{R}$ the morphisms in the homotopy category of commutative G-ring spectra from E to $H\underline{R}$ are in bijection with the maps of G-Tambara functors from $\underline{\pi}_0 E$ to $\underline{R}$. For $E = H\underline{R} \wedge H\underline{R}$, we have $\underline{\pi}_0(H\underline{R} \wedge H\underline{R}) \cong \underline{R} \Box \underline{R}$ and therefore the multiplication $\underline{R} \Box \underline{R} \to \underline{R}$ gives rise to a multiplication $H\underline{R} \wedge H\underline{R} \to H\underline{R}$. In particular, for all G-Tambara functors $\underline{R}$ Propositon 6.2 yields as a special case

$$\underline{\pi}_0(\mathcal{L}^G_X H\underline{R}) \cong \mathcal{L}^G_X(\underline{R}).$$

Remark 6.3. Let R be a cofibrant connective genuine commutative G-ring spectrum and let X be a finite simplicial G-set. If we knew that the Loday construction, $\mathcal{L}^G_X(R)$, were a proper simplicial G-spectrum, then by an equivariant analogue of [EKMM97, Theorem X.2.9], as used for instance in [BGHL19, Theorem 5.2] and [AKGH, Theorem 6.20], we would get for our simplicial connective G-spectrum $\mathcal{L}^G_X(R)$ and $\underline{\pi}_*$ as a G-homology theory a spectral sequence of the form

$$E^2_{p,q} = H_p\underline{\pi}_q(\mathcal{L}^G_X(R)) \Rightarrow \underline{\pi}_{p+q}|\mathcal{L}^G_X(R)| \tag{6.3}$$

together with an edge homomorphism

$$\underline{\pi}_p|\mathcal{L}^G_X(R)| \to H_p\underline{\pi}_0\mathcal{L}^G_X(R).$$

By Proposition 6.2 the target of the map can be identified with $H_p\mathcal{L}^G_X(\underline{\pi}_0(R))$. So we would get a map of G-Mackey functors

$$\underline{\pi}_k|\mathcal{L}^G_X(R)| \to \pi_k\mathcal{L}^G_X(\underline{\pi}_0(R)) \tag{6.4}$$

that would rightly be called a linearization map.

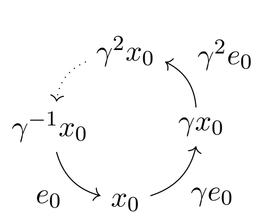
To this end, we would need that the degeneracy maps behave well in the sense that the map from the nth latching object to the nth simplicial object is a cofibration for all n. This is true in the unstable context if all degeneracies are G-cofibrations by [MMO25, Lemma 1.11].

The degeneracies induce injective maps $s_i\colon X_n \to X_{n+1}$, so the building blocks for them are isomorphisms $G/H \to G/g^{-1}Hg$ and $\varnothing \to G/H$. The first type of map induces a cofibration, and the second type corresponds to $S \to N^G_H i^*_H R$ that we can factor as $S \to N^G_H i^*_H S \to N^G_H i^*_H R$. As N^G_H is a left Quillen functor, it preserves cofibrations. The restriction functors i^*_H preserve cofibrations in the underlying category by [MM02, Lemma V.2.2]. These results point in the right direction, but we were not able to find a reference that ensures properness so that the spectral sequence exists as in (6.3) so that a linearization map as in (6.4) can be deduced.

7. Spheres and suspensions

In the non-equivariant setting the Loday construction for the circle $X_\cdot = S^1$ gives Hochschild homology. We describe the equivariant Loday construction for some circles with group action and for some unreduced suspensions where we either flip the suspension apices with a C_2-action or we fix them. We also discuss the relationship of Real topological Hochschild homology to our equivariant Loday construction.

7.1. **Circle with rotation action.** Let C_n be the cyclic group of order n, $C_n = \langle\gamma\rangle$. We let C_n act on the circle S^1_{rot} by letting γ induce a rotation by $2\pi/n$. Then S^1_{rot} has a simplicial model with non-degenerate cells being one free 0-cell $C_n \cdot x_0 = \{x_0, \gamma x_0, \cdots, \gamma^{n-1}x_0\}$ and one free 1-cell $C_n \cdot e_0$.



We have $(S^1_{\mathrm{rot}})_k = \{C_n \cdot x^0_k, C_n \cdot x^1_k, \cdots, C_n \cdot x^k_k\}$, where

$$x^0_k = s^k_0 x_0,\ x^i_k = s^{i-1}_0 s^{k-i}_1 e_0 \text{ for } 1 \leqslant i \leqslant k.$$

The simplicial identities imply that

$$\begin{aligned} d_j(x^0_k) &= x^0_{k-1}, \\ d_j(x^i_k) &= \begin{cases} x^{i-1}_{k-1} & 0 \leqslant j \leqslant i-1 \\ x^i_{k-1} & i \leqslant j \leqslant k \text{ and } i \neq k \end{cases} \\ d_k(x^k_k) &= \gamma^{-1}x^0_{k-1}. \end{aligned}$$

So for a C_n-Tambara functor $\underline{R}$ with $R := i^*_e\underline{R}$, there is

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}}(\underline{R})_k = \square_{0\leqslant i\leqslant k}(C_n \otimes \underline{R}) = (N^{C_n}_e R)^{\square(k+1)},$$

and $d_i\colon (N_e^{C_n}R)^{\Box(k+1)} \to (N_e^{C_n}R)^{\Box k}$ is

$$d_i = \mathrm{id}^i \Box \mu \Box \mathrm{id}^{k-i} \qquad \text{for } 0 \leqslant i < k$$
$$d_k = (\mu \Box \mathrm{id}^{k-1}) \circ (\gamma^{-1} \Box \mathrm{id}^k) \circ \tau$$

where $\mu\colon (N_e^{C_n}R)^{\Box 2} \to N_e^{C_n}R$ is the multiplication and $\tau\colon (N_e^{C_n}R)^{\Box(k+1)} \to (N_e^{C_n}R)^{\Box(k+1)}$ moves the last coordinate to the front.

As $i_e^*\underline{R}$ is an e-Tambara functor, it can be identified with its value on e/e and that is $\underline{R}(C_n/e)$.

We can identify the Loday construction with the twisted cyclic nerve $\underline{\mathrm{HC}}^{C_n}$ defined in [BGHL19, Definition 2.20] and its free level corresponds to a subdivision of the ordinary Loday construction.

Theorem 7.1. *The C_n-equivariant Loday construction for S^1_{rot} is*

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}}(\underline{R}) \cong \underline{\mathrm{HC}}^{C_n}(N_e^{C_n} i_e^* \underline{R}). \tag{7.1}$$

Proof. The claim follows by direct inspection of [BGHL19, Definition 2.20]. □

Remark 7.2. Note that in the Loday construction for S^1_{rot} we don't use the full structure of a C_n-Tambara functor, because the multiplicative norm maps are not used at all. As all simplices in S^1_{rot} are cyclically ordered, one can actually use associative Green functors instead of C_n-Tambara functors. This is the setting of [BGHL19].

The isomorphism (7.2) can be generalized to the relative case:

Proposition 7.3. *Let $K \leqslant C_n$ be a finite subgroup of S^1 and let S^1_{rot}/K be the circle with rotation action by C_n such that the action by K is fixed. Then the C_n-equivariant Loday construction on S^1_{rot}/K can be identified with the C_n-twisted cyclic nerve relative to K of* [BGHL19, Definition 3.19]*:*

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}/K}(\underline{R}) \cong \underline{\mathrm{HC}}^{C_n}_K(i_K^* \underline{R}). \tag{7.2}$$

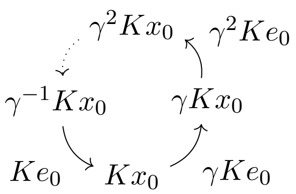
In particular, taking $K = C_n$ or $K = e$, there are isomorphisms

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}/C_n}(\underline{R}) \cong \underline{\mathrm{HC}}^{C_n}_{C_n}(\underline{R}) \cong \underline{\mathrm{HC}}^{C_n}(\underline{R}),$$
$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}}(\underline{R}) \cong \underline{\mathrm{HC}}^{C_n}_e(i_e^* \underline{R}) \cong \underline{\mathrm{HC}}^{C_n}(N_e^{C_n} i_e^* \underline{R}).$$

Proof. Again, we choose a generator γ so that $C_n = \langle \gamma \rangle$ and take the following simplicial model of S^1_{rot}/K. The non-degenerate cells are one orbit of 0-cells

$$C_n/K \cdot x_0 = \{Kx_0, \gamma K x_0, \cdots, \gamma^{|C_n/K|-1} K x_0 = \gamma^{-1} K x_0\}$$

and one orbit of 1-cells $C_n/K \cdot e_0$.



We have $(S^1_{\mathrm{rot}})_k = \{C_n/K \cdot x_k^0, C_n/K \cdot x_k^1, \cdots, C_n/K \cdot x_k^k\}$, where

$$x_k^0 = s_0^k x_0,\ x_k^i = s_0^{i-1} s_1^{k-i} e_0 \text{ for } 1 \leqslant i \leqslant k.$$

So for a C_n-Tambara functor $\underline{R}$ we obtain

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}/K}(\underline{R})_k = \Box_{0 \leqslant i \leqslant k}(C_n/K \otimes \underline{R}) = (N_K^{C_n} i_K^* \underline{R})^{\Box(k+1)}.$$

This is precisely $\underline{\mathrm{HC}}^{C_n}_K(i_K^* \underline{R})_k$. The compatibility of this identification with the simplicial structure maps can be seen similarly to the absolute case. □

If we choose the action of γ^{-1} on $(N_e^{C_n} i_e^* \underline{R})(C_n/e) = \underline{R}(C_n/e)^{\otimes n}$ such that it brings the last coordinate to the front (see Remark 2.5), then we can identify the free-orbit level of the equivariant Loday construction with the n-fold subdivision ([BHM93, §1]) of the Loday construction for the commutative ring $\underline{R}(C_n/e)$:

Theorem 7.4. *There is an isomorphism*

$$\mathcal{L}^{C_n}_{S^1_{\mathrm{rot}}}(\underline{R})(C_n/e) \cong \mathrm{sd}_n \mathcal{L}_{S^1}(\underline{R}(C_n/e)). \tag{7.3}$$

Proof. Note that by the definition of the norm at the free level we obtain that

$$\mathcal{L}_{S^1_{\mathrm{rot}}}(\underline{R})_k(C_n/e) = (\underline{R}(C_n/e)^{\otimes n})^{\otimes k+1}.$$

We send an element

$$(r_{0,1} \otimes r_{0,2} \otimes \cdots \otimes r_{0,n}) \otimes \cdots \otimes (r_{k,1} \otimes \cdots \otimes r_{k,n}) \in \mathcal{L}_{S^1_{\mathrm{rot}}}(\underline{R})_k(C_n/e) = (\underline{R}(C_n/e)^{\otimes n})^{\otimes k+1}$$

to

$$(r_{0,1} \otimes r_{1,1} \otimes \cdots \otimes r_{k,1}) \otimes (r_{0,2} \otimes \cdots \otimes r_{k,2}) \otimes \cdots \otimes (r_{0,n} \otimes \cdots \otimes r_{k,n}) \in (\mathrm{sd}_n \mathcal{L}_{S^1}\underline{R}(C_n/e))_k.$$

We have to check the compatibility of this isomorphism with the simplicial structure maps. The only non-trivial step is to compare

$$\begin{aligned} &d_k((r_{0,1} \otimes r_{0,2} \otimes \cdots \otimes r_{0,n}) \otimes \cdots \otimes (r_{k,1} \otimes \cdots \otimes r_{k,n})) \\ &= (r_{k,n}r_{0,1} \otimes r_{k,1}r_{0,2} \otimes \cdots \otimes r_{k,n-1}r_{0,n}) \otimes (r_{1,1} \otimes \cdots \otimes r_{1,n}) \otimes \cdots \otimes (r_{k-1,1} \otimes \cdots \otimes r_{k-1,n}) \end{aligned}$$

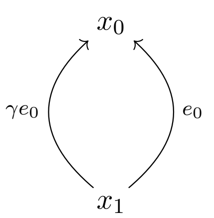
and

$$\begin{aligned} &d_k((r_{0,1} \otimes r_{1,1} \otimes \cdots \otimes r_{k,1}) \otimes (r_{0,2} \otimes \cdots \otimes r_{k,2}) \otimes \cdots \otimes (r_{0,n} \otimes \cdots \otimes r_{k,n})) \\ &= (r_{k,n}r_{0,1} \otimes r_{1,1} \otimes \cdots \otimes r_{k-1,1}) \otimes (r_{k,1}r_{0,2} \otimes \cdots \otimes r_{k-1,2}) \otimes \cdots \otimes (r_{k,n-1}r_{0,n} \otimes \cdots \otimes r_{k-1,n}). \end{aligned}$$

The isomorphism maps the first term to the second one. □

Remark 7.5. A similar relationship between the H-relative topological Hochschild homology THH_H for $H \leqslant C_n$ defined in [ABG+18] and the Loday construction for S^1_{rot} can be proven in the setting of C_n-equivariant commutative ring spectra.

7.2. **Circle with reflection action.** Let S^σ be the circle with reflection action. It has a C_2-simplicial model with non-degenerate cells being two trivial 0-cells and one free 1-cell.



We have $(S^\sigma)_k = \{x^0_k, C_2 \cdot x^1_k, \cdots, C_2 \cdot x^k_k, x^{k+1}_k\}$, where

$$x^0_k = s^k_0 x_0,\ x^{k+1}_k = s^k_0 x_1,\ x^i_k = s^{i-1}_0 s^{k-i}_1 e_0 \text{ for } 1 \leqslant i \leqslant k.$$

The simplicial identities imply that

$$\begin{aligned} d_j(x^0_k) &= x^0_{k-1}, \\ d_j(x^{k+1}_k) &= x^k_{k-1}, \\ d_j(x^i_k) &= \begin{cases} x^{i-1}_{k-1} & 0 \leqslant j \leqslant i-1, \\ x^i_{k-1} & i \leqslant j \leqslant k. \end{cases} \end{aligned}$$

So for a C_2-Tambara functor $\underline{R}$, there is an isomorphism of C_2-simplicial Tambara functors between the Loday construction and the bar construction

$$\mathcal{L}^{C_2}_{S^\sigma}(\underline{R}) \cong \mathrm{B}(\underline{R}, N^{C_2}_e i^*_e \underline{R}, \underline{R}), \tag{7.4}$$

where the C_2-Tambara structure of $\underline{R}$ endows $\underline{R}$ with an $N^{C_2}_e i^*_e \underline{R}$-algebra structure, coming from collapsing the free orbit to the trivial one.

Proposition 7.6. *Assume that R is a commutative solid ring,* i.e.*, that the multiplication map $\mu\colon R \otimes R \to R$ induces an isomorphism. If 2 is invertible in $R = i^*_e \underline{R}^c$, then $\underline{R}^c$ is a projective $N^{C_2}_e i^*_e(\underline{R}^c)$-module and*

$$\underline{\pi}_* \mathcal{L}^{C_2}_{S^\sigma}(\underline{R}^c) \cong \underline{R}^c \Box_{N^{C_2}_e i^*_e(\underline{R}^c)} \underline{R}^c$$

concentrated in degree zero.

Proof. We use the explicit formula

$$N_e^{C_2} i_e^* \underline{R}^c = N_e^{C_2} R = \begin{cases} \big(\mathbb{Z}\{R\} \oplus (R \otimes R)/\mathrm{Weyl}\big)/\mathrm{TR} & \text{at } C_2/C_2 \\ R \otimes R & \text{at } C_2/e \end{cases}$$

in order to construct a splitting

$$\sigma \colon \underline{R}^c \to N_e^{C_2} R$$

of the map $N_e^{C_2} R \to \underline{R}^c$ by sending $r \in R$ to $r \otimes 1 \in R \otimes R$ at the C_2/e-level and sending $r \in R$ to $\left[\frac{r \otimes 1}{2}\right] \in (R \otimes R)/\mathrm{Weyl}$ at the C_2/C_2-level.

We need to show that this is a morphism of $N_e^{C_2} R$-modules and to this end we have to understand the multiplication in $N_e^{C_2} R$. We denote a generator belonging to $a \in R$ in $\mathbb{Z}\{R\}$ by $N(a)$. By Frobenius reciprocity we obtain

$$N(a) \cdot [x \otimes y] = N(a)\mathsf{tr}(x \otimes y) = \mathsf{tr}(\mathsf{res}(N(a)) \cdot (x \otimes y)) = \mathsf{tr}((a \otimes a) \cdot (x \otimes y)) = [ax \otimes ay].$$

Similarly, we get $[a \otimes b] \cdot [x \otimes y] = [ax \otimes by] + [bx \otimes ay]$. As the norm is multiplicative and as $N(a) = \mathsf{norm}(a \otimes 1)$, we have $N(a) \cdot N(b) = N(ab)$.

At the C_2/e-level the map σ is compatible with the $N_e^{C_2}(R)$-module structure since R is solid. At the C_2/C_2-level, an element $N(r_1) \otimes r_2$ is mapped by $\mathrm{id} \otimes \sigma$ to $N(r_1) \otimes [\frac{r_2 \otimes 1}{2}]$ and the module action sends this to $[\frac{r_1 r_2 \otimes r_1}{2}]$. Applying first the module action, however, yields $r_1 r_2 r_1$ and σ maps this to $[\frac{r_1 r_2 r_1 \otimes 1}{2}]$. If R is a solid commutative ring, the multiplication in R identifies $r_1 r_2 r_1 \otimes 1$ and $r_1 r_2 \otimes r_1$ with each other.

Similarly, $[a \otimes b] \otimes r_2$ is mapped by $\mathrm{id} \otimes \sigma$ to $[a \otimes b] \otimes [\frac{r_2 \otimes 1}{2}]$ which the module action sends to $[\frac{a r_2 \otimes b}{2}] + [\frac{b r_2 \otimes a}{2}]$ and since R is a solid commutative ring this agrees with

$$\sigma([a \otimes b] r_2) = \sigma(a r_2 b + b r_2 a).$$

Hence $\underline{R}^c$ is a projective $N_e^{C_2} i_e^*(\underline{R}^c)$-module. The section σ gives rise to a contraction of $B(\underline{R}, N_e^{C_2} i_e^* \underline{R}, \underline{R})$ by sending $\underline{R} \square N_e^{C_2} i_e^* \underline{R}^{\square n} \square \underline{R}$ in simplicial degree n with the map $\eta \square \sigma \square \mathrm{id}^{\square n+1}$ to degree $n+1$. Here $\eta \colon \underline{A} \to \underline{R}$ is the unit map of $\underline{R}$. $\square$

Remark 7.7. Bousfield and Kan classified all solid commutative rings [BK72]. Typical building blocks are rings of the form $\mathbb{Z}/n\mathbb{Z}$ or subrings of the rationals $\mathbb{Z}[J^{-1}]$ for some set of primes J.

We will come back later to the C_2-Loday construction on S^σ, when we identify Real topological Hochschild homology with a suitable equivariant Loday construction in Theorem 7.9.

7.3. **Unreduced suspension of a G-simplicial set.** Let $SY.$ be the unreduced suspension of a finite G-simplicial set $Y.$. Using the standard simplicial model $\Delta^1_k = \{x^0_k, \cdots, x^{k+1}_k\}$ with $d_j(x^0_k) = x^0_{k-1}$, $d_j(x^{k+1}_k) = x^k_{k-1}$ for all $0 \leqslant j \leqslant k$, and

$$d_j(x^i_k) = \begin{cases} x^{i-1}_{k-1}, & \text{if } 0 \leqslant j \leqslant i-1, \\ x^i_{k-1}, & \text{if } i \leqslant j \leqslant k, \end{cases}$$

for $1 \leqslant i \leqslant k$ we get $SY_k = \{x^1_k, \cdots, x^k_k\} \times Y_k \cup \{x^0_k, x^{k+1}_k\}$ and hence for any G-Tambara functor $\underline{R}$,

$$SY_k \otimes \underline{R} \cong \underline{R} \square (Y_k \otimes \underline{R}) \square \underline{R}.$$

Keeping track of the structure maps shows that $\mathcal{L}_{SY.}(\underline{R})$ is isomorphic to the diagonal of the bisimplicial Tambara functor $\mathrm{B}(\underline{R}, \mathcal{L}_{Y.}(\underline{R}), \underline{R})$.

7.4. **Unreduced suspension of a C_2-simplicial set.** Let $S^\sigma Y.$ be the unreduced suspension of a finite C_2-simplicial set $Y.$ so that C_2 flips the suspension coordinate. The interval $[-1,1]$ with reflection action has model as a C_2-simplicial set $[-1,1]_k = \{x_k^0, C_2 \cdot x_k^1, \cdots, C_2 \cdot x_k^k, C_2 \cdot x_k^{k+1}\}$ with $d_j(x_k^0) = x_{k-1}^0$ and $d_j(x_k^{k+1}) = x_{k-1}^k$ for all $0 \leqslant j \leqslant k$, and

$$d_j(x_k^i) = \begin{cases} x_{k-1}^{i-1}, & \text{if } 0 \leqslant j \leqslant i-1, \\ x_{k-1}^i, & \text{if } i \leqslant j \leqslant k, \end{cases} \quad \text{for } 1 \leqslant i \leqslant k.$$

We therefore get

$$S^\sigma Y_k = \{x_k^0\} \times Y_k \cup \{C_2 \cdot x_k^1, \cdots, C_2 \cdot x_k^k\} \times Y_k \cup \{C_2 \cdot x_k^{k+1}\}$$

and for a C_2-Tambara functor $\underline{R}$,

$$SY_k^\sigma \otimes \underline{R} = (Y_k \otimes \underline{R}) \Box ((C_2 \times Y_k) \otimes \underline{R}) \Box (N_e^{C_2} i_e^* \underline{R}).$$

Again we get that $\mathcal{L}^{C_2}_{S^\sigma Y.}(\underline{R})$ is isomorphic to the diagonal of the bisimplicial Tambara functor $\mathrm{B}(\mathcal{L}^{C_2}_{Y.}(\underline{R}), \mathcal{L}^{C_2}_{C_2 \times Y.}(\underline{R}), \mathcal{L}^{C_2}_{C_2}(\underline{R}))$. Note also that $S^\sigma Y.$ is the simplicial join $C_2 \star Y.$, which is also the homotopy pushout of $Y. \leftarrow C_2 \times Y. \to C_2$.

7.5. **Real topological Hochschild homology.** Hesselholt and Madsen developed Real algebraic K-theory, a variant of algebraic K-theory that accepts as input algebras with anti-involution [HM]. The corresponding Real variant of topological Hochschild homology, THR, was investigated in Dotto's thesis and in [DMPR21] where the authors also identified $\mathsf{THR}(A)$ in good cases with a two-sided bar construction [DMPR21, Prop. 2.11, Theorem 2.23] analogous to (7.4). Horev proved a similar result in the context of equivariant factorization homology [Hor, Proposition 7.11]. Angelini-Knoll, Gerhardt, and Hill [AKGH, Definitions 4.2 and 4.5] constructed two $O(2)$-equivariant spectra: the norm $N_{C_2}^{O(2)}(A)$ and the tensor $A \otimes_{C_2} O(2)$ using a simplicial model of $O(2)$ for a genuine commutative C_2-ring spectrum A. They showed [AKGH, Propositions 4.6 and 4.9] that there are (zig-zag of) maps of $O(2)$-spectra $\mathsf{THR}(A) \simeq N_{C_2}^{O(2)} A$ and $N_{C_2}^{O(2)}(A) \to A \otimes_{C_2} O(2)$ such that the first one is a C_2-equivalence when A is flat ([AKGH, Definition 3.22]) and that the second one is a C_2-equivalence when A is well-pointed ([AKGH, Definition 3.24]).

We claim that there is an equivalence of simplicial C_2-spectra

$$A \otimes_{C_2} O(2)_\bullet \simeq \mathcal{L}^{C_2}_{S^\sigma}(A). \tag{7.5}$$

In fact, writing D_{2n} for the dihedral group of order $2n$, so that $C_2 = D_2$, the k-simplices of $O(2)_\bullet$ are given by $O(2)_k = D_{4k+4}$ viewed as a D_2-set [AKGH, Definition 4.4]. As $D_{4k+4} = \mu_{2k+2} \rtimes D_2$, we have a split short exact sequence of groups

$$1 \longrightarrow \mu_{2k+2} \longrightarrow D_{4k+4} \longrightarrow D_2 \longrightarrow 1$$

and the induced D_2-action on μ_{2k+2} maps the generator $\zeta = (1, 2, \ldots, 2k+2)$ to its inverse. The D_2-action on D_{4k+4} is free and as D_2-sets $D_{4k+4}/D_2 \cong \mu_{2k+2}$. Then,

$$A \otimes_{D_2} D_{4k+4} \cong A \otimes \mu_{2k+2}.$$

If we choose an ordering of the D_2-set μ_{2k+2} as $1 < \zeta < \zeta^2 < \ldots < \zeta^{2k+1}$, then we always get two trivial orbits generated by 1 and ζ^{k+1} and k free orbits generated by $\zeta, \ldots, \zeta^k$. Hence we get that

$$A \otimes \mu_{2k+2} \cong \mu_{2k+2} \otimes A$$

where now the tensor product of μ_{2k+2} with A is the one that uses that genuine commutative C_2-spectra are C_2-commutative monoids [Hil20].

We can identify μ_{2k+2} with the k-simplices of the reflection circle S^σ in Section 7.2 and this identification is compatible with the simplicial structure maps. This shows (7.5).

Remark 7.8. Here, we view D_{4k+4} only as a D_2-set. In [AKGH], the group structure of D_{4k+4} is used to set up $A \otimes_{D_2} O(2)_\bullet$ as a Real cyclic object in C_2-spectra. This way, the geometric realization becomes an $O(2)$-spectrum indexed on a S^1-trivial $O(2)$-universe, and $A \otimes_{D_2} O(2)$ is defined to be this realization after changing to a complete universe.

The following result summarizes the above arguments.

Theorem 7.9. *If A is a flat and well-pointed C_2-commutative ring spectrum, then there is an equivalence of C_2-spectra*
$$\mathsf{THR}(A) \simeq A \otimes_{C_2} O(2) \simeq |\mathcal{L}^{C_2}_{S^\sigma}(A)|.$$

8. Relative equivariant Loday constructions

In the non-equivariant context the Loday construction from (2.9) has a relative variant: if A is a commutative k algebra for k an arbitrary commutative ring, we can define $\mathcal{L}^k_{X.}(A)$ by setting

$$\mathcal{L}^k_{X.}(A) = \{[n] \mapsto \bigotimes_{x \in X_n, k} A\} \tag{8.1}$$

so the tensor product over the integers is replaced by the tensor product over k.

Assume that $f\colon \underline{R} \to \underline{T}$ is a map of G-Tambara functors. In the equivariant context it does *not* work to replace the $\Box$-product in Definition 2.2 by the relative box product $\Box_{\underline{R}}$. The norm terms $N^G_H i^*_H \underline{T}$ for instance don't carry an $\underline{R}$-module structure in general. We propose the following definition.

Definition 8.1. Let G be a finite group, $\underline{R}$ and $\underline{T}$ be two G-Tambara functors, and let $f\colon \underline{R} \to \underline{T}$ be a map of Tambara functors. Then for any G-simplicial set $X.$ we define the equivariant Loday construction of $\underline{T}$ relative to $\underline{R}$ on $X.$ as
$$\mathcal{L}^{G,\underline{R}}_{X.}(\underline{T}) = \mathcal{L}^G_{X.}(\underline{T}) \Box_{\mathcal{L}^G_{X.}(\underline{R})} \underline{R}.$$
Here, the map $\mathcal{L}^G_{X.}(\underline{R}) \to \mathcal{L}^G_{X.}(\underline{T})$ is induced by f and the map $\mathcal{L}^G_{X.}(\underline{R}) \to \underline{R}$ is induced by sending $X.$ to a point. The box product over a Tambara functor is defined as the usual coequalizer.

In [ABG+18, Definition 1.7] and [BGHL19, §8] the authors define a relative norm in a similar manner.

In spectra, if we have a cofibrant commutative S-algebra A and a cofibrant commutative A-algebra B, the Loday construction of B on a simplicial set $X.$ is defined as
$$\mathcal{L}_{X.}(B) = \{[n] \mapsto \bigwedge_{x \in X_n} B\}$$
and the Loday construction of B on $X.$ over A is defined by replacing the smash products with smash products over A,
$$\mathcal{L}^A_{X.}(B) = \{[n] \mapsto \bigwedge_{x \in X_n, A} B\}.$$
In [HHL+18, §3] we show that
$$\mathcal{L}^A_{X.}(B) \simeq \mathcal{L}_{X.}(B) \wedge_{\mathcal{L}_{X.}(A)} A$$
which is analogous to the definition above in the equivariant setting.

Remark 8.2. Assume that $f\colon \underline{R} \to \underline{T}$ is a morphism of G-Tambara functors that turns $\underline{T}$ into a projective $\underline{R}$-module, hence it is a retract of a free $\underline{R}$-module. As the norm functor N^G_H preserves free modules [HMQ23, Proposition 4.0.5] and as N^G_H – as a functor – sends retracts to retracts and takes values in an abelian category, $N^G_H(\underline{T})$ is a projective $N^G_H(\underline{R})$-module and in this sense, Definition 8.1 is derived.

In some cases we can identify the relative Loday construction with an absolute one:

Proposition 8.3. *For any finite group G and commutative ring R the map of constant Tambara functors $\underline{\mathbb{Z}}^c \to \underline{R}^c$ induced by the unit map $\mathbb{Z} \to R$ gives rise to an isomorphism*

$$\mathcal{L}_{X.}^{G,\underline{\mathbb{Z}}^c}(\underline{R}^c) \cong \mathcal{L}_{X.}^{G}(\underline{R}^c)\Box\underline{\mathbb{Z}}^c$$

for any finite simplicial G-set X.

Proof. As before, since $R \cong \mathbb{Z} \otimes R$, by [LRZ24, Lemma 5.1] we get that $\underline{R}^c \cong \underline{\mathbb{Z}}^c\Box\underline{R}^c$. We then apply part (1) of Proposition 2.4 to deduce $\mathcal{L}_{X.}^{G}(\underline{R}^c) \cong \mathcal{L}_{X.}^{G}(\underline{R}^c)\Box\mathcal{L}_{X.}^{G}(\underline{\mathbb{Z}}^c)$ where the map induced by the unit map $\mathbb{Z} \to R$ exactly sends $\mathcal{L}_{X.}^{G}(\underline{\mathbb{Z}}^c)$ to the second factor. $\Box$

Note that in the case when $G = C_p$ for a prime p and $X.^{C_p} \neq \varnothing$, by Corollary 5.4 and the result above this in fact means that

$$\mathcal{L}_{X.}^{C_p,\underline{\mathbb{Z}}^c}(\underline{R}^c) \cong \mathcal{L}_{X.}^{C_p}(\underline{R}^c). \tag{8.2}$$

We also get a relative version of Proposition 5.2:

Proposition 8.4. *Let $X.$ be a simplicial set with trivial G-action and let B be any commutative A-algebra. Then*

$$\mathcal{L}_{X.}^{G,\underline{A}^c}(\underline{B}^c) \cong \underline{\mathcal{L}_{X.}^{A}(B)}^c,$$

where $\mathcal{L}_{X.}^{A}(B)$ is the nonequivariant relative Loday construction from (8.1).

Proof. As we know by [LRZ24, Lemma 5.1] that the box product of two constant Tambara functors corresponding to commutative rings is the constant Tambara functor corresponding to the tensor product of these rings, this also holds for the coequalizers $\underline{B}^c\Box_{\underline{A}^c}\underline{B}^c \cong \underline{(B \otimes_A B)}^c$ and iterations of these. So in every simplicial degree n of $\mathcal{L}_{X.}^{G,\underline{A}^c}(\underline{B}^c)$, we have $\Box_{x\in X_n,\underline{A}^c}\underline{B}^c \cong \underline{(\otimes_{x\in X_n,A}B)}^c$ and these isomorphisms are compatible with the simplicial structure maps. $\Box$

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