GALOIS AND SEPARABLE EXTENSIONS OF TAMBARA FUNCTORS

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ABSTRACT. For a map of Tambara functors $\underline{R} \to \underline{T}$ we define when $\underline{T}$ is separable over $\underline{R}$ and if $\underline{T}$ carries an action by some finite group H that fixes $\underline{R}$ we also define when $\underline{T}$ is an H-Galois extension of Tambara functors. We show that flat and separable extensions are (formally) étale in the sense of Hill and we prove that Galois extensions are separable. We express Nullstellensatzian Tambara functors in the context of Galois theory.

1. INTRODUCTION

In classical algebra, the difference between ordinary commutative rings and fields is huge. Their intrinsic algebraic properties are very different and they differ drastically with respect to their category of modules: modules over a ring can behave pretty erratic whereas linear algebra tells us how to control modules (aka vector spaces) over fields. In analogy with the Galois theory of fields there is a Galois theory for commutative rings, where one has to require more properties from an inclusion $R \subset T$ of commutative rings so that it behaves similar to a H-Galois extension of fields for a finite group H. In particular, on top of the usual condition that R agrees with T^H, the H-fixed points of T, one has to add a condition that ensures that the extension is unramified by asking that $T \otimes_R T$ is isomorphic to $\prod_H T$ via the map that sends $t_1 \otimes t_2$ to $(t_1 \cdot h(t_2))_{h \in H}$. For instance the extension $\mathbb{Z} \subset \mathbb{Z}[i]$ is *not* C_2-Galois because the above map fails to be surjective, but if one inverts the ramified prime 2, then $\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}, i]$ *is* a C_2-Galois extension.

For Tambara functors the distinction between "fields" and ordinary Tambara functors is less prominent. Nakaoka defined field-like G-Tambara functors [Nak12]. They can be characterized as those G-Tambara functors whose free level $\underline{T}(G/e)$ has no non-trivial G-invariant ideals and that have injective restriction maps associated to equivariant maps of finite transitive G-sets [Nak12, Theorem 4.32]. However, this does *not* imply that all levels are fields or that modules over such a field-like Tambara functor are necessarily free.

Nonetheless, there are classification results for Tambara fields. For instance Wisdom classifies field-like Tambara functors for cyclic group of prime power order [Wisa]. We also understand Tambara functors that behave like algebraically closed fields thanks to work of Schuchardt, Spitz, and Wisdom [SSW]. But it was not clear how to relate these notions to Galois theory. Can we interpret such an algebraically closed object as sitting at the end of a chain of Galois extensions?

This paper aims to study Galois extensions of Tambara functors and to relate them to étale and separable extensions.

In [Hil17] Mike Hill defined formal étaleness in the context of Tambara functors by introducing an equivariant version of the module of Kähler differentials. These differentials are linear and satisfy a Leibniz rule as in the non-equivariant case, but their equivariant nature is reflected in the fact that they also have to obey a twisted Leibniz rule that involves norms, transfers and restrictions. Examples of formally étale extensions mostly originating in non-equivariant Galois extensions were established in [LRZ24]. Noah Wisdom extended this range of examples drastically in [Wisc] and proved several structural properties about (formally) étale maps [Wisc,

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§4]. In [MQS] Kähler differentials were related to an equivariant version of Hochschild homology. We describe this relation from a slightly different angle in Section 3.

Non-equivariantly, extensions which are flat and separable are formally étale, and Galois extensions of commutative rings are separable. Hence these types of algebras give natural examples for formally étale maps. In this paper we define separability (see Section 4) and Galois extensions for Tambara functors with analogous properties. As separability is defined via the module category of a ring, there is a straightforward generalization in the equivariant context. We use the concept of separability for Tambara functors to prove that some equivariant Loday constructions in the sense of [LRZ25] have vanishing homotopy groups in positive degrees (see Section 5).

There is a well-developed notion of Galois extensions of commutative rings [AG60, CHR65, Gre92]. We define Galois extensions of Tambara functors in a similar manner in Section 6 and prove that these extensions are always separable. For some other properties that hold for Galois extensions of commutative rings it is less clear that they carry over to the setting of Tambara functors because some of the classical tools are not (yet) available, such as local-to-global arguments.

We study examples of such Galois extensions. In the non-equivariant setting the above mentioned extension of commutative rings $\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}, i]$ is a fundamental example. If one wants to mimic its construction in the world of C_2-Tambara functors by taking a suitable quotient of a free Tambara functors on a generator at the trivial level, then this does not work, see Section 7.

Schuchardt, Spitz, and Wisdom [SSW] investigate and classify Nullstellensatzian objects in the sense of Burklund, Schlank, and Yuan [BSY] in the category of G-Tambara functors for any finite group G. They identify them as co-inductions $\mathsf{Coind}_e^G(\mathbb{F})$ of algebraically closed fields $\mathbb{F}$. We interpret these objects in the context of Galois theory of Tambara functors in Section 8.

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2. Fixed point Tambara functors and box products

We start with a small, but very helpful technical result that will allow us to control box products of fixed point Tambara functors.

Lewis noted several facts about fixed point and orbit Mackey functors [Lew88, §1]. We will use one of them to prove a generalization of a formula for fixed point Tambara functors from [LR, Lemma 3.2]: There we showed that there is an isomorphism of C_2-Mackey functors $\underline{M}^{\mathrm{fix}} \Box \underline{N}^{\mathrm{fix}} \cong \underline{(M \otimes N)}^{\mathrm{fix}}$, if M and N are abelian groups with a C_2-action and if 2 is invertible in one of them. We also proved an upgrade to an isomorphism of C_2-Tambara functors if both M and N are in fact commutative C_2-rings. It was also clear that the arugments could be extended to arbitrary cyclic groups of prime order C_p. In the following we extend the result to arbitrary finite groups as long as $|G|$ is invertible in one of the factors.

In the following, G is an arbitrary but fixed finite group. Let M be an abelian group with a G-action. We denote by $\underline{L(M)}$ the orbit Mackey functor. This has $\underline{L(M)}(G/H) = M_H$, the coinvariants of M with respect to H. For $K < H$ the transfer map $\mathsf{tr}_K^H\colon M_K \to M_H$ is the canonical projection map and the restriction $\mathsf{res}_K^H\colon M_H \to M_K$ is given by sending an equivalence class $[m] \in M_H$ to the class of $\sum_{\gamma \in H/K} \gamma m \in M_K$. Here, the sum is taken over representatives for K in H.

Note that $\underline{L}$ is a functor from the category of abelian groups with G-action to the category of G-Mackey functors and is left adjoint to the evaluation functor that takes a G-Mackey functor and evaluates it at the free level G/e.

Lemma 2.1. *Let M and N be abelian groups with a G-action. If $|G|$ is invertible in M or N, then there is an isomorphism of G-Mackey functors*

$$\underline{M}^{\mathrm{fix}} \Box \underline{N}^{\mathrm{fix}} \cong \underline{(M \otimes N)}^{\mathrm{fix}}. \tag{2.1}$$

Proof. Lewis states in [Lew88, Example 1.2 (b)] that for all G-Mackey functors $\underline{T}$ and all abelian groups with G-action M

$$\underline{L(M)} \Box \underline{T} \cong \underline{L(M \otimes \underline{T}(G/e))}$$

where one considers the diagonal G-action on $M \otimes \underline{T}(G/e)$.

The map $\underline{L(M \otimes \underline{T}(G/e))} \to \underline{L(M)} \Box \underline{T}$ is determined by the adjoint map

$$M \otimes \underline{T}(G/e) \to \underline{L(M)} \Box \underline{T}(G/e) = M \otimes \underline{T}(G/e)$$

which we take to be the identity map. For the converse map we use the description of maps out of a box product as componentwise maps

$$\begin{tikzcd}
\underline{L(M)}(G/H) \otimes \underline{T}(G/H) \arrow[r] \arrow[d, equal] & \underline{L(M \otimes \underline{T}(G/e))}(G/H) \arrow[d, equal] & \\
M_H \otimes \underline{T}(G/H) \arrow[rr, bend right, "\psi_H"'] & (M \otimes \underline{T}(G/e))_H \arrow[r, "\cong"] & M_H \otimes \underline{T}(G/e)_H
\end{tikzcd}$$

that are compatible with the structure maps, see [Maz13, Remark 1.2.3 and p. 41] for the explicit list of conditions. We set ψ_H to be $\mathrm{id}_{M_H} \otimes (\pi \circ \mathsf{res}^H_e)$ where $\pi \colon \underline{T}(G/e) \to \underline{T}(G/e)_H$ is the projection map. These maps are inverse to each other.

If $|G|$ is invertible in M, then $\underline{L(M)}$ is isomorphis to $\underline{M}^{\mathrm{fix}}$ as G-Mackey functors: At level G/H

$$M_H \to M^H \tag{2.2}$$

is the map $[m] \mapsto \sum_{h \in H} hm$ and conversely, the map $M^H \to M_H$ is given by sending an H-fixed point m to $\frac{1}{|H|}[m]$.

Taking these two isomorphisms together yields the desired isomorphism of G-Mackey functors:

$$\begin{aligned} \underline{M}^{\mathrm{fix}} \Box \underline{N}^{\mathrm{fix}} \cong & \underline{L(M)} \Box \underline{N}^{\mathrm{fix}} \\ \cong & \underline{L(M \otimes N)} \\ \cong & \underline{(M \otimes N)}^{\mathrm{fix}}. \end{aligned} \tag{2.3}$$

□

Corollary 2.2. *If $\underline{T}^{\mathrm{fix}}$ and $\underline{R}^{\mathrm{fix}}$ are G-Tambara functors and if $|G|$ is invertible in R or T, then the isomorphism* (2.1) *is an isomorphism of G-Tambara functors.*

Proof. Without loss of generality we assume that $|G|$ is invertible in T. The isomorphism (2.3) uses the isomorphism $\underline{L(T)} \Box \underline{R}^{\mathrm{fix}} \cong \underline{L(T \otimes R)}$ together with the map (2.2) and its inverse.

Explicitly, it sends $t \otimes r$ at level G/K with $t \in T^K$ and $r \in R^K$ to

$$\begin{aligned} t \otimes r &\mapsto \frac{1}{|K|}[t] \otimes r \in T_K \otimes R^K \\ &\mapsto \frac{1}{|K|}[t] \otimes [\mathsf{res}_e^K(r)] \in T_K \otimes R_K \\ &\mapsto \sum_{k \in K} \frac{1}{|K|} kt \otimes k\mathsf{res}_e^K(r) \in (T \otimes R)^K, \end{aligned}$$

but as t and r are K-fixed points and as the restriction map in fixed point Tambara functors is the inclusion, this agrees with $t \otimes r$, so in total the map includes $T^K \otimes R^K$ into $(T \otimes R)^K$ and this is a ring map and is compatible with the norm. $\square$

If the order of the group G is neither invertible in M nor in N, then the box product of two fixed point G-Tambara functors does not necessarily simplify to the fixed point functor of the tensor product:

Example 2.3. Let $M = \mathbb{Z}[i] = N$. We claim that

$$\underline{\mathbb{Z}[i]}^{\mathrm{fix}} \Box \underline{\mathbb{Z}[i]}^{\mathrm{fix}} \ncong \underline{(\mathbb{Z}[i] \otimes \mathbb{Z}[i])}^{\mathrm{fix}}.$$

The two Tambara functors agree at the free level:

$$\underline{\mathbb{Z}[i]}^{\mathrm{fix}} \Box \underline{\mathbb{Z}[i]}^{\mathrm{fix}}(C_2/e) = \mathbb{Z}[i] \otimes \mathbb{Z}[i] = \underline{(\mathbb{Z}[i] \otimes \mathbb{Z}[i])}^{\mathrm{fix}}(C_2/e).$$

However, at the trivial level we have

$$\underline{(\mathbb{Z}[i] \otimes \mathbb{Z}[i])}^{\mathrm{fix}}(C_2/C_2) = (\mathbb{Z}[i] \otimes \mathbb{Z}[i])^{C_2}$$

and the fixed points in the tensor product are the abelian group spanned by $1 \otimes 1$ and $i \otimes i$.

For $\underline{\mathbb{Z}[i]}^{\mathrm{fix}} \Box \underline{\mathbb{Z}[i]}^{\mathrm{fix}}(C_2/C_2)$ we obtain

$$\Big(\mathbb{Z} \otimes \mathbb{Z} \oplus (\mathbb{Z}[i] \otimes \mathbb{Z}[i])/W \Big)/\mathrm{FR}$$

where W denotes the Weyl group action that sends $a \otimes b$ to $\bar{a} \otimes \bar{b}$. We denote the Weyl equivalence class of $a \otimes b$ by $[a \otimes b]$ and as $\bar{i} = -i$ the terms $[1 \otimes i]$ and $[i \otimes 1]$ vanish.

Frobenius reciprocity (FR) identifies $[\mathsf{res}(x) \otimes b]$ with $x \otimes \mathsf{tr}(b)$ and $[a \otimes \mathsf{res}(y)]$ with $\mathsf{tr}(a) \otimes y$ and therefore $[1 \otimes 1] = 2 \cdot 1 \otimes 1$, so we only need to keep the generators $1 \otimes 1$ and $[i \otimes i]$.

Assume that there is an isomorphism of Tambara functors $\phi \colon \underline{\mathbb{Z}[i]}^{\mathrm{fix}} \Box \underline{\mathbb{Z}[i]}^{\mathrm{fix}} \to \underline{(\mathbb{Z}[i] \otimes \mathbb{Z}[i])}^{\mathrm{fix}}$. Then at both levels we have an isomorphism of rings and thus $1 \otimes 1$ has to map to $1 \otimes 1$ in both levels.

As $i \otimes i$ is a square root of $1 \otimes 1$, ϕ_e has to send it again to such a square root. As we assume that ϕ_e is injective, it cannot map it to $-1 \otimes 1$, so it has to sent it to $\pm i \otimes i$. This results in

$$\begin{aligned} \phi_{C_2}([i \otimes i]) &= \phi_{C_2}(\mathsf{tr}(i \otimes i)) \\ &= \mathsf{tr}(\phi_e(i \otimes i)) \\ &= \pm \mathsf{tr}(i \otimes i) \\ &= \pm(i \otimes i + (-i) \otimes (-i)) = \pm 2 \cdot i \otimes i. \end{aligned}$$

As 2 is not invertible, this implies that the map ϕ_{C_2} is not surjective.

Later, in the context of Galois extensions, we will be able to generalize Corollary 2.2 to contexts where the group order is not necessarily invertible; see Remark 6.8 (1).

3. Hochschild homology and genuine Kähler differentials

Mike Hill defined a notion of genuine Kähler differentials [Hil17, Definition 5.4]. In [MQS, Theorem 4.19] the authors relate them to the first equivariant Hochschild homology group for Green functors. We first recall the definition of equivariant Hochschild homology:

Definition 3.1. Let $\underline{R}$ be a commutative Green functor and let $\underline{R} \to \underline{T}$ be a morphism of Green functors and let $\underline{M}$ be a $\underline{T}$-bimodule over $\underline{R}$. The *Hochschild homology of* $\underline{T}$ *over* $\underline{R}$ *with coefficients in* $\underline{M}$, $\mathsf{HH}_*^{\underline{R}}(\underline{T};\underline{M})$, is the homology of the chain complex associated with the simplicial Mackey functor whose part in degree q is

$$C_q^{\underline{R}}(\underline{T};\underline{M}) = \underline{M}\Box_{\underline{R}}\underline{T}^{\Box_{\underline{R}}q}$$

and whose differential is $b^{(q)}\colon C_q^{\underline{R}}(\underline{T};\underline{M}) \to C_{q-1}^{\underline{R}}(\underline{T};\underline{M})$ with $b^{(q)} = \sum_{i=0}^{q}(-1)^i b_i$ where

$$b_i = \mathrm{id}^{\Box_{\underline{R}}i-1}\Box_{\underline{R}}\mu\Box_{\underline{R}}\mathrm{id}^{\Box_{\underline{R}}q-i-1} \text{ for } 0 \leqslant i < q$$

Here μ denotes the left and right $\underline{T}$-module structure of $\underline{M}$ and the multiplication in $\underline{T}$. The last face map b_q mimics the last Hochschild face map, so

$$b_q = (\mu\Box_{\underline{R}}\mathrm{id}^{\Box_{\underline{R}}q-1}) \circ \gamma,$$

where $\gamma\colon \underline{M}\Box_{\underline{R}}\underline{T}^{\Box_{\underline{R}}q} \to \underline{T}\Box_{\underline{R}}\underline{M}\Box_{\underline{R}}\underline{T}^{\Box_{\underline{R}}q-1}$ cyclically permutes the factors by bringing the last factor to the front.

The face maps give rise to a semi-simplicial structure. Together with the degeneracies this gives a simplicial object in Mackey functors, where the ith degeneracy map just inserts the unit $\eta\colon \underline{R} \to \underline{T}$ between the spot i and $i+1$.

Remark 3.2. For a map of G-Tambara functors $\underline{R} \to \underline{T}$ this actually agrees with the equivariant Loday construction with respect to S^1, $\mathcal{L}_{S^1}^{G,\underline{R}}(\underline{T})$, as in [LRZ25, Definition 2.2] where $S^1 = \Delta_1/\partial\Delta_1$ is the small model of the simplicial circle viewed as a finite G-simplicial set with trivial G-action. The triviality of the G-action on S^1 and the fact that there is a cyclic ordering on S^1_n for all n ensure that we can actually get away with fewer assumptions: We don't need multiplicative norms and we do not need $\underline{R}$ to be commutative.

In the non-equivariant commutative case the first Hochschild homology group also coincides with the module of Kähler differentials: For a commutative k-algebra A and a symmetric A-bimodule M over k: $\mathsf{HH}_1^k(A;M) \cong M \otimes_A \Omega^1_{A|k}$.

Recall from [Hil17, Definition 4.1] that a genuine $\underline{R}$-derivation $d\colon \underline{T} \to \underline{M}$ is a morphism of G-Mackey functors such that

(1) d satisfies a Leibniz rule: for all finite G-sets S and all $a, b \in \underline{T}(S)$
$$d(ab) = ad(b) + d(a)b.$$
(2) For all $a \in \underline{T}(G/H)$ and for all $H < K < G$:
$$d(\mathsf{norm}_H^K(a)) = \mathsf{tr}_H^K(N_{\pi_2}R_{\pi_1}(a) \cdot d(a)).$$
Here $\pi_i\colon G/H \times_{G/K} G/H \setminus \Delta \to G/H$ is the projection to the ith factor and Δ is the diagonal.
(3) $d \circ \eta = 0$.

Conditions (1) and (3) agree with the non-equivariant conditions for a derivation. Condition (2) can be thought of as a *twisted Leibniz rule*. The genuine Kähler differentials are defined as $\underline{I}/\underline{I}^{>1}$ where $\underline{I}$ denotes the kernel of the multiplication map $\underline{T}\Box_{\underline{R}}\underline{T} \to \underline{T}$ and $\underline{I}^{>1}$ is the ideal generated by the image of norms on 2-surjective maps. This contains the image of the multiplication map on $\underline{I}$. The module of genuine Kähler differentials, $\Omega^{1,G}_{\underline{T}^{\mathrm{fix}}/\underline{R}^c}$, then represents genuine derivations [Hil17, Theorem 5.7].

In the equivariant case, the relationship between the first Hochschild homology group and the module of genuine Kähler differentials is similar to the non-equivariant setting. A proof of the following result can also be found in [MQS, Theorem 4.19].

Theorem 3.3. *If $\eta\colon \underline{R} \to \underline{T}$ is a morphism of G-Tambara functors, then*
$$\underline{\mathsf{HH}}_1^{\underline{R}}(\underline{T}) \cong \underline{I}/\underline{I}^2.$$

The following proof is a direct transfer of the non-equivariant proof.

Proof. There is a canonical map $\varphi\colon \underline{T} \to \underline{I}$ that takes the difference of the two maps
$$\underline{T} \cong \underline{T}\Box_{\underline{R}}\underline{R} \xrightarrow{\mathrm{id}\Box\eta} \underline{T}\Box_{\underline{R}}\underline{T} \text{ and } \underline{T} \cong \underline{R}\Box_{\underline{R}}\underline{T} \xrightarrow{\eta\Box\mathrm{id}} \underline{T}\Box_{\underline{R}}\underline{T}.$$
We prolong this map with the projection and obtain $\underline{T} \to \underline{I}/\underline{I}^2$. As $\underline{I}$ is a $\underline{T}$-module (and so is $\underline{I}/\underline{I}^2$), this yields a morphism of Mackey functors
$$\phi\colon \underline{T}\Box_{\underline{R}}\underline{T} \to \underline{T}\Box_{\underline{R}}\underline{I}/\underline{I}^2 \to \underline{I}/\underline{I}^2.$$
We claim that ϕ factors through $\underline{\mathsf{HH}}_1^{\underline{R}}(\underline{T})$ and induces the desired isomorphism.

We have to show that ϕ vanishes on the image of the Hochschild boundary
$$b^{(2)} = \mu\Box_{\underline{R}}\mathrm{id} - \mathrm{id}\Box_{\underline{R}}\mu + (\mu\Box_{\underline{R}}\mathrm{id}) \circ \gamma.$$
To that end we observe that the commutativity of $\underline{T}$ ensures that we can use the permutation τ instead of γ, where τ interchanges the third box product factor with the second one. The resulting map is the same. But now we can rewrite $b^{(2)}$ as a composite
$$\begin{tikzcd}
\underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \arrow[r, "\mathrm{id}\Box\psi"] & \underline{T}\Box_{\underline{R}}\underline{R}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \arrow[d, "\mathrm{id}\Box\eta\Box\mathrm{id}\Box\mathrm{id}"] \\
& \underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \arrow[d, "\mathrm{id}\Box b^{(2)}"] \\
& \underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \arrow[d, "\mu\Box\mathrm{id}"] \\
& \underline{T}\Box_{\underline{R}}\underline{T}
\end{tikzcd}$$
where ψ is the canonical isomorphism $\underline{T}\Box_{\underline{R}}\underline{T} \cong \underline{R}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T}$. The composite $b^{(2)} \circ (\eta\Box\mathrm{id}\Box\mathrm{id})$ is in the image of $\underline{I}^2$ because up to a degenerate summand it corresponds to the product $\mu\circ(\varphi\Box\varphi)$. $\Box$

Using the identification $\underline{M}\Box_{\underline{R}}\underline{T}^{\Box_{\underline{R}}n} \cong \underline{M}\Box_{\underline{T}}(\underline{T}^{\Box_{\underline{R}}n+1})$ and an analogous proof as above we immediatly obtain the following:

Corollary 3.4. *If $\eta\colon \underline{R} \to \underline{T}$ is a morphism of G-Tambara functors and if $\underline{M}$ is a $\underline{T}$-module, then*
$$\underline{\mathsf{HH}}_1^{\underline{R}}(\underline{T};\underline{M}) \cong \underline{M}\Box_{\underline{T}}\underline{I}/\underline{I}^2.$$

In the special case of fixed point Tambara functors, the classical identification of the first Hochschild homology group in terms of the module of Kähler differentials still holds:

Proposition 3.5. *Let R be a commutative ring with trivial G-action and let T be a commutative ring with G-action together with a map of commutative rings $R \to T$. Then for every $\underline{T}^{\mathrm{fix}}$-module $\underline{M}$:*
$$\underline{\mathsf{HH}}_1^{\underline{R}^c}(\underline{T}^{\mathrm{fix}};\underline{M}) \cong \underline{M}\Box_{\underline{T}^{\mathrm{fix}}}\Omega^{1,G}_{\underline{T}^{\mathrm{fix}}/\underline{R}^c}.$$
Here, $\Omega^{1,G}_{\underline{T}^{\mathrm{fix}}/\underline{R}^c}$ denotes the $\underline{T}^{\mathrm{fix}}$-module of genuine Kähler differentials [Hil17, Definition 5.4].

Note that a ring map $R \to T$ as above induces a map of G-Tambara functors $\eta\colon \underline{R}^c \to \underline{T}^{\mathrm{fix}}$.

Proof. It suffices to consider the case where $\underline{M} = \underline{T}^{\mathrm{fix}}$. For a standard projection $\pi_H^K\colon G/H \to G/K$ the associated norm in a fixed point Tambara functor is given by

$$\mathsf{norm}_H^K(a) = \prod_{g \text{ rep. for } K/H} ga. \tag{3.1}$$

Hence in this case the twisted Leibniz rule follows from the ordinary Leibniz rule. □

Remark 3.6. Nakaoka shows [Nak12, Proposition 4.21] that a G-Tambara functor $\underline{B}$ is a sub-Tambara functor of a Tambara functor of the form $\underline{T}^{\mathrm{fix}}$ if and only if $\underline{B}$ has injective restriction maps for all maps between transitive G-sets and he calls this property (MRC). In sub-Tambara functors of fixed point Tambara functors, we again have the formula for the norm as in 3.1, so here again, the twisted Leibniz rule follows from the ordinary one.

In particular, a cohomological G-Tambara functor $\underline{T}$ with the property that for all $H < G$ the commutative ring $\underline{T}(G/H)$ has no $|G|$-torsion satisfies this condition [Nak12, Example 4.20]. Nakaoka also shows [Nak12, Theorem 4.32] that field-like Tambara functors satisfy (MRC). This class of Tambara functors is extensively studied in [Wisa]. See also Section 8 below.

4. Étale and separable Tambara functors

Recall from [Hil17] that for a map of G-Tambara functors $\underline{R} \to \underline{T}$ we call $\underline{T}$ *formally étale* over $\underline{R}$ if $\underline{T}$ is flat as an $\underline{R}$-module and if $\Omega^{1,G}_{\underline{T}/\underline{R}} = 0$. In the following we relate the notion of separability to formal étaleness. As the definition of separability is only based on the module category, we can define it as usual:

Definition 4.1. Let $\underline{R} \to \underline{T}$ be a map of G-Tambara functors. Then $\underline{T}$ is *separable over* $\underline{R}$ if the multiplication map of $\underline{T}$, $\mu\colon \underline{T}\Box_{\underline{R}}\underline{T} \to \underline{T}$ has a section $s\colon \underline{T} \to \underline{T}\Box_{\underline{R}}\underline{T}$ which is a map of $\underline{T}$-bimodules over $\underline{R}$.

Example 4.2. If a commutative ring T is separable over a commutative ring R, then the constant G-Tambara functor $\underline{T}^c$ is separable over $\underline{R}^c$. This follows because $\underline{T}^c\Box_{\underline{R}^c}\underline{T}^c \cong \underline{(T \otimes_R T)}^c$ (see [LRZ24, Lemma 5.1]).

Example 4.3. Let $R \to T$ be a map of commutative rings and assume that a group G acts trivially on R and acts on T by R-algebra maps. If $|G|$ is invertible in T and if T is separable over R, then Corollary 2.2 implies that $\underline{T}^{\mathrm{fix}}$ is separable over $\underline{R}^c$.

In non-equivariant algebra the ring of integers $\mathbb{Z}$ is the inital object and is separably closed [Rog08, Proposition 10.3.2], *i.e.*, every non-trivial separable extension of $\mathbb{Z}$ has non-trivial idempotents. We obtain an analogous statement in equivariant algebra.

Proposition 4.4. *The Burnside Tambara functor $\underline{A}$ is separably closed for all finite groups G,* i.e.*, if $\underline{A} \to \underline{T}$ turns $\underline{T}$ into a separable commutative $\underline{A}$-algebra G-Tambara functor with $\underline{A} \subsetneq \underline{T}$, then $\underline{T}$ has a non-trivial idempotent, so that $\underline{T} \cong \underline{T}_0 \times \underline{T}_1$ as Tambara functors.*

Proof. The ring of integers is separably closed. At the free level, we get a map of commutative rings $\underline{A}(G/e) = \mathbb{Z} \to \underline{T}(G/e)$ and the separability of $\underline{A} \to \underline{T}$ turns $\underline{T}(G/e)$ into a separable commutative $\mathbb{Z}$-algebra. Hence there is a non-trivial idempotent $e(G/e) = (e(G/e))^2$ at the free level. As in [Wisa, §3] this idempotent can be extended to a non-trivial idempotent element of $\underline{T}$. □

Proposition 4.5. *Let $\underline{R} \to \underline{T}$ be a map of Tambara functors and assume that $\underline{T}$ is separable and flat over $\underline{R}$. Then the canonical map $\underline{T} \to \underline{\mathsf{HH}}_*^{\underline{R}}(\underline{T})$ is an isomorphism.*

Proof. Thanks to flatness we have

$$\underline{\mathsf{HH}}^{\underline{R}}_*(\underline{T}) \cong \mathsf{Tor}_*^{\underline{T}\Box_{\underline{R}}\underline{T}}(\underline{T}, \underline{T})$$

but the separability of $\underline{T}$ over $\underline{R}$ ensures that $\underline{T}$ is a projective $\underline{T}\Box_{\underline{R}}\underline{T}$-module. □

As there is a surjection $\underline{I}/\underline{I}^2 \to \underline{I}/\underline{I}^{>1}$, Theorem 3.3 implies the following fact:

Corollary 4.6. *If $\underline{T}$ is separable and flat over $\underline{R}$, then $\underline{T}$ is formally étale over $\underline{R}$.*

We now study how restrictions and norms interact with separability. Recall that for a subgroup $H < G$, the restriction functor

$$i_H^G \colon G\text{-}\mathsf{Tamb} \to H\text{-}\mathsf{Tamb}$$

defined as $i_H^G\underline{R}(H/K) := \underline{R}(G/H \times_H H/K) = \underline{R}(G/K)$ has both a left adjoint (which is the norm functor N_H^G) and a right adjoint (which is co-induction).

Proposition 4.7.
(1) *Restriction preserves separability.*
(2) *If $\underline{R} \to \underline{T}$ is separable, then so is $N_H^G\underline{R} \to N_H^G\underline{T}$.*

Proof. Assume that $\underline{R} \to \underline{T}$ is separable in the category of G-Tambara functors. Let $s \colon \underline{T} \to \underline{T}\Box_{\underline{R}}\underline{T}$ be a $\underline{T}$-bimodule section of the multiplication map of $\underline{T}$. As i_H^G is strong symmetric monoidal and as it preserves coequalizers because it is a left adjoint, the relative box product $i_H^G(\underline{T})\Box_{i_H^G(\underline{R})}i_H^G(\underline{T})$ is isomorphic to $i_H^G(\underline{T}\Box_{\underline{R}}\underline{T})$ and

$$i_H^G(s) \colon i_H^G\underline{T} \to i_H^G(\underline{T}\Box_{\underline{R}}\underline{T})$$

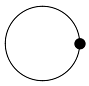
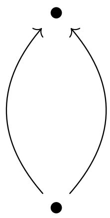
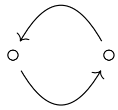
is a section of the multiplication map on $i_H^*\underline{T}$. The fact that i_H^* is strong symmetric monoidal then also ensures that $i_H^*(s)$ is a map of $i_H^*(\underline{T})$-bimodules.

As the norm is a left adjoint and is strong symmetric monoidal, the analogue of the above proof applied to a map $\underline{R} \to \underline{T}$ of H-Tambara functors also shows that $N_H^G(\underline{R}) \to N_H^G(\underline{T})$ is separable if $\underline{R} \to \underline{T}$ was separable. □

5. Separability and Loday constructions

In the non-equivariant setting, algebraic homology theories like Hochschild homology interpret coeffients to be located at the base point: For Hochschild homology of an associative k-algebra A one would place an A-bimodule M at the base point of the simplicial model of the 1-sphere, S^1, and copies of A at the other points of S^1. In the equivariant setting for a finite group G the trivial orbit G/G is the terminal object in the category of finite G-sets and G-equivariant maps, but note that G/G is *not* an initial object.

There are natural examples of equivariant homology theories where one might want to glue the coefficients to a non-trivial orbit. For instance for the group or order two there are three naturally occurring finite C_2-simplicial sets, whose underlying simplicial set models S^1: We can consider $S^1 = \Delta_1/\partial\Delta_1$ with the trivial C_2-action, there is the simplicial model of the one-point compactification of the real sign-representation, S^σ, and there is a rotation circle $S(\pi)$ for the rotation by 180 degrees which is the unit sphere in the 2-dimensional rotation representation on $\mathbb{R}^2$:

$S^1 =$  $S^\sigma =$  $S(\pi) =$ 

Note that S^σ is equivalent to the Segal-Quillen subdivision [Seg73, Appendix 1] of S^1. In the first two examples the zero simplices carry a trivial C_2-action, whereas in the third example they give rise to a free orbit. Following the definition of equivariant Loday constructions from [LRZ25] it is therefore natural to associate to a Tambara functor $\underline{R}$ and to a symmetric $\underline{R}$-bimodule $\underline{M}$ the Loday construction which has as zero simplices

$$\mathcal{L}^{C_2}_{S(\pi)}(\underline{R};\underline{M})_0 = N_e^{C_2} i_e^* \underline{M}.$$

Here, $N_e^{C_2}$ denotes the norm functor $N_e^{C_2}\colon e\text{-}\mathsf{Mack} \to C_2\text{-}\mathsf{Mack}$, constructed for instance in [Maz13, Hoy14].

For S^1 with a trivial C_2-action we get the equivariant version of Hochschild homology studied above (compare Definition 3.1) where the tensor products are replaced by the box product. In particular,

$$\mathcal{L}^{C_2}_{S^1}(\underline{R};\underline{M})_0 = \underline{M}.$$

As S^σ is the Segal-Quillen subdivision of S^1 with $S^\sigma_0 = sq(S^1)_0 = S^1_1$ one might place $\underline{M}$ at one of the trivial orbits and $\underline{R}$ at the other:

$$(\mathcal{L}^{C_2}_{S^\sigma}(\underline{R};\underline{M}))_0 = \mathcal{L}^{C_2}_{S^1}(\underline{R};\underline{M}))_1 = \underline{M}\square\underline{R}.$$

This is the convention that we used in [LR].

In the non-equivariant context if a map $A \to B$ is étale, then the canonical map from B to the Hochschild homology of B over A, $B \to \mathsf{HH}^A_*(B)$, is an isomorphism [WG91]. One can view this map as the map that sends B located at a point to B located at the basepoint of $\Delta_1/\partial\Delta_1$. If we aim at a corresponding result in equivariant algebra, then we might have to adapt this to several flavors of basepoints. So one might expect that in the case of $S(\pi)$ a formally étale map $\underline{R} \to \underline{T}$ of G-Tambara functors in the sense of Hill [Hil17] induces an isomorphism on homotopy groups

$$N_e^{C_2,\underline{R}} i_e^{C_2}\underline{T} \to \mathcal{L}^{C_2,\underline{R}}_{S(\pi)}(\underline{T})$$

where the left hand side is the constant simplicial Tambara functor on the relative norm restriction:

$$N_e^{C_2,\underline{R}} i_e^{C_2}\underline{T} = N_e^{C_2} i_e^{C_2,}\underline{T}\square_{N_e^{C_2} i_e^{C_2}\underline{R}}\underline{R}.$$

We know by Proposition 4.5 that separability ensures the vanishing of equivariant Hochschild homology groups in positive degrees. We show that this also holds for the Loday construction on the flip circle:

Proposition 5.1. *Assume that $\underline{R} \to \underline{T}$ is a map of C_2-Tambara functors that turns $\underline{T}$ into a separable $\underline{R}$-algebra and assume that $\underline{T}(C_2/e) =: T$ is flat as an $\underline{R}(C_2/e)$-module. Then*

$$\underline{\pi}_i \mathcal{L}^{C_2,\underline{R}}_{S^\sigma}(\underline{T}) \cong 0 \textit{ for all } i > 0.$$

Proof. For the Loday construction on S^σ we show in [LRZ, Proposition 3.4] that it doesn't matter which Weyl group action one uses on the norm-restriction terms. For the proof we use the diagonal action. We first show that

$$\pi_i \mathcal{L}^{C_2,\underline{R}}_{S^\sigma}(\underline{T})(C_2/e) \cong \begin{cases} T, & i = 0, \\ 0, & i > 0. \end{cases}$$

At the free level, the relative norm-restriction $N_e^{C_2,\underline{R}} i_e^{C_2}(\underline{T})$ is just

$$N_e^{C_2,\underline{R}} i_e^{C_2}(\underline{T})(C_2/e) \cong T \otimes_R T.$$

where $R = \underline{R}(C_2/e)$. As $\underline{T}$ is separable over $\underline{R}$, the commutative ring T is separable over R. The $N_e^{C_2,\underline{R}} i_e^{C_2}(\underline{T})(C_2/e)$-module structure of T is just given by the augmentation map followed by the multiplication in T and hence we can identify the chain complex associated with the simplical object $\mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})(C_2/e)$ with the two-sided bar construction $B_*^R(T, T \otimes_R T, T)$ and as T is flat over R, its homology groups calculate $\mathsf{Tor}_i^{T\otimes_R T}(T,T)$. The separability of T over R ensures that T is projective as a $T \otimes_R T$-module, so the homology groups in positive degrees vanish and in degree zero we obtain $T \otimes_{T\otimes_R T} T \cong T$.

As the homotopy groups $\underline{\pi}_* \mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})$ form a graded Tambara functor, the triviality of the homotopy groups $\pi_i \mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})(C_2/e)$ for positive i ensure the triviality of $\pi_i \mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})(C_2/C_2)$ for positive i: the norm map sends $1 = 0$ in the free level to 1 in the C_2/C_2-level, so we also get $1 = 0$ at C_2/C_2.

□

Note that the constant simplicial Tambara functor with value $\underline{T}$ splits off $\mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})$: There are two fixed points in S^σ that correspond to the 0 in the real sign representation and the compactification point $\infty \in S^\sigma$, so we get two inclusions of the constant simplicial C_2-set that has C_2/C_2 in every simplicial degree into $\mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})$. As C_2/C_2 is the terminal object in the category of finite C_2-sets, we also get a canonical projection map $S^\sigma_n \to C_2/C_2$ that assembles into a map from S^σ to the constant simplicial finite C_2-set with value C_2/C_2. On the level of Loday constructions this splits off a constant $\underline{T}$.

In the above result we couldn't really pin down that $\underline{\pi}_0 \mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})(C_2/C_2)$ agrees with $\underline{T}(C_2/C_2)$. We know that we get at least a summand $\underline{T}$ in $\underline{\pi}_0 \mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})$, but a priori it could happen that $\underline{\pi}_0 \mathcal{L}_{S^\sigma}^{C_2,\underline{R}}(\underline{T})(C_2/C_2) \cong \underline{T}(C_2/C_2) \oplus M$ for some non-trivial M. However, for fixed point Tambara functors we can rule that out:

Corollary 5.2. *Let $R \to T$ be a map of commutative C_2-rings where R has trivial C_2-action, such that $\underline{T}^{\mathrm{fix}}$ is separable and flat over $\underline{R}^c$. Then*

$$\underline{\pi}_0 \mathcal{L}_{S^\sigma}^{C_2,\underline{R}^c}(\underline{T}^{\mathrm{fix}}) \cong \underline{T}^{\mathrm{fix}}$$

as Tambara functors.

Proof. In general, the restriction functor $i_e^{C_2}$ from the category of C_2-Tambara functors to e-Tambara functors (*i.e.*, to commutative rings) is not faithful. However, if we restrict $i_e^{C_2}$ to the subcategory of fixed point Tambara functors then it *is* faithful, because the behaviour at the free level determines the behaviour at C_2/C_2 in these cases. Thus we know that the counit map $N_e^{C_2} i_e^{C_2} \underline{T}^{\mathrm{fix}} \to \underline{T}^{\mathrm{fix}}$ is surjective. As the counit induces a well defined map on the relative norm-restriction

$$N_e^{C_2} i_e^{C_2} \underline{T}^{\mathrm{fix}} \Box_{N_e^{C_2} i_e^{C_2} \underline{R}^{\mathrm{fix}}} \underline{R}^{\mathrm{fix}} \to \underline{T}^{\mathrm{fix}}$$

this implies that the augmentation map on the relative norm is also surjective and hence we can calculate $\underline{\pi}_0$ of the Loday construction for S^σ as the cokernel of the map

$$\mu \Box \mathrm{id} - \mathrm{id} \Box \mu \colon \underline{T}^{\mathrm{fix}} \Box_{\underline{R}^c} \underline{T}^{\mathrm{fix}} \Box_{\underline{R}^c} \underline{T}^{\mathrm{fix}} \to \underline{T}^{\mathrm{fix}} \Box_{\underline{R}^c} \underline{T}^{\mathrm{fix}}$$

which is isomorphic to $\underline{T}^{\mathrm{fix}}$.

□

We relate the vanishing result above to a vanishing result of reflexive homology, HR^+.

Corollary 5.3. *Assume that T is a commutative and separable R-algebra with a C_2-action by R-algebra maps. If 2 is invertible in R and if T is flat over R, then*

$$\mathsf{HR}_i^{+,R}(T) \cong \begin{cases} T^{C_2}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Proof. We actually give two proofs:

By Example 4.3 we obtain that $\underline{T}^{\mathrm{fix}}$ is a commutative and separable $\underline{R}^c$-algebra Tambara functor. By [LR, Theorem 6.5] we know that under the above assumptions

$$\mathsf{HR}_i^{+,R}(T) \cong \underline{\pi}_i \mathcal{L}_{S^\sigma}^{C_2,\underline{R}^c}(\underline{T}^{\mathrm{fix}})(C_2/C_2)$$

and therefore Corollary 5.2 implies the claim.

The second proof argues directly with reflexive homology: Daniel Graves developed a spectral sequence converging to $\mathsf{HR}^{+,R}(T)$ in [Gra24] by constructing a bicomplex whose total complex has $\mathsf{HR}^{+,R}(T)$ as its homology groups. Calculating the vertical homology of this bicomplex results in the Hochschild homology groups $\mathsf{HH}_*^R(T)$ in every column. The horizontal homology then results in the homology groups $H_*(C_2; \mathsf{HH}_*^R(T))$ of the group C_2 with coefficients in $\mathsf{HH}_*^R(T)$. By assumption the groups $\mathsf{HH}_*^R(T)$ are concentrated in the zero line with value $\mathsf{HH}_0^R(T) = T$. As 2 is invertible in R, the group homology is trivial in positive degrees. Thus in total we obtain

$$\mathsf{HR}_n^{+,R}(T) \cong \begin{cases} T_{C_2}, & n = 0, \\ 0, & n > 0. \end{cases}$$

Here, T_{C_2} denotes the C_2-coinvariants of T and in our case these are isomorphic to the C_2-invariants. $\square$

Remark 5.4. In the non-equivariant context formal étaleness of a commutative algebra A over k implies that there is a weak equivalence $A \simeq \mathcal{L}_X^k(A)$ and in particular $\pi_0 \mathcal{L}_X^k(A) \cong A$ and $\pi_{i>0} \mathcal{L}_X^k(A) = 0$ for all connected finite simplicial sets X [LR22, Proposition 2.11]. There are equivariant homology theories for which this result does not carry over, as the following example shows.

Example 5.5. We claim that the zeroth homology of the twisted cyclic nerve of [BGHL19, Definition 2.20] vanishes at the free level for the C_2-Galois extension $\underline{\mathbb{Q}}^c \to \underline{\mathbb{Q}(i)}^{\mathrm{fix}}$. The simplicial structure of the twisted cyclic nerve almost agrees with the one of equivariant Hochschild homology, but the last face map is twisted by the group action. This homology theory is therefore modelled on S^1.

For the proof we show by brute force that the differential

$$d^\tau \colon \underline{\mathbb{Q}(i)}^{\mathrm{fix}} \Box_{\underline{\mathbb{Q}}^c} \underline{\mathbb{Q}(i)}^{\mathrm{fix}} \to \underline{\mathbb{Q}(i)}^{\mathrm{fix}}$$

is surjective at the free level.

We can write down the differential explicitly as

$$d^\tau \colon \mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i) \to \mathbb{Q}(i),$$
$$d^\tau((a+bi) \otimes (u+vi)) = (a+bi)(u+iv) - (u-vi)(a+bi),$$

and this is $-2bv + 2avi$. A general element $x + yi$ is then hit by the differential for arbitrary u, $v = 1$, $b = -\frac{x}{2}$ and $a = \frac{y}{2}$.

6. Galois extensions of Tambara functors

In non-equivariant algebra Galois extensions are an important source of separable and étale extensions. We study the corresponding notion for Tambara functors. Here, the definition is based on Galois extensions of commutative rings, developed originally by Auslander and Goldman [AG60, Appendix].

Before we state our definition, we note the following fact:

Lemma 6.1. *Assume that $\underline{R} \to \underline{T}$ is a morphism of G-Tambara functors. If a finite group H acts trivially on $\underline{R}$ and acts on the left on $\underline{T}$ through $\underline{R}$-algebra maps in G-Tambara functors, then the fixed points $\underline{T}^H$ defined levelwise as $\underline{T}^H(G/K) = \underline{T}(G/K)^H$ constitute a sub-G-Tambara functor of $\underline{T}$ and $\underline{T}^H$ is still a commutative $\underline{R}$-algebra.*

Proof. As H acts via maps of G-Tambara functors, the structure maps of $\underline{T}$ commute with the H-action. Therefore the fixed points inherit a G-Tambara structure. As H fixes $\underline{R}$, the fixed points are still a commutative $\underline{R}$-algebra. $\square$

For a map of G-Tambara functors $\varphi\colon \underline{R} \to \underline{T}$ we denote the multiplication map of $\underline{T}$ over $\underline{R}$ by $\mu\colon \underline{T}\Box_{\underline{R}}\underline{T} \to \underline{T}$.

Definition 6.2. Let H and G be finite groups. Let $\varphi\colon \underline{R} \to \underline{T}$ be a levelwise injective morphism of G-Tambara functors. We call $\varphi\colon \underline{R} \to \underline{T}$ an *H-Galois extension of G-Tambara functors* if the following conditions hold:

(1) The group H acts on the left on $\underline{T}$ through $\underline{R}$-algebra maps in G-Tambara functors.
(2) The morphism $\varphi\colon \underline{R} \to \underline{T}$ induces an isomorphism
$$\varphi\colon \underline{R} \cong \underline{T}^H,$$
i.e., for all $K < G$
$$\varphi_K\colon \underline{R}(G/K) \cong \underline{T}(G/K)^H.$$
(3) The morphism
$$\chi\colon \underline{T}\Box_{\underline{R}}\underline{T} \to \prod_H \underline{T}$$
is an isomorphism of G-Tambara functors, where χ is defined as
$$\begin{tikzcd} \underline{T}\Box_{\underline{R}}\underline{T} \arrow[r, "(\mathrm{id}\Box h)_{h\in H}"] \arrow[rr, bend right, "\chi"'] & \prod_H \underline{T}\Box_{\underline{R}}\underline{T} \arrow[r, "\prod_H \mu"] & \prod_H \underline{T} \end{tikzcd}$$

Remark 6.3. If $\varphi\colon \underline{R} \to \underline{T}$ is an H-Galois extension of G-Tambara functors, then the underlying extension of commutative rings
$$\varphi_e\colon R = \underline{R}(G/e) \to \underline{T}(G/e) = T$$
is an H-Galois extension of commutative rings because $\varphi_e\colon R \to T$ is a injective map of commutative rings, so that H acts on T by R-algebra maps, fixing R, such that $\varphi_e\colon R \cong T^H$ and $\chi_e\colon (\underline{T}\Box_{\underline{R}}\underline{T})(G/e) = T \otimes_R T \to \prod_H T$ is an isomorphism, given by $t_1 \otimes t_2 \mapsto (t_1 h(t_2))_{h\in H}$, thus φ_e satisfies the conditions of [CHR65, Definition 1.4].

We can also transform every Galois extension of commutative rings into a Galois extensions of G-Tambara functors via the constant Tambara functor:

Proposition 6.4. *Assume that $R \to T$ is an H-Galois extension of commutative rings. Then $\underline{R}^c \to \underline{T}^c$ is an H-Galois extension of G-Tambara functors for every finite group G and $\underline{T}^c$ is flat as an $\underline{R}^c$-module.*

Proof. We let H act on $\underline{T}$ by acting with H on every level $\underline{T}^c(G/K) = T$. As the Weyl action on $\underline{R}^c$ and $\underline{T}^c$ is trivial, it commutes with the H-action on $\underline{T}^c$. As the restriction maps are the identity, they also commute with the H-action. The transfer just multiplies by the index and the norm takes an element to the power of that element by the index, thus they also commute with the H-action. The H-fixed points in every level are precisely R.

Finally, as $\underline{T}^c \Box_{\underline{R}^c} \underline{T}^c \cong \underline{(T \otimes_R T)}^c$ by [LRZ24, Lemma 5.1], we get that

$$\underline{T}^c \Box_{\underline{R}^c} \underline{T}^c \cong \underline{(T \otimes_R T)}^c \cong \underline{\prod_H T}^c.$$

As constant Tambara functors are fixed point Tambara functors and as the latter are right adjoints, they commute with products, so in total we obtain

$$\underline{T}^c \Box_{\underline{R}^c} \underline{T}^c \cong \prod_H \underline{T}^c.$$

It remains to show that $\underline{T}^c$ is flat over $\underline{R}^c$. To that end we prove that for every $\underline{R}^c$-module $\underline{M}$ we have at every level G/K that

$$\underline{M} \Box_{R^c} \underline{T}^c(G/K) \cong \underline{M}(G/K) \otimes_R T. \tag{6.1}$$

This proves the claim because for every H-Galois extension $R \to T$ of commutative rings, T is flat (even projective) as an R-module [CHR65, Theorem 1.3].

In order to show (6.1) we use [Str, Proposition 3.14] where Strickland shows that we can represent every element in the box product $\underline{M} \Box \underline{T}^c(G/K)$ as $T_p(m \otimes t)$ with

$$m \otimes t \in \underline{M}(G/L) \otimes \underline{T}^c(G/L) = \underline{M}(G/L) \otimes T$$

and where $p \colon G/L \to G/K$ a map of finite G-sets. With the help of Frobenius reciprocity [Str, Lemma 3.13] and by writing every $t \in \underline{T}^c(G/L)$ as $R_p(t) = \mathrm{id}(t) = t$ we can rewrite $T_p(m \otimes t)$ as

$$T_p(m \otimes t) = T_p(m \otimes R_p(t)) = T_p(m) \otimes t.$$

Therefore we can push every representative in $\underline{M} \Box \underline{T}^c(G/K)$ to an element $T_p(m) \otimes t \in \underline{M}(G/K) \otimes T$. Using that $\underline{M} \Box_{\underline{R}^c} \underline{T}^c$ is the coqualizer of a diagram involving

$$\underline{M} \Box \underline{R}^c \Box \underline{T}^c \cong \underline{M} \Box \underline{(R \otimes T)}^c$$

and $\underline{M} \Box \underline{T}^c$, we get the claim.

□

As for commutative rings we always get non-connected examples (see for instance [Gre92, Example (b) in §1]):

Example 6.5. If $\underline{R}$ is an arbitrary G-Tambara functor for any finite group G and if H is a finite group, then

$$\underline{R} \to \prod_H \underline{R}$$

is an H-Galois extension. Here, H acts on the indexing set of the product and $\underline{R}$ embeds into $\prod_H \underline{R}$ as a diagonal copy. As H is finite, $\prod_H \underline{R}$ is free over $\underline{R}$; in particular, it is flat.

The next result is a key property of Galois extensions:

Theorem 6.6. *If $\varphi \colon \underline{R} \to \underline{T}$ is an H-Galois extension of G-Tambara functors, then $\underline{R} \to \underline{T}$ is separable.*

Proof. We define $\sigma \colon \underline{T} \to \underline{T} \Box_{\underline{R}} \underline{T}$ as $\chi^{-1} \circ i^{(e)}$ where $i^{(e)} \colon \underline{T} \to \prod_H \underline{T}$ is the inclusion into the factor corresponding to the neutral element $e \in H$. Note that $i^{(e)}$ does not preserve the multiplicative unit, but it is a map of Mackey functors and preserves the multiplication and commutes with the norm.

We have to show that $\mu \circ \sigma = \mathrm{id}_{\underline{T}}$ and that σ is a $\underline{T}$-bimodule map relative $\underline{R}$. For the first property, consider the following commutative diagram:

$$\begin{tikzcd}
\underline{T} \arrow[r, "i^{(e)}"] \arrow[dr, equal] \arrow[rrr, bend left, "\sigma"] & \prod_H \underline{T} \arrow[d, "p^{(e)}"] & \prod_H \underline{T}\Box_{\underline{R}}\underline{T} \arrow[l, "\prod_H \mu"'] \arrow[d, "p^{(e)}"] & \underline{T}\Box_{\underline{R}}\underline{T} \arrow[l, "(\mathrm{id}\Box h)_{h\in H}"'] \arrow[dl, equal] \\
& \underline{T} & \underline{T}\Box_{\underline{R}}\underline{T} \arrow[l, "\mu"] &
\end{tikzcd}$$

where $p^{(e)}$ is the projection to the e-coordinate. This shows $\mu \circ \sigma = p^{(e)} \circ i^{(e)} = \mathrm{id}_{\underline{T}}$.

For the compatibility with the bimodule structure, we adapt the classical proof, see for instance [Tak65, Proof of Lemma 2], to the equivariant setting: Note that the map $\chi\colon \underline{T}\Box_{\underline{R}}\underline{T} \to \prod_H \underline{T}$ is equivariant if we use the H-action on the right factor of $\underline{T}$ in $\underline{T}\Box_{\underline{R}}\underline{T}$ and the action on $\prod_H \underline{T}$ on the indexing set by right multiplication. For every $h \in H$ composing $i^{(e)}$ with the action by h gives the inclusion into the factor with label h^{-1}, $i^{h^{-1}}$. Therefore the following diagram commutes:

$$\begin{tikzcd}
& \prod_H \underline{T} \arrow[r, "h"] & \prod_H \underline{T} \arrow[dr, "p^{(e)}"] & \\
\underline{T} \arrow[ur, "i^{(e)}"] \arrow[urr, bend left=60, "i^{(h^{-1})}"] \arrow[r, "\sigma"] & \underline{T}\Box_{\underline{R}}\underline{T} \arrow[u, "\chi"] \arrow[r, "\mathrm{id}\Box h"] & \underline{T}\Box_{\underline{R}}\underline{T} \arrow[u, "\chi"] \arrow[r, "\mu"] & \underline{T}.
\end{tikzcd}$$

Here, we have used that the multiplication μ coincides with the composite $p^{(e)} \circ \chi$.

With $\delta_{h,e}$ denoting the Kronecker delta for h and e this yields:

$$\mu \circ (\mathrm{id}\Box h) \circ \sigma = p^{(e)} \circ i^{(h^{-1})} = \delta_{h,e} \cdot \mathrm{id}_{\underline{T}}. \tag{6.2}$$

In order to avoid a glaring clash of notation we denote by s_H what is usually called the norm or trace of the finite group H:

$$s_H = \sum_{h\in H} h. \tag{6.3}$$

With (6.2) we can rewrite the identity map on $\underline{T}$ as follows:

$$\mathrm{id}_{\underline{T}} = \mu \circ (\mathrm{id}\Box s_H) \circ \sigma. \tag{6.4}$$

With a similar argument one shows

$$\mathrm{id}_{\underline{T}} = \mu \circ (s_H\Box\mathrm{id}) \circ \sigma. \tag{6.5}$$

We construct two auxiliary maps from $\underline{T}$ to $\underline{T}\Box_{\underline{R}}\underline{T}$. Let $\eta\colon \underline{R} \to \underline{T}$ denote the unit of $\underline{T}$ and let $\nu_L\colon \underline{T} \cong \underline{R}\Box_{\underline{R}}\underline{T}$ denote the monoidal isomorphism. We let ψ_L denote the composite

$$\underline{T} \xrightarrow[\nu_L]{\cong} \underline{R}\Box_{\underline{R}}\underline{T} \xrightarrow{\eta\Box\mathrm{id}} \underline{T}\Box_{\underline{R}}\underline{T} \xrightarrow{\sigma\Box\mathrm{id}} \underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \xrightarrow{\mathrm{id}\Box\mu} \underline{T}\Box_{\underline{R}}\underline{T} \tag{6.6}$$

and dually, let ψ_R be the composite

$$\underline{T} \xrightarrow[\nu_R]{\cong} \underline{T}\Box_{\underline{R}}\underline{R} \xrightarrow{\mathrm{id}\Box\eta} \underline{T}\Box_{\underline{R}}\underline{T} \xrightarrow{\mathrm{id}\Box\sigma} \underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \xrightarrow{\mu\Box\mathrm{id}} \underline{T}\Box_{\underline{R}}\underline{T} \tag{6.7}$$

In the following we abbreviate $\sigma \circ \eta$ by ξ so that we can write

$$\psi_L = (\mathrm{id}\Box\mu) \circ (\xi\Box\mathrm{id}) \circ \nu_L \text{ and } \psi_R = (\mu\Box\mathrm{id}) \circ (\mathrm{id}\Box\xi) \circ \nu_R.$$

We claim that $\psi_L = \psi_R = \sigma$ which ensures that σ is a bimodule map. In the following diagrams we omit the unit isomorphisms ν_R and ν_L, unadorned box products are over $\underline{R}$, id_n denotes an identity map on n box product factors and μ_n denotes an iterated multiplation map.

For the claim that ψ_L and ψ_R agree we consider the composite

$$\theta = \left(\begin{tikzcd} \underline{T} \arrow[r, "\xi\square\mathrm{id}\square\xi"] & \underline{T}^{\square 5} \arrow[r, "\mathrm{id}\square\mu_3\square\mathrm{id}"] & \underline{T}^{\square 3} \arrow[r, "\mathrm{id}\square s_H\square\mathrm{id}"] & \underline{T}^{\square 3} \arrow[r, bend left, "\mu\square\mathrm{id}"] \arrow[r, bend right, "\mathrm{id}\square\mu"'] & \underline{T}\square_{\underline{R}}\underline{T} \end{tikzcd}\right)$$

Note that for defining θ it is irrelevant whether we use $\mu\square\mathrm{id}$ or $\mathrm{id}\square\mu$ as the last map in the composite because the image of s_H is contained in $\underline{R}$. We compare ψ_R with the θ using the map $\mu\square\mathrm{id}$ at the end of the composition and leave the dual proof of comparing ψ_L with the other variant to the diligent reader.

We embed the above composite θ as the top row of the following enlarged diagram

$$\begin{tikzcd}
\underline{T} \arrow[r, "\xi\square\mathrm{id}\square\xi"] \arrow[d, "\mathrm{id}\square\xi"'] & \underline{T}^{\square 5} \arrow[r, "\mathrm{id}\square\mu_3\square\mathrm{id}"] \arrow[d, "\mathrm{id}_2\square\mu\square\mathrm{id}"'] & \underline{T}^{\square 3} \arrow[r, "\mathrm{id}\square s_H\square\mathrm{id}"] & \underline{T}^{\square 3} \arrow[r, "\mu\square\mathrm{id}"] & \underline{T}\square_{\underline{R}}\underline{T} \\
\underline{T}\square_{\underline{R}}\underline{T}\square_{\underline{R}}\underline{T} \arrow[d, "\mu\square\mathrm{id}"'] & \underline{T}^{\square 4} \arrow[ur, "\mathrm{id}\square\mu\square\mathrm{id}"] \arrow[r, "\mathrm{id}\square s_{\Delta H}\square\mathrm{id}"] & \underline{T}^{\square 4} \arrow[ur, "\mathrm{id}\square\mu\square\mathrm{id}"] \arrow[r, "\mu\square\mathrm{id}_2"] & \underline{T}^{\square 3} \arrow[ur, "\mu\square\mathrm{id}"] & \\
\underline{T}\square_{\underline{R}}\underline{T} \arrow[ur, "\xi\square\mathrm{id}_2"] \arrow[r, "\eta\square\mathrm{id}_2"] \arrow[rrrruu, bend right, "\mathrm{id}_2"'] & \underline{T}^{\square 3} \arrow[u, "\sigma\square\mathrm{id}_2"'] \arrow[rru, bend right, "\mathrm{id}_3"'] & & &
\end{tikzcd}$$

The pentagon on the left commutes because the maps don't interfere. The associativity of the multiplication ensures that the upper small triangle and the right-most parallelogram commute. The lower left triangle commutes thanks to the definition of ξ.

In the diagram $s_{\Delta H}$ denotes the trace on $\underline{T}\square_{\underline{R}}\underline{T}$ that uses the diagonal H-action on both $\underline{T}$-factors. As the H-action is multiplicative, we get that the parallelogram containing s_H commutes.

Due to (6.4) we also get that the curved triangle commutes because only the terms in the trace that act via the unit element are left. The bottom part of the diagram commutes because η is a unit for μ.

In total we get that the composite on the left which is ψ_R coincides with the composite θ. As ψ_L also coincides with θ we get that $\psi_R = \psi_L$.

As ψ_R uses the right unit isomorphism ν_R and then applies ξ and a multiplication, it does not interfere when we multiply on the left and hence the diagram

$$\begin{tikzcd}
\underline{T}\square_{\underline{R}}\underline{T} \arrow[r, "\mathrm{id}\square\psi_R"] \arrow[d, "\mu"'] & \underline{T}\square_{\underline{R}}\underline{T}\square_{\underline{R}}\underline{T} \arrow[d, "\mu\square\mathrm{id}"] \\
\underline{T} \arrow[r, "\psi_R"] & \underline{T}\square_{\underline{R}}\underline{T}
\end{tikzcd}$$

commutes. Hence ψ_R is a left $\underline{T}$-module map. Dually, ψ_L is a right $\underline{T}$-module map. As $\psi_R = \psi_L$, this map is compatible with the $\underline{T}$-bimodule structure.

Last but not least we show that $\psi_R = \sigma$ which then of course also implies $\psi_L = \sigma$. To that end consider the diagram

$$\begin{tikzcd}
\underline{T} \arrow[rr, "i^{(e)}"] \arrow[d, "\nu_R"'] & & \prod_H \underline{T} \arrow[r, "\chi^{-1}"] & \underline{T}\Box_{\underline{R}}\underline{T} \\
\underline{T}\Box_{\underline{R}}\underline{R} \arrow[r, "\mathrm{id}\Box\eta"] & \underline{T}\Box_{\underline{R}}\underline{T} \arrow[r, "\mathrm{id}\Box i^{(e)}"] & \underline{T}\Box_{\underline{R}}\prod_H \underline{T} \arrow[u, "m"'] \arrow[r, "\mathrm{id}\Box\chi^{-1}"] & \underline{T}\Box_{\underline{R}}\underline{T}\Box_{\underline{R}}\underline{T} \arrow[u, "\mu\Box\mathrm{id}"']
\end{tikzcd}$$

Here, m is the left $\underline{T}$-module structure on $\prod_H \underline{T}$ that is given by the multiplication in $\underline{T}$ in every factor. By direct inspection one sees that $m \circ (\mathrm{id}\Box\chi) = \chi \circ (\mu \otimes \mathrm{id})$ and therefore the right square commutes and the left diagram commutes because $i^{(e)}$ only includes $\underline{T}$ into the e-factor of the product and η is the unit map of the multiplication.

Therefore the top row describing σ coincides with the other composite which is ψ_R. $\square$

We will now provide some examples of Galois extensions of Tambara functors.

Proposition 6.7. *If $K \to L$ is a C_2-Galois extension of fields, then $\underline{K}^c \to \underline{L}^{\mathrm{fix}}$ is a C_2-Galois extension of C_2-Tambara functors where we define the C_2-Galois action as the Weyl-action on $\underline{L}^{\mathrm{fix}}$.*

Proof. As C_2 is abelian, the C_2-Galois action commutes with the C_2-Weyl action of the C_2-Tambara functor.

We first consider the case where K is a field of characteristic prime to 2. We claim that the isomorphism of C_2-Tambara functors from Corollary 2.2 $\underline{L}^{\mathrm{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathrm{fix}} \cong \underline{(L \otimes_K L)}^{\mathrm{fix}}$ can be combined with the isomorphism of commutative rings $\chi\colon L \otimes_K L \cong \prod_{C_2} L$ to induce an isomorphism of C_2-Tambara functors

$$\underline{L}^{\mathrm{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathrm{fix}} \cong \prod_{C_2} \underline{L}^{\mathrm{fix}}.$$

A priori the map χ is equivariant with respect to the C_2-action that acts on the right hand tensor factor in $L \otimes_K L$ and acts on the indexing set in $\prod_{C_2} L$. However, in $\underline{(L \otimes_K L)}^{\mathrm{fix}}$ the C_2-Weyl action is diagonally on $L \otimes_K L$ and acts on the L in $\prod_{C_2} L$.

We show that χ is also equivariant with respect to this action: If $C_2 = \langle \tau | \tau^2 = e \rangle$, then $L = K(\alpha)$ with $\alpha^2 = a \in K$ and $\tau(\alpha) = -\alpha$. We denote elements in $\prod_{C_2} L$ by (b, c) where b is in the 1-coordinate, and c is in the τ-coordinate. The generator $1 \otimes 1$ is fixed under τ. The generators $1 \otimes \alpha$, $\alpha \otimes 1$ and $\alpha \otimes \alpha$ sit in commutative diagrams

(6.8)

$$\begin{tikzcd}
1 \otimes \alpha \arrow[r, mapsto, "\chi"] \arrow[d, "\tau"'] & (\alpha, -\alpha) \arrow[d, "\tau"] & \alpha \otimes 1 \arrow[r, mapsto, "\chi"] \arrow[d, "\tau"'] & (\alpha, \alpha) \arrow[d, "\tau"] & \alpha \otimes \alpha \arrow[r, mapsto, "\chi"] \arrow[d, "\tau"'] & (\alpha^2, -\alpha^2) \arrow[r, equal] & (a, -a) \arrow[d, "\tau"] \\
-1 \otimes \alpha \arrow[r, mapsto, "\chi"] & (-\alpha, \alpha) & -\alpha \otimes 1 \arrow[r, mapsto, "\chi"] & (-\alpha, -\alpha) & (-\alpha) \otimes (-\alpha) \arrow[r, mapsto, "\chi"] & (\alpha^2, -\alpha^2) \arrow[r, equal] & (a, -a).
\end{tikzcd}$$

Therefore $\chi\colon L \otimes_K L \cong \prod_{C_2} L$ as commutative C_2-K-algebras with the above action and this shows

$$\underline{L}^{\mathrm{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathrm{fix}} \cong \underline{(L \otimes_K L)}^{\mathrm{fix}} \cong \underline{\Big(\prod_{C_2} L\Big)}^{\mathrm{fix}} \cong \prod_{C_2} \underline{L}^{\mathrm{fix}}.$$

The last isomorphism is due to the fact that $(-)^{\mathrm{fix}}$ is a right adjoint functor.

The fixed point property $\left(\underline{L}^{\mathrm{fix}}\right)^{C_2} \cong \underline{K}^c$ is easy to see, because $L^{C_2} = K$ and $K^{C_2} = K$.

Assume now that the characteristic of K is 2. Then L is an Artin-Schreier extension and is of the form $L = K(\alpha)$ with $\alpha^2 + \alpha + a = 0$ for some $a \in K$ and $\tau(\alpha) = \alpha + 1$. By [LRZ24, Proposition

3.4] we know that generators in $\underline{L}^{\mathrm{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathrm{fix}}$ are of the form $1\otimes 1\in K\otimes_K K\cong K$ and $[\lambda\alpha\otimes\alpha]$ with $\lambda\in K$. We claim that also in this case we get an isomorphism of C_2-Tambara functors

$$\underline{L}^{\mathrm{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathrm{fix}}\cong \underline{\left(L\otimes_K L\right)}^{\mathrm{fix}}. \tag{6.9}$$

Again, at the free level we take the identity map whereas at level C_2/C_2 we send $1\otimes 1$ to $1\otimes 1$ but

$$[\lambda\alpha\otimes\alpha]\mapsto \lambda\alpha\otimes 1+\lambda\otimes\alpha+\lambda\otimes 1.$$

We need to show that this is an isomorphism of commutative rings which together with the identity at the free level constitutes an isomorphism of C_2-Tambara functors. The λ-factor only contributes notational complexity. The product

$$[\alpha\otimes\alpha]^2=[\alpha\otimes\alpha\cdot\mathsf{restr}(\alpha\otimes\alpha)]$$

can be calculated to give $[\alpha\otimes\alpha]+a[\alpha\otimes 1]+a[1\otimes\alpha]$ but as $[\alpha\otimes 1]\sim 1\otimes 1\sim[1\otimes\alpha]$ we are left with $[\alpha\otimes\alpha]$. In the target

$$(\alpha\otimes 1+1\otimes\alpha+1\otimes 1)^2=\alpha\otimes 1+1\otimes\alpha+1\otimes 1$$

thus the map respects the multiplication. A direct calculation shows that the map is also compatible with restriction, transfer and norm.

As in the first case we have to show that the map $\chi\colon L\otimes_K L\cong\prod_{C_2}L$ is also equivariant with respect to the diagonal action in the source and the action on L in the target. Again, this is checked on generators. We present the most complicated case and chase $\alpha\otimes\alpha$:

$$\begin{tikzcd}
\alpha\otimes\alpha \arrow[r, mapsto, "\chi"] \arrow[d, "\tau"] & (\alpha^2,\alpha(\alpha+1)) \arrow[r, equal] & (\alpha+a,a) \arrow[d, "\tau"] \\
(\alpha+1)\otimes(\alpha+1) \arrow[r, mapsto, "\chi"] & ((\alpha+1)^2,(\alpha+1)\alpha) \arrow[r, equal] & (\alpha^2+1,a)=(\alpha+a+1,a)
\end{tikzcd} \tag{6.10}$$

The rest of the argument agrees with the one in the first case. ☐

Remark 6.8.

(1) Note that we proved in (6.9) that

$$\underline{L}^{\mathrm{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathrm{fix}}\cong \underline{\left(L\otimes_K L\right)}^{\mathrm{fix}}$$

if $K\subset L$ is a C_2-Galois extension, even if K has characteristic 2, so in particular $|C_2|=2$ is *not* invertible.

(2) If $K\subset L$ is an H-Galois extension of fields, then in general $\underline{K}^c\to\underline{L}^{\mathrm{fix}}$ won't be an H-Galois extension of H-Tambara functors: If H is not abelian, then the Galois action does not have to commute with the Weyl action despite the fact that they agree. Even if H is abelian, the diagrams analogous to (6.8) and (6.10) won't necessarily commute.

The above examples of C_2-Galois extension of C_2-Tambara functors also shows that the map $s_H=\sum_{h\in H}h\colon\underline{T}\to\underline{R}$ does not have to be surjective:

Proposition 6.9. *There are H-Galois extensions of G-Tambara functors $\underline{R}\to\underline{T}$ such that the map $s_H\colon\underline{T}\to\underline{R}$ is not surjective at every level.*

Proof. Consider a C_2-Galois extension of fields $K\to L$ of characteristic 2 and the corresponding C_2-Galois extension of C_2-Tambara functors $\underline{K}^c\to\underline{L}^{\mathrm{fix}}$. At level C_2/C_2 the map s_H is

$$(e+\tau)\colon\underline{L}^{\mathrm{fix}}(C_2/C_2)=L^{C_2}=K\to\underline{K}^c(C_2/C_2)=K$$

and hence it sends any $x\in K$ to $2x=0$. ☐

Remark 6.10. In the Galois theory of commutative rings the surjectivity of the trace $s_H\colon T \to R$ is an important means to ensure the projectivity and faithful flatness of T as an R-module. If $\underline{R} \to \underline{T}$ is an H-Galois extension of G-Tambara functors, then $\underline{R}(G/e) \to \underline{T}(G/e)$ is an H-Galois extension of commutative rings and therefore we know that $s_H\colon \underline{T}(G/e) \to \underline{R}(G/e)$ is surjective, $\underline{T}(G/e)$ is projective and faithfully flat as an $\underline{R}(G/e)$-module.

For $\underline{K}^c \to \underline{L}^{\text{fix}}$ coming from a C_2-Galois extension $K \subset L$ we showed in [LRZ24, Lemma 3.8] by different methods that $\underline{L}^{\text{fix}}$ is projective over $\underline{K}^c$. We don't have a general argument for the projectivity of Galois extensions of Tambara functors.

In analogy with the corresponding notion for modules over ring spectra [Rog08, Definition 4.3.1] we define faithfulness for modules over a Tambara functor as follows:

Definition 6.11. Let $\underline{R}$ be a G-Tambara functor and let $\underline{M}$ be an $\underline{R}$-module. Then $\underline{M}$ is *faithful*, if for all $\underline{R}$-modules $\underline{N}$ the triviality of $\underline{N}\Box_{\underline{R}}\underline{M}$ implies the triviality of $\underline{N}$.

Note that this clashes with the classical notion of faithfulness in the category of modules over a commutative ring where one requires the annihilator of the module to be trivial.

If the group order $|H|$ is invertible in $\underline{R}$, then we do get faithfulness with the help of the usual proof (see [Rog08, Lemma 6.4.3]):

Proposition 6.12. *Assume that $\underline{R} \to \underline{T}$ is an H-Galois extension of G-Tambara functors and assume that $|H|$ is invertible in $\underline{R}$, then $\underline{T}$ is a faithful $\underline{R}$-module.*

Proof. Consider the composite $\underline{R} \xrightarrow{\eta} \underline{T} \xrightarrow{s_H} \underline{R}$ which sends an $x \in \underline{R}(G/K)$ to $|H|x$. As $|H|$ is invertible, this shows that $\underline{R}$ splits off $\underline{T}$ via maps of $\underline{R}$-modules. Assume that for an $\underline{R}$-module $\underline{M}$ we get $\underline{M}\Box_{\underline{R}}\underline{T} \cong 0$. Then $\underline{M} \cong \underline{M}\Box_{\underline{R}}\underline{R}$ splits of 0, and hence $\underline{M} \cong 0$ and $\underline{T}$ is faithful as an $\underline{R}$-module. $\Box$

Example 6.13. If $R \to T$ is an H-Galois extension of commutative rings, then we saw in Proposition 6.4 that $\underline{R}^c \to \underline{T}^c$ is an H-Galois extension in G-Tambara functors such that $\underline{T}^c$ is flat over $\underline{R}^c$. As the trace map $s_H\colon T \to R$ is a surjective map, we get surjectivity of the trace map at every level. This insures faithfulness because for every $\underline{R}^c$-module $\underline{M}$ the map

$$\underline{M}\Box_{\underline{R}^c}\underline{T}^c \xrightarrow{\text{id}\Box s_H} \underline{M}\Box_{\underline{R}^c}\underline{R}^c \cong \underline{M}$$

is surjective.

Theorem 6.14. *Assume that G and H are finite groups and that $|G|$ is prime to the characteristic of a field k. If $j\colon k \to L$ is a $G\times H$-Galois extension of fields and $K = L^H$, then the H-Galois extension $\phi\colon K \to L$ induces an H-Galois extension of G-Tambara functors $\varphi\colon \underline{K}^{\text{fix}} \to \underline{L}^{\text{fix}}$ with $\varphi_e = \phi$.*

Proof. As the G and H action on L commute, the H action on L gives a well-defined action on the G-Tambara functor $\underline{L}^{\text{fix}}$. At every level G/U we get

$$\underline{L}(G/U)^H = (L^U)^H \cong (L^H)^U = K^U = \underline{K}^{\text{fix}}(G/U).$$

Thus we only have to check the unramified condition, thus we consider

$$\chi\colon \underline{L}^{\text{fix}}\Box_{\underline{K}^{\text{fix}}}\underline{L}^{\text{fix}} \to \prod_H \underline{L}^{\text{fix}}.$$

As $|G|$ is invertible in k (and hence in K and L) we know that $\underline{L}^{\text{fix}}\Box\underline{L}^{\text{fix}} \cong \underline{(L\otimes L)}^{\text{fix}}$ and $\underline{L}^{\text{fix}}\Box\underline{K}^{\text{fix}}\Box\underline{L}^{\text{fix}} \cong \underline{(L\otimes K\otimes L)}^{\text{fix}}$. We showed in the proof of [LR, Proposition 4.1] that taking the fixed point Tambara functor commutes with coequalizers and therefore

$$\underline{L}^{\text{fix}}\Box_{\underline{K}^{\text{fix}}}\underline{L}^{\text{fix}} \cong \underline{(L\otimes_K L)}^{\text{fix}}.$$

But as $K \subset L$ is an H-Galois extension the map of commutative rings $\chi\colon L \otimes_K L \to \prod_H L$, $\chi(a \otimes b) = (a \cdot h(b))_{h\in H}$ is an isomorphism. Thus we obtain

$$\underline{(L \otimes_K L)}^{\mathrm{fix}} \cong \underline{\Big(\prod_H L\Big)}^{\mathrm{fix}}.$$

As the functor that takes an abelian group with a G-action to its fixed point G-Mackey functor is right adjoint to the functor that evaluates a G-Mackey functor at the free level, it commutes with products, and therefore we obtain that h is an isomorphism of G-Tambara functors

$$\underline{L}^{\mathrm{fix}} \Box_{\underline{K}^{\mathrm{fix}}} \underline{L}^{\mathrm{fix}} \cong \prod_H \underline{L}^{\mathrm{fix}}.$$

□

Example 6.15. The C_4-Galois extension of fields $\mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{Q}(\sqrt{2}, \zeta_5)$ induces a C_4-Galois extension of C_2-Tambara functors $\underline{\mathbb{Q}(\sqrt{2})}^{\mathrm{fix}} \to \underline{\mathbb{Q}(\sqrt{2}, \zeta_5)}^{\mathrm{fix}}$.

7. No naive Tambara analogue of the Gaussian integers

The Gaussian integers $\mathbb{Z}[i]$ provide an important example of a ramified extension. The conjugation action has $\mathbb{Z}$ as fixed points but the extension $\mathbb{Z} \to \mathbb{Z}[i]$ is *not* a C_2-Galois extension of commutative rings because the map $\xi\colon \mathbb{Z}[i] \otimes \mathbb{Z}[i] \to \prod_{C_2} \mathbb{Z}[i]$ is not surjective. But the extension $\mathbb{Z} \to \mathbb{Z}[i]$ is only ramified at 2 and

$$\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}][i]$$

is a C_2-Galois extension of commutative rings. This works in broader generality: for a Galois extension of number fields, inverting all the ramified primes yields a Galois extension of the corresponding rings of integers [Gre92, Theorem 4.1].

If one wants to mimic the construction of the Gaussian integers $\mathbb{Z}[i] = \mathbb{Z}[x]/x^2 + 1$ in the setting of C_2-Tambara functors, then one would rewrite them as a pushout

$$\mathbb{Z}[x]/x^2 + 1 = \mathbb{Z}[x] \otimes_{\mathbb{Z}[y]} \mathbb{Z}$$

where y is mappd to $x^2 \in \mathbb{Z}[x]$ and to -1 in $\mathbb{Z}$.

Blumberg and Hill provide very helpful explicit formulas for free C_2-Tambara functors in [BH19, §3]. Adjoining a free generator at the trivial level C_2/C_2, x_{C_2/C_2}, to the Burnside Tambara functor for C_2 yields

$$\begin{tikzcd} \underline{A}[x_{C_2/C_2}]: & \mathbb{Z}[t]/t^2 - 2t[x_*, n_*]/tn_* = tx_*^2 \arrow[d, bend right=50, "\mathsf{res}"'] \\ & \mathbb{Z}[x] \arrow[u, bend right=30, "\mathsf{tr}"'] \arrow[u, bend right=70, "\mathsf{norm}"'] \end{tikzcd}$$

with $\mathsf{res}(x_*) = x$, $\mathsf{res}(n_*) = x^2$, $\mathsf{tr}(p(x)) = tp(x_*)$ for any $p(x) \in \mathbb{Z}[x]$ and the norm is determined by the norm in the Burnside Tambara functor, by $\mathsf{norm}(x) = n_*$ and by an explicit inductive formular for polynomials of higher degree [BH19, Proposition 3.5].

A – probably too naive – attempt to build a Tambara version of the Gaussian integers is to form a relative box product

$$\underline{G} = \underline{A}[x_{C_2/C_2}] \Box_{\underline{A}[y_{C_2/C_2}]} \underline{A}$$

where at level C_2/e we send y to x^2 in $\underline{A}[x_{C_2/C_2}](C_2/e) = \mathbb{Z}[x]$ and to $-1 \in \underline{A}(C_2/e) = \mathbb{Z}$. We then obtain $\mathbb{Z}[x]/x^2+1 = \mathbb{Z}[i]$ at the free level. At both levels we have (quotients of) polynomial algebras, so we can talk about the degrees of homogeneous elements.

As in the non-equivariant case we assume the following about the maps in the pushout diagram

$$\underline{A}[x_{C_2/C_2}] \xleftarrow{\varphi} \underline{A}[y_{C_2/C_2}] \xrightarrow{\psi} \underline{A}:$$

(1) The C_2-action is $\underline{A}$-linear and is trivial on $\underline{A}[y_{C_2/C_2}]$.
(2) The map φ is a morphism of Tambara functors that doubles the degree.
(3) The map ψ is a morphism of Tambara functors.

However, we show the following no-go result:

Proposition 7.1. *There is no way to define a C_2-action on $\underline{G}$ which satisfies the above conditions (1), (2) and (3), such that $\underline{A} \cong \underline{G}^{C_2}$.*
In particular, inverting 2 does not *yield a C_2-Galois extension $\underline{A}[\frac{1}{2}] \to \underline{G}[\frac{1}{2}]$.*

Proof. As we define the C_2-action on the free level as

$$\tau(x) = -x$$

this implies

$$\tau(n_*) = \tau(\mathsf{norm}(x)) = \mathsf{norm}(-x) = \mathsf{norm}(-1)n_* = (t-1)n_*.$$

Note that this already implies that elements of the form $\beta t n_*$ are fixed by τ.

We claim that we have no choice but to define $\tau(x_*)$ as $(1-t)x_*$: As we assume that the C_2-action is degree-preserving, we know that

$$\tau(x_*) = (a+bt)x_* \text{ with } a, b \in \mathbb{Z}.$$

As τ has to act via morphisms of Tambara functors we get

$$\tau(tx_*) = \tau(\mathsf{tr}(x)) = \mathsf{tr}(-x) = -tx_*$$

and this results in the condition

$$a + 2b = -1 \tag{7.1}$$

The equality

$$-x = \tau(x) = \tau(\mathsf{res}(x_*)) = \mathsf{res}\tau(x_*) = \mathsf{res}((a+bt)x_*) = (a+2b)x$$

yields the same condition.
Using $\tau^2(x_*) = x_*$ further implies that

$$a^2 + (2ab + 2b^2)t = 1$$

and by comparing this with (7.1) we get $b = -1$ and $a = 1$, hence

$$\tau(x_*) = (1-t)x_*. \tag{7.2}$$

Therefore the C_2-action is completely determined by condition (1).
It is straightforward to check that elements of the form $(-2d + dt)x_*^{2n+1}$ are fixed points for all $d \in \mathbb{Z}$ and all $n \geqslant 0$, because $(-2d + dt)(1-t) = -2d + 2dt + dt - 2dt = -2d + dt$, so we have fixed points in all odd degrees.
As we assume that φ doubles the degree, these fixed points are not in the image of φ_{C_2} and hence they give non-trivial fixed points in $\underline{G}^{C_2}$, proving that $\underline{A}$ is properly contained in $\underline{G}^{C_2}$. □

8. Nullstellensatzian Tambara functors and Galois extensions

For Tambara funtors there is a notion of being field-like due to Nakaoka [Nak12]. Field-like G-Tambara functors $\underline{T} \neq 0$ can be characterized as those G-Tambara functors $\underline{T}$ for which $\underline{T}(G/e)$ has no non-trivial G-invariant ideals and that have injective restriction maps associated to equivariant maps of finite transitive G-sets and [Nak12, Theorem 4.32]. Typical examples arise from G-Galois extensions $K \subset L$ of fields. In this case the fixed point G-Tambara functor $\underline{L}^{\mathrm{fix}}$ is field-like. Wisdom shows [Wisb, Corollary B] that every field-like G-Tambara functor $\underline{k}$ is of the form $\mathsf{Coind}_H^G(\underline{\ell})$ for some field-like H-Tambara functor $\underline{\ell}$ such that $\underline{\ell}(H/e)$ is a field. Here, Coind_H^G denotes the co-induction functor $\mathsf{Coind}_H^G \colon H\text{-}\mathsf{Tamb} \to G\text{-}\mathsf{Tamb}$. This functor is right adjoint to the restriction functor $i_H^G \colon G\text{-}\mathsf{Tamb} \to H\text{-}\mathsf{Tamb}$.

Schuchardt, Spitz, and Wisdom investigate which Tambara functors behave like algebraically closed objects. They do this using the concept of *Nullstellensatzian objects*, introduced in [BSY] and show that Nullstellensatzian G-Tambara functors $\underline{k}$ are isomorphic to the fixed point Tambara functor $\underline{\underline{k}(G/e)}^{\mathrm{fix}}$ and can be expressed as the co-induction of an algebraically closed field $\mathbb{F}$, $\underline{k} \cong \mathsf{Coind}_e^G(\mathbb{F})$ [SSW, §5]. So it is a natural question to ask how co-induction from the trivial group to some finite group G interacts with Galois extensions:

Proposition 8.1. *If $R \to T$ is an H-Galois extension of commutative rings for some finite group H, then $\mathsf{Coind}_e^G(R) \to \mathsf{Coind}_e^G(T)$ is an H-Galois extension of G-Tambara functors for any finite group G.*

Proof. As Coind_e^G is a functor from the category of commutative rings to the category of G-Tambara functors, the H-action on T by R-algebra maps is transformed into an H-action on $\mathsf{Coind}_e^G(T)$ by $\mathsf{Coind}_e^G(R)$-algebra maps.

As $\mathsf{Coind}_e^G(T)(G/K) = T(\bigsqcup_{[G:K]} e/e) = \prod_{[G:K]} T$, the componentwise H-action yields

$$\mathsf{Coind}_e^G(T)(G/K)^H \cong \mathsf{Coind}_e^G(R)(G/K).$$

The co-induction functor is lax symmetric monoidal and the left $\mathsf{Coind}_e^G(R)$-module structure of $\mathsf{Coind}_e^G(T)$ is induced by the composite

$$\mathsf{Coind}_e^G(R)\Box\mathsf{Coind}_e^G(T) \to \mathsf{Coind}_e^G(R \otimes T) \to \mathsf{Coind}_e^G(T)$$

where the first map comes from the lax symmetric monoidality of Coind_e^G, noting that the tensor product is the box product of e-Tambara functors. The second map is the map induced by the R-module structure of T. The right module structure can be described similarly.

We claim that $\mathsf{Coind}_e^G(T)\Box_{\mathsf{Coind}_e^G(R)}\mathsf{Coind}_e^G(T) \cong \mathsf{Coind}_e^G(T \otimes_R T)$. The relative box product is the coequalizer of

$$\mathsf{Coind}_e^G(T)\Box\mathsf{Coind}_e^G(R)\Box\mathsf{Coind}_e^G(T) \underset{r}{\overset{\ell}{\rightrightarrows}} \mathsf{Coind}_e^G(T)\Box\mathsf{Coind}_e^G(T).$$

Again, we use Strickland's description of elements in box products, so at level G/U for some $U < G$ of $\mathsf{Coind}_e^G(T)\Box\mathsf{Coind}_e^G(T)$ we can represent elements as $T_p(t \otimes t')$ with $t, t' \in \mathsf{Coind}_e^G(T)(G/V)$ for some $V < U$ and $p\colon G/V \to G/U$. Starting at the free level we have

$$\mathsf{Coind}_e^G(T)\Box\mathsf{Coind}_e^G(T)(G/e) = (\prod_G T) \otimes (\prod_G T)$$

and we consider the element $(t, t, \ldots, t) \otimes (t', 0, \ldots, 0) \in \prod_G T \otimes \prod_G T$. As the transfer in a co-induction Tambara functor just induces addition and as restriction gives a diagonal map, we obtain with Frobenius reciprocity

$$t \otimes t' = t \otimes \mathsf{tr}(t', 0, \ldots, 0) = \mathsf{tr}(\mathsf{res}(t) \otimes (t', 0, \ldots, 0)) = \mathsf{tr}((t, t, \ldots, t) \otimes (t', 0, \ldots, 0))$$

and hence we can represent every element in every level of the box product as a transfer coming from the free level and at the free level, we just have $(\prod_G T) \otimes (\prod_G T)$.

A similar argument shows that every element at every level of the three-fold box product $\mathsf{Coind}_e^G(T)\Box\mathsf{Coind}_e^G(R)\Box\mathsf{Coind}_e^G(T)$ can be written as the transfer of an element coming from the free level and there we get

$$(\prod_G T) \otimes (\prod_G R) \otimes (\prod_G T).$$

The coequalizer at the free level is

$$(\prod_G T) \otimes_{(\prod_G R)} \otimes (\prod_G T)$$

and coequalizing the $\prod_G R$-module structure kills mixed terms, so we are left with $\prod_G (T \otimes_R T)$.

As $\chi\colon T \otimes_R T \to \prod_H T$ is an isomorphism of commutative rings, $\prod_G \chi$ is an isomorphism as well. As the free level determines the box products for co-inductions from the trivial group to G, we get

$$\mathsf{Coind}_e^G(T)\Box_{\mathsf{Coind}_e^G(R)}\mathsf{Coind}_e^G(T) \cong \mathsf{Coind}_e^G(T \otimes_R T) \cong \mathsf{Coind}_e^G(\prod_H R)$$

and as co-induction is a right adjoint, it commutes with products. $\Box$

Assume now that $\mathbb{F}$ is an algebraically closed field and that K is a field such that $K \subset \mathbb{F}$ is a (possibly non-finite) Galois extension. Then we can write $\mathbb{F}$ as a filtered colimit

$$\mathbb{F} = \bigcup_{\substack{K \subset L \text{Galois} \\ [L:K]<\infty}} L.$$

For all these *finite* extensions $K \subset L$ with Galois group $H = \mathrm{Gal}(L/K)$ we obtain a corresponding $\mathrm{Gal}(L/K)$-Galois extension of G-Tambara functors $\mathsf{Coind}_e^G(K) \to \mathsf{Coind}_e^G(L)$ for all finite groups G. Filtered colimits are built levelwise in Tambara functors and the co-induction functor Coind_e^G commutes with filtered colimits. Therefore we can express $\mathsf{Coind}_e^G\mathbb{F}$ as a filtered colimit

$$\mathsf{Coind}_e^G\mathbb{F} = \mathrm{colim}_{\substack{K \subset L \text{Galois} \\ [L:K]<\infty}} \mathsf{Coind}_e^G(L).$$

Using [SSW, Theorem 5.7] this expresses every Nullstellensatzian G-Tambara functor $\underline{k} = \mathsf{Coind}_e^G(\mathbb{F})$ as a filtered colimit of finite Galois extensions.

Remark 8.2. In the non-equivariant Galois theory of commutative rings, H-Galois extensions of the form $R \to \prod_H R$ are frowned upon: the target ring has non-trivial idempotents and this non-connectedness creates problems for instance if one wants to define what a separable closure is. But Schuchardt, Spitz, and Wisdom show that a Nullstellensatzian object in the category of commutative H-rings is precisely an H-ring of the form $\mathrm{Fun}(H, \mathbb{F}) = \prod_H \mathbb{F}$ where $\mathbb{F}$ is algebraically closed [SSW, Theorem 5.13]. Adjunction gives a canonical map $\mathbb{F} \to \prod_H \mathbb{F}$ where we forget about the H-action in the target. The map, however, is the diagonal map which yields the H-Galois extension of commutative rings $\mathbb{F} \to \prod_H \mathbb{F}$.

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