Properties of the homology of algebraic n-fold loop spaces

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Lille, October 2012

The operad of little n-cubes

The rational case

Structure in characteristic two

Little n-cubes

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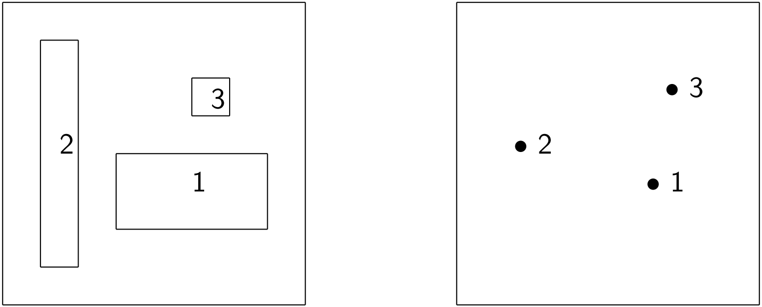

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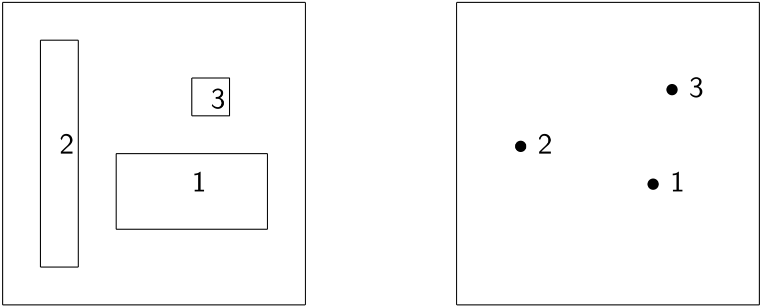

This equivalence is Σ_r-equivariant.

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$$Q_i \colon H_q(X; \mathbb{F}_2) \to H_{2q+i}(X; \mathbb{F}_2)$$

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Browder (1960): Description of $H_*(\Omega^n \Sigma^n Z; \mathbb{F}_2)$ as an algebra in terms of $H_*(Z; \mathbb{F}_2)$. Construction of a new operation (Browder operation).

Some history – continued

Dyer-Lashof (1962): Extension of (some of) the Q_i's to odd primes. Partial results about $H_*(\Omega^n\Sigma^n Z; \mathbb{F}_p)$.

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Milgram (1966): $H_*(\Omega^n\Sigma^n Z; \mathbb{F}_p)$ as an algebra, depending only on the homology of Z and n.
Cohen (1976): Complete description of the homology operations on iterated loop spaces, and of $H_*(\Omega^n\Sigma^n Z; k)$ for $k = \mathbb{Q}$ and $k = \mathbb{F}_p$.

Where do the operations come from?

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But in general there is more, unless we have $k = \mathbb{Q}$...

n-Gerstenhaber algebras

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Cohen showed that the rational homology of any space $X = \Omega^{n+1}Y$ is an n-Gerstenhaber algebra and that

$$H_*(C_{n+1}Z; \mathbb{Q}) \cong nG(\bar{H}_*(Z; \mathbb{Q}))$$

for any space Z.

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Think of this as being 'half the circle' giving rise to 'half the Lie bracket $[x, x]$', aka the restriction on x.

Dyer-Lashof operations

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There are also relations between the Q^i's and the action of the duals of the Sq^j's (Nishida relations).

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Take the standard resolution of $\Sigma_2 = \mathbb{Z}/2\mathbb{Z}$, W_*, and compose

$$\theta_* \colon W_* \otimes C_*(X; \mathbb{F}_2)^{\otimes 2} \to C_*(C_\infty(2); \mathbb{F}_2) \otimes C_*(X; \mathbb{F}_2)^{\otimes 2}$$

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$Q_i(x) = \theta_*(e_i \otimes x \otimes x)$ ($e_i \in W_i$) is the induced map on homology and $Q^s(x) := Q_{s-|x|}(x)$ if $s - |x| \geq 0$ (0 otherwise).

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We get additional relations wrt the mod-p Bockstein.

Homology of $\Omega^{n+1}\Sigma^{n+1}Z$

Cohen: Complete descriptions of $H_*(C_{n+1}Z;\mathbb{F}_p)$ and $H_*(\Omega^{n+1}\Sigma^{n+1}Z;\mathbb{F}_p)$ as free objects built out of the reduced homology of Z.

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- a compatible coalgebra structure.

We get a Hopf algebra with a compatible Dyer-Lashof action and a restricted n-Lie algebra structure.

On chain level

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b) Over $\mathbb{F}_2$:

$$H_*(E_2(C_*)) \cong 1rG(H_*C_*).$$