Gabriel-Zisman homology and homotopy colimits for diagrams in chain complexes

Birgit Richter

The Legacy of Peter Gabriel, Bielefeld August 2025

P . Gabriel and M. Zisman

CALCULUS OF FRACTIONS AND HOMOTOPY THEORY

ERGEBNISSE DER MATHEMATIK
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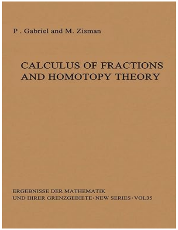

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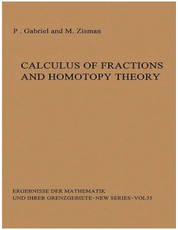

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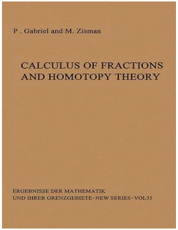

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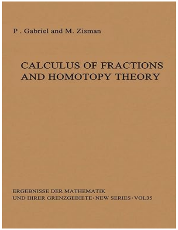

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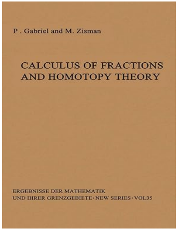

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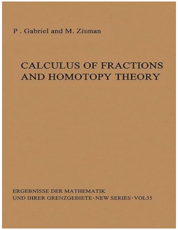

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This started a flurry of work by Quillen, Thomason and others.

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Let $N_n(\mathcal{C})$ be the set

$$\{\ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} C_n \ \mid C_i \text{ an object of } \mathcal{C}, f_i \in \mathcal{C}(C_{i-1}, C_i)\}$$

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$$d_i[f_n|\dots|f_1] = \begin{cases} [f_n|\dots|f_2], & i = 0, \\ [f_n|\dots|f_{i+1} \circ f_i|\dots|f_1], & 0 < i < n, \\ [f_{n-1}|\dots|f_1], & i = n. \end{cases}$$

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We then take the associated chain complex $C_*(\mathcal{C}; L)$ with

$$C_n(\mathcal{C}; L) := \bigoplus_{[f_n|\ldots|f_1]\in N_n(\mathcal{C})} L(C_0) \text{ and differential } \delta = \sum_{i=0}^{n} (-1)^i d_i.$$

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$$HC_*^k(A) \cong H_*(\Delta C^{op}, \mathcal{L}^k(A; A)).$$

Theorem [Gabriel-Zisman, Proposition II.3.3] For any small category $\mathcal{C}$ and any functor $L\colon \mathcal{C} \to \mathcal{A}$ where $\mathcal{A}$ is a cocomplete abelian category with exact coproducts, the homology groups of $\mathcal{C}$ with coefficients in L are the left derived functors of $\mathrm{colim}_{\mathcal{C}} L$:

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- $H_0(\mathcal{C}; L) \cong \operatorname{colim}_{\mathcal{C}} L$, Theorem [Gabriel-Zisman, Proposition II.3.3] For any small category $\mathcal{C}$ and any functor $L\colon \mathcal{C} \to \mathcal{A}$ where $\mathcal{A}$ is a cocomplete abelian category with exact coproducts, the homology groups of $\mathcal{C}$ with coefficients in L are the left derived functors of $\mathrm{colim}_{\mathcal{C}} L$:

- $H_0(\mathcal{C}; L) \cong \mathrm{colim}_{\mathcal{C}} L$,
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Let $L\colon \mathcal{C} \to \mathsf{Ch}_{\geq 0}(k)$ be a functor from a small category $\mathcal{C}$ to the category of non-negatively graded chain complexes. Consider the simplicial chain complex $\mathrm{srep}(L)$ whose simplicial degree n-part is

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This is a very explicit Bousfield-Kan type model of the homotopy colimit.

It is the total complex associated with the bicomplex

$$
\begin{array}{ccccc}
\cdots & & \cdots & & \\
\downarrow \delta & & \downarrow -\delta & & \\
\bigoplus_{[f_2|f_1]\in N_2(\mathcal{C})} L(s(f_1))_0 & \xleftarrow{\bigoplus d} & \bigoplus_{[f_2|f_1]\in N_2(\mathcal{C})} L(s(f_1))_1 & \xleftarrow{\bigoplus d} & \cdots \\
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where $s(g)$ denotes the source of a morphism g.

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For $F, G \in \mathrm{Ch}(k)^{\mathcal{C}}$:

$$\begin{array}{ccccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{F \times G} & \mathrm{Ch}(k) \times \mathrm{Ch}(k) & \xrightarrow{\otimes} & \mathrm{Ch}(k) \\ \downarrow{\scriptstyle \sqcup} & & & \nearrow{\scriptstyle F \boxtimes G} & \\ \mathcal{C} & & & & \end{array}$$

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These homotopy colimits give strictly commutative models for the cochains on topological spaces.

Theorem [R-Sagave '20]

1. For every topological space X and every commutative ring k, there is a commutative monoid $A^{\mathcal{I}}(X;k)$ in $\mathrm{Ch}(k)^{\mathcal{I}}$ such that $C^*(X;k)$ and $\mathrm{hocolim}_{\mathcal{I}} A^{\mathcal{I}}(X;k)$ are naturally quasi-isomorphic as dg E_∞-algebras.

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3. If X, Y are nilpotent topological spaces of finite type, then X is weakly equivalent to Y if and only if $A^{\mathcal{I}}(X;\mathbb{Z})$ is weakly equivalent to $A^{\mathcal{I}}(Y;\mathbb{Z})$ as commutative monoids in $\mathrm{Ch}(\mathbb{Z})$.

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A morphism from x to y therefore corresponds to

$$\begin{array}{ccc} \Delta(-,[n]) & \xrightarrow{\theta} & \Delta(-,[m]) \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle y} & \\ & X & \end{array}$$

For a simplicial set X and an arbitrary functor $L\colon (\Delta/X)^{op} \to \mathcal{A}$ define the Gabriel-Zisman chain complex $C_*^{GZ}(X;L)$ as

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Imma Gálvez-Carillo, Frank Neumann and Andrew Tonks (2021) study $H_*^{GZ}(N\mathcal{C}; L)$ as a homology theory for $\mathcal{C}$.

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  Here, $L_q((\Delta/f)^{op})_*$ is the p-th left satellite of the left Kan extension along $(\Delta/f)^{op}\colon (\Delta/X)^{op} \to (\Delta/Y)^{op}$.