

An interpretation of E_n -homology as functor homology

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Abstract We prove that E_n -homology of non-unital commutative algebras can be described as functor homology when one considers functors from a certain category of planar trees with n levels. For different n these homology theories are connected by natural maps, ranging from Hochschild homology and its higher order versions to Gamma homology.

Keywords Functor homology · Iterated bar construction · E_n -homology · Hochschild homology · Operads

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1 Introduction

By neglect of structure, any commutative and associative algebra can be considered as an associative algebra. More generally, we can view such an algebra as an E_n -algebra, i.e., an algebra over an operad in chain complexes that is weakly equivalent to the chain complex of the little- n -cubes operad of [4] for $1 \leq n \leq \infty$. Hochschild homology is a classical homology theory for associative algebras and hence it can be applied to commutative algebras as well.

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Less classically, Gamma homology [15] is a homology theory for E_∞ -algebras and Gamma homology of commutative algebras plays an important role in the obstruction theory for E_∞ structures on ring spectra [1, 7, 14] and its structural properties are rather well understood [13].

It is desirable to have a good understanding of the appropriate homology theories in the intermediate range, i.e., for $1 < n < \infty$. A definition of E_n -homology for augmented commutative algebras is due to Fresse [6] and the main topic of this paper is to prove that these homology theories possess an interpretation in terms of functor homology. We extend the range of E_n -homology to functors from a suitable category Epi_n to modules in such a way that it coincides with Fresse’s theory when we consider a functor that belongs to an augmented commutative algebra and show in Theorem 4.1 that E_n -homology can be described as functor homology, so that the homology groups are certain Tor-groups.

As a warm-up we show in Sect. 2 that bar homology of a non-unital algebra can be expressed in terms of functor homology for functors from the category of order-preserving surjections to k -modules. In Sect. 3 we introduce our categories of epimorphisms, Epi_n , and their relationship to planar trees with n -levels. We introduce a definition of E_n -homology for functors from Epi_n to k -modules that coincides with Fresse’s definition of E_n -homology of a non-unital commutative algebra, \bar{A} , when we apply our version of E_n -homology to a suitable functor, $\mathcal{L}(\bar{A})$. We describe a spectral sequence that has tensor products of bar homology groups as input and converges to E_2 -homology. Section 4 is the technical heart of the paper. Here we prove that E_n -homology has a Tor interpretation. The proof of the acyclicity of a family of suitable projective generators is an inductive argument that uses homology of small categories.

For varying n , the E_n -homology theories are related to each other via a sequence of maps

$$H_*^{E_1} \rightarrow H_*^{E_2} \rightarrow H_*^{E_3} \rightarrow \dots$$

In a different context it is well known that the stabilization map from Hochschild homology to Gamma homology can be factored over so called higher order Hochschild homology [9]: for a commutative algebra A there is a sequence of maps connecting Hochschild homology of A , $HH_*(A)$, to Hochschild homology of order n of A and finally to Gamma homology of A , $H\Gamma_{*-1}(A)$. We explain how higher order Hochschild homology is related to E_n -homology for n ranging from 1 to ∞ in Sect. 3.1.

In the following we fix a commutative ring with unit, k . For a set S we denote by $k[S]$ the free k -module generated by S . If $S = \{s\}$, then we write $k[s]$ instead of $k[\{s\}]$.

2 Tor interpretation of bar homology

We interpret the bar homology of a functor from the category of finite sets and order-preserving surjections to the category of k -modules as a Tor-functor.

For unital k -algebras, the complex for the Hochschild homology of the algebra can be viewed as the chain complex associated to a simplicial object. In the absense of units, this is no longer possible.

Let \bar{A} be a non-unital k -algebra. The bar-homology of \bar{A} , $H_*^{\text{bar}}(\bar{A})$, is defined as the homology of the complex

$$C_*^{\text{bar}}(\bar{A}) : \dots \rightarrow \bar{A}^{\otimes n+1} \xrightarrow{b'} \bar{A}^{\otimes n} \xrightarrow{b'} \dots \xrightarrow{b'} \bar{A} \otimes \bar{A} \xrightarrow{b'} \bar{A}$$

with $C_n^{\text{bar}}(\bar{A}) = \bar{A}^{\otimes n+1}$ and $b' = \sum_{i=0}^{n-1} (-1)^i d_i$ where d_i applied to $a_0 \otimes \dots \otimes a_n \in \bar{A}^{\otimes n+1}$ is $a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$.

The category of non-unital associative k -algebras is equivalent to the category of augmented k -algebras. If one replaces \bar{A} by $A = \bar{A} \oplus k$, then $C_n^{\text{bar}}(\bar{A})$ corresponds to the reduced Hochschild complex of A with coefficients in the trivial module k , shifted by one: $H_*^{\text{bar}}(\bar{A}) = HH_{*+1}(A, k)$, for $* \geq 0$.

Definition 2.1 Let Δ^{epi} be the category whose objects are the sets $[n] = \{0, \dots, n\}$ for $n \geq 0$ with the ordering $0 < 1 < \dots < n$ and whose morphisms are order-preserving surjective functions. We will call covariant functors $F: \Delta^{\text{epi}} \rightarrow k\text{-mod}$ Δ^{epi} -modules.

We have the basic order-preserving surjections $d_i: [n] \rightarrow [n - 1]$, $0 \leq i \leq n - 1$ that are given by

$$d_i(j) = \begin{cases} j, & j \leq i, \\ j - 1, & j > i. \end{cases}$$

Any order-preserving surjection is a composition of these basic ones. The generating morphisms d_i in Δ^{epi} correspond to the face maps in the standard simplicial model of the 1-sphere with the exception of the last face map. The standard simplicial identities imply that $b' = \sum_{i=0}^{n-1} (-1)^i F(d_i)$ satisfies $b'^2 = 0$. This justifies the following definition.

Definition 2.2 We define the *bar-homology of a Δ^{epi} -module F* as the homology of the complex $C_*^{\text{bar}}(F)$ with $C_n^{\text{bar}}(F) = F[n]$ and differential $b' = \sum_{i=0}^{n-1} (-1)^i F(d_i)$.

For a non-unital algebra \bar{A} the functor $\mathcal{L}(\bar{A})$ that assigns $\bar{A}^{\otimes(n+1)}$ to $[n]$ and $\mathcal{L}(d_i)(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$ ($0 \leq i \leq n - 1$) is a Δ^{epi} -module. In that case, $C_*^{\text{bar}}(\mathcal{L}(\bar{A})) = C_*^{\text{bar}}(\bar{A})$.

In the following we use the machinery of functor homology as in [11]. Note that the category of Δ^{epi} -modules has enough projectives: the representable functors $(\Delta^{\text{epi}})^n: \Delta^{\text{epi}} \rightarrow k\text{-mod}$ with $(\Delta^{\text{epi}})^n[m] = k[\Delta^{\text{epi}}([n], [m])]$ are easily seen to be projective objects and each Δ^{epi} -module receives a surjection from a sum of representables. The analogous statement is true for contravariant functors from Δ^{epi} to the category of k -modules where we can use the functors Δ_n^{epi} with $\Delta_n^{\text{epi}}[m] = k[\Delta^{\text{epi}}([m], [n])]$ as projective objects.

We call the cokernel of the map between contravariant representables

$$(d_0)_*: \Delta_1^{\text{epi}} \rightarrow \Delta_0^{\text{epi}}$$

b^{epi} . Note that $\Delta_0^{\text{epi}}[n]$ is free of rank one for all $n \geq 0$ because there is just one map in Δ^{epi} from $[n]$ to $[0]$ for all n . Furthermore, $\Delta_1^{\text{epi}}[0]$ is the zero module, because $[0]$ cannot surject onto $[1]$. Therefore

$$b^{\text{epi}}[n] \cong \begin{cases} 0 & \text{for } n > 0, \\ k & \text{for } n = 0. \end{cases}$$

Proposition 2.3 For any Δ^{epi} -module F

$$H_p^{\text{bar}}(F) \cong \text{Tor}_p^{\Delta^{\text{epi}}}(b^{\text{epi}}, F) \quad \text{for all } p \geq 0. \tag{2.1}$$

For the proof recall that a sequence of Δ^{epi} -modules and natural transformations

$$0 \rightarrow F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \rightarrow 0 \tag{2.2}$$

is *short exact* if it gives rise to a short exact sequence of k -modules

$$0 \rightarrow F'[n] \xrightarrow{\phi[n]} F[n] \xrightarrow{\psi[n]} F''[n] \rightarrow 0$$

for every $n \geq 0$.

Proof We have to show that $H_*^{\text{bar}}(-)$ maps short exact sequences of Δ^{epi} -modules to long exact sequences, that $H_*^{\text{bar}}(-)$ vanishes on projectives in positive degrees and that $H_0^{\text{bar}}(F)$ and $b^{\text{epi}} \otimes_{\Delta^{\text{epi}}} F$ agree for all Δ^{epi} -modules F .

A short exact sequence as in (2.2) is sent to a short exact sequence of chain complexes

$$0 \longrightarrow C_*^{\text{bar}}(F') \xrightarrow{C_*^{\text{bar}}(\phi)} C_*^{\text{bar}}(F) \xrightarrow{C_*^{\text{bar}}(\psi)} C_*^{\text{bar}}(F'') \longrightarrow 0$$

and therefore the first claim is true.

In order to show that $H_*^{\text{bar}}(P)$ is trivial in positive degrees for any projective Δ^{epi} -module P it suffices to show that the representables $(\Delta^{\text{epi}})^n$ are acyclic. In order to prove this claim we construct an explicit chain homotopy.

Let $f \in (\Delta^{\text{epi}})^n[m]$ be a generator, i.e., a surjective order-preserving map from $[n]$ to $[m]$. Note that $f(0) = 0$. We can codify such a map by its fibres, i.e., by an $(m + 1)$ -tuple of pairwise disjoint subsets (A_0, \dots, A_m) with $A_i \subset [n]$, $0 \in A_0$ and $\bigcup_{i=0}^m A_i = [n]$ such that $x < y$ for $x \in A_i$ and $y \in A_j$ with $i < j$. With this notation $d_i(A_0, \dots, A_m) = (A_0, \dots, A_{i-1}, A_i \cup A_{i+1}, \dots, A_m)$.

We define the chain homotopy $h: \Delta^{\text{epi}}([n], [m]) \rightarrow \Delta^{\text{epi}}([n], [m + 1])$ as

$$h(A_0, \dots, A_m) := \begin{cases} 0 & \text{if } A_0 = \{0\}, \\ (0, A'_0, A_1, \dots, A_m) & \text{if } A_0 = \{0\} \cup A'_0, A'_0 \neq \emptyset. \end{cases} \tag{2.3}$$

If $A_0 = \{0\}$, then

$$\begin{aligned} (b' \circ h + h \circ b')(\{0\}, \dots, A_m) &= 0 + h \circ b'(\{0\}, \dots, A_m) \\ &= h(\{0\} \cup A_1, \dots, A_m) \\ &= (\{0\}, \dots, A_m). \end{aligned}$$

In the other case a direct calculation shows that $(b' \circ h + h \circ b')(\{0\}, \dots, A_m) = \text{id}(A_0, \dots, A_m)$.

It remains to show that both homology theories coincide in degree zero. By definition $H_0^{\text{bar}}(F)$ is the cokernel of the map

$$F(d_0): F[1] \longrightarrow F[0].$$

A Yoneda-argument [16, 17.7.2(a)] shows that the tensor product $\Delta_n^{\text{epi}} \otimes_{\Delta^{\text{epi}}} F$ is naturally isomorphic to $F[n]$ and hence the above cokernel is the cokernel of the map

$$((d_0)_* \otimes_{\Delta^{\text{epi}}} \text{id}): \Delta_1^{\text{epi}} \otimes_{\Delta^{\text{epi}}} F \longrightarrow \Delta_0^{\text{epi}} \otimes_{\Delta^{\text{epi}}} F.$$

As tensor products are right-exact [16, 17.7.2 (d)], the cokernel of the above map is isomorphic to

$$\text{coker}((d_0)_*: \Delta_1^{\text{epi}} \rightarrow \Delta_0^{\text{epi}}) \otimes_{\Delta^{\text{epi}}} F = b^{\text{epi}} \otimes_{\Delta^{\text{epi}}} F = \text{Tor}_0^{\Delta^{\text{epi}}}(b^{\text{epi}}, F).$$

□

Remark 2.4 Note that the previous proof applies in a graded context. More precisely, assume every element $i \in [n]$ comes with a grading $d(i) \in \mathbb{N}_0$. In the sequel we will consider surjective maps $\phi: X \rightarrow [n]$ where X is a graded set and $d(i) = \sum_{x|\phi(x)=i} d(x)$.

The degree of a subset A of $[n]$ is $d(A) = \sum_{i \in A} d(i)$. To this grading one can associate the complex $(C_m^{X,\phi}((\Delta^{\text{epi}})^n) = \bigoplus_{f \in \Delta^{\text{epi}}([n],[m])} k[f], b')$, with $d_i(A_0, \dots, A_m) = (-1)^{\sum_{j=0}^i d(A_j)} (A_0, \dots, A_i \cup A_{i+1}, \dots, A_m)$ and $b' = \sum_{i=0}^{m-1} (-1)^i d_i$. As the d_i 's still satisfy the simplicial identities, b' is of square 0. The complexes $C_*^{X,\phi}((\Delta^{\text{epi}})^n)$ are acyclic, as one can see using the homotopy

$$h(A_0, \dots, A_m) := \begin{cases} 0 & \text{if } A_0 = \{0\}, \\ (-1)^{d(0)} (0, A'_0, A_1, \dots, A_m) & \text{if } A_0 = \{0\} \cup A'_0, A'_0 \neq \emptyset. \end{cases}$$

3 Epimorphisms and trees

Planar level trees are used in [2,6] and [3, 3.15] as a means to codify E_n -structures. An n -level tree is a planar level tree with n levels. We will use categories of planar level trees in order to gain a description of E_n -homology as functor homology. If \mathcal{C} is a small category, then we denote the nerve of \mathcal{C} by $N\mathcal{C}$.

Definition 3.1 Let $n \geq 1$ be a natural number. The category Epi_n has as objects the elements of $N_{n-1}(\Delta^{\text{epi}})$, i.e., sequences

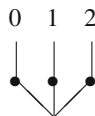
$$[r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1] \tag{3.1}$$

with $[r_i] \in \Delta^{\text{epi}}$ and surjective order-preserving maps f_i . A morphism in Epi_n from the above object to an object $[r'_n] \xrightarrow{f'_n} [r'_{n-1}] \xrightarrow{f'_{n-1}} \dots \xrightarrow{f'_2} [r'_1]$ consists of surjective maps $\sigma_i : [r_i] \rightarrow [r'_i]$ for $1 \leq i \leq n$ such that $\sigma_1 \in \Delta^{\text{epi}}$ and for all $2 \leq i \leq n$ the map σ_i is order-preserving on the fibres $f_i^{-1}(j)$ for all $j \in [r_{i-1}]$ and such that the diagram

$$\begin{array}{ccccccc} [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \dots & \xrightarrow{f_2} & [r_1] \\ \downarrow \sigma_n & & \downarrow \sigma_{n-1} & & & & \downarrow \sigma_1 \\ [r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \dots & \xrightarrow{f'_2} & [r'_1] \end{array}$$

commutes.

As an example, consider the object $[2] \xrightarrow{\text{id}} [2]$ in Epi_2 which can be viewed as the 2-level tree



Possible maps from this object to $[2] \xrightarrow{d_0} [1]$ are

$$\begin{array}{ccc} \begin{array}{ccc} [2] & \xrightarrow{\text{id}} & [2] \\ \text{id} \downarrow & & \downarrow d_0 \\ [2] & \xrightarrow{d_0} & [1] \end{array} & \text{and} & \begin{array}{ccc} [2] & \xrightarrow{\text{id}} & [2] \\ (0,1) \downarrow & & \downarrow d_0 \\ [2] & \xrightarrow{d_0} & [1] \end{array} \end{array}$$

where $(0, 1)$ denotes the transposition that permutes 0 and 1. For $\sigma_1 = d_1$ there is no possible σ_2 to fill in the diagram.

If $n = 1$, then Epi_1 coincides with the category Δ^{epi} . Note that there is a functor $\iota_n : \Delta^{\text{epi}} = \text{Epi}_1 \rightarrow \text{Epi}_n$ for all $n \geq 1$ with

$$\iota_n([m]) := [m] \longrightarrow [0] \longrightarrow \dots \longrightarrow [0].$$

We call trees of the form $\iota_n([m])$ *palm trees with $m + 1$ leaves*. More generally we have functors connecting the various categories of planar level trees.

Lemma 3.2 *For all $n > k \geq 1$ there are functors $\iota_n^k : \text{Epi}_k \rightarrow \text{Epi}_n$, with*

$$\iota_n^k \left([r_k] \xrightarrow{f_k} \dots \xrightarrow{f_2} [r_1] \right) = [r_k] \xrightarrow{f_k} \dots \xrightarrow{f_2} [r_1] \longrightarrow [0] \longrightarrow \dots \longrightarrow [0]$$

on objects, with the canonical extension to morphisms.

Remark 3.3 The maps ι_n^k correspond to iterated suspension morphisms in [2, 4.1]. There is a different way of mapping a planar tree with n levels to one with $n + 1$ levels, by sending

$$[r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \text{ to } [r_n] \xrightarrow{\text{id}_{[r_n]}} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1].$$

Definition 3.4 We call trees of the form $[r_n] \xrightarrow{\text{id}_{[r_n]}} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$ *fork trees*.

Fork trees will need special attention later when we prove that representable functors are acyclic.

For any Σ_* -cofibrant operad \mathcal{P} there exists a homology theory for \mathcal{P} -algebras which is denoted by $H_*^{\mathcal{P}}$ and is called \mathcal{P} -homology. Fresse studies the particular case of $\mathcal{P} = E_n$ a differential graded operad quasi-isomorphic to the chain operad of the little n -disks operad. He proves that for any commutative algebra the E_n -homology coincides with the homology of its n -fold bar construction. In fact, his result is more general since he defines an analogous n -fold bar construction for E_n -algebras and proves the result for any E_n -algebra in [6, theorem 7.26].

We consider the n -fold bar construction of a non-unital commutative k -algebra \bar{A} , $B^n(\bar{A})$, as an n -complex, such that

$$B^n(\bar{A})_{(r_n, \dots, r_1)} = \bigoplus_{[r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \in \text{Epi}_n} \bar{A}^{\otimes(r_n+1)}.$$

The differential in $B^n(\bar{A})$ is the total differential associated to n -differentials $\partial_1, \dots, \partial_n$ such that ∂_n is built out of the multiplication in \bar{A} , ∂_{n-1} corresponds to the shuffle multiplication on $B(\bar{A})$ and so on. We describe the precise setting in a slightly more general context.

In order to extend the Tor-interpretation of bar homology of Δ^{epi} -modules to functors from Epi_n to modules (alias Epi_n -modules) we describe the n kinds of face maps for Epi_n in detail by considering diagrams of the form

$$\begin{array}{cccccccccccccccc}
 [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \dots & \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{f_{j+1}} & [r_j] & \xrightarrow{f_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} & \dots & \xrightarrow{f_2} & [r_1] \\
 \tau_n^{i,j} \downarrow & & \tau_{n-1}^{i,j} \downarrow & & & \tau_{j+1}^{i,j} \downarrow & & d_i \downarrow & & \text{id} \downarrow & & & & & \text{id} \downarrow & \\
 [r_n] & \xrightarrow{g_n} & [r_{n-1}] & \xrightarrow{g_{n-1}} & \dots & \xrightarrow{g_{j+2}} & [r_{j+1}] & \xrightarrow{g_{j+1}} & [r_j - 1] & \xrightarrow{g_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} & \dots & \xrightarrow{f_2} & [r_1].
 \end{array} \tag{3.2}$$

Given the object in the first row, it is not always possible to extend $(d_i : [r_j] \rightarrow [r_j - 1], \text{id}_{[r_{j-1}]}, \dots, \text{id}_{[r_1]})$ to a morphism in Epi_n : we have to find order-preserving surjective maps g_k for $j \leq k \leq n$ and bijections $\tau_k^{i,j} : [r_k] \rightarrow [r_k]$ that are order-preserving on the fibres of f_k for $j + 1 \leq k \leq n$ such that the diagram commutes.

By convention we denote the constant map $[r_1] \rightarrow [0]$ by f_1 .

- Lemma 3.5** (a) *There is a unique order-preserving surjection $g_j : [r_j - 1] \rightarrow [r_{j-1}]$ with $g_j \circ d_i = f_j$ if and only if $f_j(i) = f_j(i + 1)$. When it exists, g_j is denoted by $f_j|_{i=i+1}$.*
 (b) *If $f_j(i) = f_j(i + 1)$ then we can extend the diagram to one of the form (3.2) so that $\tau_{j+1}^{i,j}$ is a shuffle of the fibres $f_{j+1}^{-1}(i)$ and $f_{j+1}^{-1}(i + 1)$. Each choice of a $\tau_{j+1}^{i,j}$ uniquely determines the maps $\tau_k^{i,j}$ for all $j + 1 < k \leq n$.*
 (c) *If $f_j(i) = f_j(i + 1)$ then each choice of a $\tau_{j+1}^{i,j}$ uniquely determines the maps g_k for $k \geq j$. The diagram (3.2) takes the following form*

$$\begin{array}{cccccccccccccccc}
 [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_{j+2}} & [r_{j+1}] & \xrightarrow{f_{j+1}} & [r_j] & \xrightarrow{f_j} & [r_{j-1}] & \xrightarrow{f_{j-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\
 \tau_n^{i,j} \downarrow & & \tau_{n-1}^{i,j} \downarrow & & & & \tau_{j+1}^{i,j} \downarrow & & d_i \downarrow & & \text{id} \downarrow & & & & \text{id} \downarrow \\
 [r_n] & \xrightarrow{g_n^r} & [r_{n-1}] & \xrightarrow{g_{n-1}^r} & \cdots & \xrightarrow{g_{j+2}^r} & [r_{j+1}] & \xrightarrow{d_i f_{j+1}} & [r_j - 1] & \xrightarrow{f_j|_{i=i+1}} & [r_{j-1}] & \xrightarrow{f_{j-1}} & \cdots & \xrightarrow{f_2} & [r_1].
 \end{array}
 \tag{3.3}$$

Proof If there is such a map g_j , then $f_j(i + 1) = g_j \circ d_i(i + 1) = g_j \circ d_i(i) = f_j(i)$. As f_j is order-preserving, it is determined by the cardinalities of its fibres. The decomposition of morphisms in the simplicial category then ensures that we can factor f_j in the desired way.

For the third claim, assume that g_j exists with the properties mentioned in (a). As g_{j+1} and $d_i \circ f_{j+1}$ are both order-preserving maps from $[r_{j+1}]$ to $[r_j - 1]$, they are determined by the cardinalities of the fibres and thus they have to agree. Then $\tau_{j+1}^{i,j} = \text{id}_{[r_{j+1}]}$ extends the diagram up to layer $j + 1$. For the higher layers we then have to choose $g_k = f_k$ and $\tau_k^{i,j} = \text{id}_{[r_k]}$.

In general, $\tau_{j+1}^{i,j}$ has to satisfy the conditions that it is order-preserving on the fibres of f_{j+1} . If $A_i = f_{j+1}^{-1}(i)$ then this implies that $\tau_{j+1}^{i,j}$ is an (A_0, \dots, A_r) -shuffle. Furthermore we have that

$$(d_i \circ f_{j+1})^{-1}(k) = \begin{cases} A_k & \text{if } k < i, \\ A_i \cup A_{i+1} & \text{if } k = i, \\ A_{k+1} & \text{if } k > i. \end{cases}$$

Therefore $\tau_{j+1}^{i,j}$ has to map $A_0, \dots, A_{i-1}, A_{i+2}, \dots, A_r$ identically and is hence an (A_i, A_{i+1}) -shuffle.

If we fix a shuffle $\tau_{j+1}^{i,j}$, then the next permutation $\tau_{j+2}^{i,j}$ has to be order-preserving on the fibres of f_{j+2} , thus it is at most a shuffle of the fibres. In addition, it has to satisfy

$$g_{j+2} \circ \tau_{j+2}^{i,j} = \tau_{j+1}^{i,j} \circ f_{j+2}.
 \tag{3.4}$$

Again, as g_{j+2} is order-preserving we have no choice but to take the order-preserving map satisfying $|g_{j+2}^{-1}(k)| = |(\tau_{j+1}^{i,j} \circ f_{j+2})^{-1}(k)|$, for all $k \in [r_{j+1}]$. By (3.4) we know that $\tau_{j+2}^{i,j}$ has to send $f_{j+2}^{-1}(k)$ to $g_{j+2}^{-1}(\tau_{j+1}^{i,j}(k))$ and this determines $\tau_{j+2}^{i,j}$. A proof by induction shows the general claim in (b). \square

In the following we will extend the notion of E_n -homology for commutative non-unital k -algebras to Epi_n -modules. Again thanks to Fresse’s theorem [6, theorem 7.26], the

E_n -homology and the homology of the n -fold bar construction of a commutative algebra coincide.

Definitions 3.6 and 3.7 describe a complex while Lemma 3.8 asserts the complex property.

Definition and Notation 3.6 Let $t : [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]$ be an n -level tree.

The degree of t , denoted by $d(t)$, is the number of its edges, that is $\sum_{i=1}^n (r_i + 1)$.

For fixed $1 \leq j \leq n$ and $i \in [r_j]$ let $t_{j,i}$ be the $(n - j)$ -level tree defined by the j -fibre of t over $i \in [r_j]$:

$$(f_{j+1} f_{j+2} \dots f_n)^{-1}(i) \xrightarrow{f_n} \dots \xrightarrow{f_{k+1}} (f_{j+1} \dots f_k)^{-1}(i) \xrightarrow{f_k} \dots \xrightarrow{f_{j+2}} f_{j+1}^{-1}(i).$$

Conversely a tree t can be recovered from its 1-fibres, that is $t = [t_{1,0}, \dots, t_{1,r_1}]$ and $d(t) = \sum_{i=0}^{r_1} (d(t_{1,i}) + 1) = r_1 + 1 + \sum_{i=0}^{r_1} d(t_{1,i})$.

Let F be an Epi_n -module. For fixed $1 \leq j \leq n$ and $i \in [r_j]$ such that $f_j(i) = f_j(i + 1)$ or $j = 1$ we define

$$d_i^j : F([r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]) \longrightarrow \bigoplus_{t' = [r_n] \xrightarrow{g_n} \dots \xrightarrow{g_{j+2}} [r_{j+1}] \xrightarrow{d_i f_{j+1}} [r_j - 1] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1] \in \text{Epi}_n} F(t')$$

as

$$\begin{cases} d_i^j = \sum_{\tau_{j+1}^{i,j} \in \text{Sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \varepsilon(\tau_{j+1}^{i,j}; t_{j,i}, t_{j,i+1}) F(\tau_n^{i,j}, \dots, \tau_{j+1}^{i,j}, d_i, \text{id}, \dots, \text{id}) & \text{if } j < n, \\ d_i^n = F(d_i, \text{id}, \dots, \text{id}) & \text{if } j = n, \end{cases} \tag{3.5}$$

where the sign $\varepsilon(\tau_{j+1}^{i,j}; t_{j,i}; t_{j,i+1})$ is defined as follows: one expresses the $(n - j)$ -level trees $t_{j,i}$ and $t_{j,i+1}$ as a sequence of $(n - j - 1)$ -level trees $t_{j,i} = [t_1, \dots, t_p]$ and $t_{j,i+1} = [t_{p+1}, \dots, t_{p+q}]$; the shuffle $\sigma = \tau_{j+1}^{i,j}$ is indeed a (p, q) -shuffle and acts on t by replacing the fibres $t_{j,i}$ and $t_{j,i+1}$ by the fibre $u_{j,i} = [t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(p+q)}]$. The sign $\varepsilon(\sigma; [t_1, \dots, t_p], [t_{p+1}, \dots, t_{p+q}])$ picks up a factor of $(-1)^{(d(t_a)+1)(d(t_b)+1)}$ whenever $\sigma(a) > \sigma(b)$ but $a < b$.

In Example 3.11 we treat an example with $n = 3$.

Definition 3.7

- If F is an Epi_n -module, then the E_n -chain complex of F is the n -fold chain complex whose (r_n, \dots, r_1) spot is

$$C_{(r_n, \dots, r_1)}^{E_n}(F) = \bigoplus_{[r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \in \text{Epi}_n} F \left([r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \right). \tag{3.6}$$

The differential in the j th coordinate is

$$\partial_j : C_{(r_n, \dots, r_j, \dots, r_1)}^{E_n}(F) \rightarrow C_{(r_n, \dots, r_{j-1}, \dots, r_1)}^{E_n}(F)$$

with

$$\partial_j := \sum_{i | f_j(i) = f_j(i+1)} (-1)^{s_{j,i}} d_i^j,$$

where $s_{j,i}$ is obtained as follows: drawing the tree t on a plane with its root at the bottom, one can label its edges—from 1 to $d(t)$ —from bottom to top and left to right; the integer $s_{j,i}$ is the label of the right most top edge of the tree $t_{j,i}$. For $j = n$ we use the convention that $s_{n,i}$ is the label of the i th leaf of t for $0 \leq i \leq r_n$.

- The E_n -homology of F , $H_*^{E_n}(F)$ is defined to be the homology of the total complex associated to (3.6).

Note that for $n = 1$ we recover Definition 2.2, but with $\partial_1 = -b'$.

Lemma 3.8 *The k -modules $C_{(r_n, \dots, r_1)}^{E_n}(F)$ constitute an n -fold chain complex.*

We postpone the proof of the lemma until after Example 3.11.

In order to prove the main Theorem 4.1, we need categories of n -trees depending on a fixed finite ordered set X of graded elements denoted Epi_n^X . For any $x \in X$, $d(x) \in \mathbb{N}_0$ will denote its degree. For any subset A of X the degree $d(A)$ is the sum of the degrees of the elements of A , thus for instance $d(X) = \sum_{x \in X} d(x)$. If A, B is a pair of disjoint subsets of X , then one defines $\epsilon(A; B) = \prod_{a \in A; b \in B; a > b} (-1)^{d(a)d(b)}$. One has

$$\epsilon(A; B)\epsilon(B; A) = (-1)^{d(A)d(B)}. \tag{3.7}$$

An object in the category Epi_n^X is an n -level tree t together with a surjection $\phi : X \rightarrow [r_n]$. Any such element is denoted by (t, ϕ) and is called an (X, n) -level tree. A morphism from (t, ϕ) to (t', ϕ') is a morphism $\sigma : t \rightarrow t'$ in the category Epi_n satisfying $\phi' = \sigma_n \phi$. The following should be considered as a graded version of Definitions 3.6 and 3.7, the consistency of the definition is stated in Lemma 3.10.

Definition 3.9 Let $(t, \phi) : X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]$ be an (X, n) -level tree in Epi_n^X .

The degree of (t, ϕ) , denoted by $d(t)$, is the sum of the number of its edges and the degrees of elements of X ,

$$d(t) = \sum_{i=1}^n (r_i + 1) + d(X).$$

For a fixed $1 \leq j \leq n - 1$ and $i \in [r_j]$ let $t_{j,i}$ be the $(X_{j,i}, n - j)$ -level tree defined by the j -fibre of t over $i \in [r_j]$:

$$X_{j,i} = (f_{j+1}f_{j+2} \dots f_n \phi)^{-1}(i) \xrightarrow{\phi} (f_{j+1}f_{j+2} \dots f_n)^{-1}(i) \xrightarrow{f_n} \dots \xrightarrow{f_{k+1}} (f_{j+1} \dots f_k)^{-1}(i) \xrightarrow{f_k} \dots \xrightarrow{f_{j+2}} f_{j+1}^{-1}(i).$$

For $j = n$, $t_{n,i}$ coincides with $X_{n,i} = \phi^{-1}(i)$.

Conversely an (X, n) -level tree (t, ϕ) can be recovered from its 1-fibres, that is $t = [t_{1,0}, \dots, t_{1,r_1}]$ and $d(t) = \sum_{i=0}^{r_1} (d(t_{1,i}) + 1) = r_1 + 1 + \sum_{i=0}^{r_1} d(t_{1,i})$.

Let F be an Epi_n^X -module. For fixed $1 \leq j \leq n$ and $i \in [r_j]$ such that $f_j(i) = f_j(i + 1)$ or $j = 1$ we define the map d_i^j

$$F(t, \phi) \xrightarrow{d_i^j} \bigoplus_{(t', \phi') = (X \xrightarrow{\phi'} [r_n] \xrightarrow{g_n} \dots \xrightarrow{g_{j+2}} [r_{j+1}] \xrightarrow{d_i f_{j+1}} [r_{j-1}] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]) \in \text{Epi}_n^X} F(t', \phi')$$

as

$$\begin{cases} d_i^j = \sum_{\tau_{j+1}^{i,j} \in \text{Sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\tau_{j+1}^{i,j}; t_{j,i}, t_{j,i+1}) F(\tau_n^{i,j}, \dots, \tau_{j+1}^{i,j}, d_i, \text{id}, \dots, \text{id}) & \text{if } j < n, \\ d_i^n = \epsilon(t_{n,i}; t_{n,i+1}) F(d_i, \text{id}, \dots, \text{id}) & \text{if } j = n. \end{cases} \tag{3.8}$$

- If F is an Epi_n^X -module, then the (E_n, X) -chain complex of F is the n -fold chain complex whose (r_n, \dots, r_1) spot is

$$C_{(r_n, \dots, r_1)}^{E_n, X}(F) = \bigoplus_{X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \in \text{Epi}_n^X} F \left(X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \right). \tag{3.9}$$

The differential in the j th coordinate is

$$\partial_j : C_{(r_n, \dots, r_j, \dots, r_1)}^{E_n, X}(F) \rightarrow C_{(r_n, \dots, r_{j-1}, \dots, r_1)}^{E_n, X}(F)$$

with

$$\partial_j := \sum_{i | f_j(i) = f_j(i+1)} (-1)^{s_{j,i}} d_i^j,$$

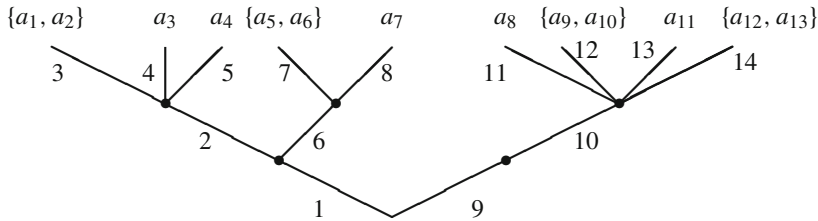
where the $s_{j,i}$ are obtained as follows: drawing the tree t on a plane with its root at the bottom, one can label its edges from bottom to top and left to right; the integer $s_{j,i}$ is the sum of the label of the right most top edge of the tree $t_{j,i}$ and the degrees of the elements in X which are in the fibre of the leaves that are to the left of the top edge so defined, including it.

- The (E_n, X) -homology of the Epi_n^X -module F , $H_*^{E_n, X}(F)$ is defined to be the homology of the total complex associated to (3.9).

Lemma 3.10 *The k -modules $C_{(r_n, \dots, r_1)}^{E_n, X}(F)$ constitute an n -fold chain complex.*

The proof of the lemma follows after Example 3.11.

Example 3.11 Let t be the following tree of degree 14 with its edges labelled, $X = \{a_1, \dots, a_{13}\}$ and ϕ can be read off the picture.



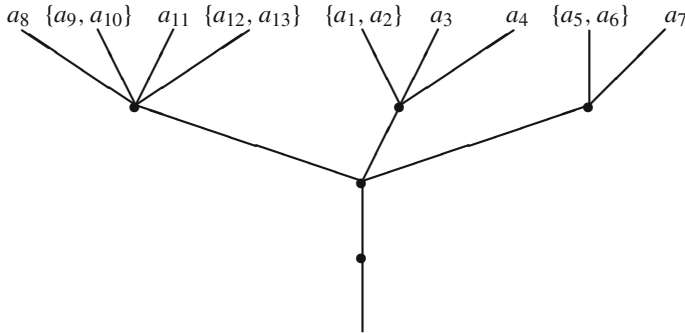
This tree represents the object $X \xrightarrow{\phi} [8] \xrightarrow{f_3} [2] \xrightarrow{f_2} [1] \in \text{Epi}_3^X$, where the map f_3 maps 0,1,2 to 0, it sends 3,4 to 1 and 5,6,7,8 to 2 and f_2 is d_0 .

With our notation the tree $t_{1,0}$ is the 2-level tree whose root is the vertex above the edge labelled by 1, the tree $t_{1,1}$ is the subtree above the edge with label 9, the tree $t_{2,0}$ is the 1-level tree above the label 2, $t_{2,1}$ the one above the label 6 and $t_{2,2}$ the one above the label 10.

We have to determine the differentials ∂_1, ∂_2 and ∂_3 .

In our example the differential ∂_1 glues the edges labelled by 1 and 9 and shuffles the subtrees $t_{1,0} = [t_{2,0}, t_{2,1}]$ and $t_{1,1} = [t_{2,2}]$. One has $\partial_1 = (-1)^{8+d(\{a_1, \dots, a_7\})} d_0^1$ where 8 is the label of the right most edge of $t_{1,0}$. In addition we have the shuffle signs. One has $d(t_{2,0}) = 3 + d(\{a_1, \dots, a_4\})$, $d(t_{2,1}) = 2 + d(\{a_5, a_6, a_7\})$, $d(t_{1,0}) = 7 + d(\{a_1, \dots, a_7\})$ and $d(t_{2,2}) = 4 + d(\{a_8, \dots, a_{13}\})$. In the expansion of d_0^1 there are 3 shuffles involved: id, (132), and (231), written in image notation. The first is coming with sign +1, the second one with

sign $(-1)^{(d(t_{2,1})+1)(d(t_{2,2})+1)}$ and the third one with sign $(-1)^{(d(t_{2,0})+1+d(t_{2,1})+1)(d(t_{2,2})+1)}$. For instance the image of the latter shuffle is in $F((t', \phi'))$ where (t', ϕ') is the following tree:

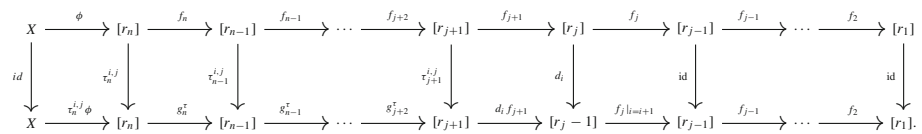


The differential ∂_2 is $(-1)^{5+d(\{a_1, \dots, a_4\})} d_0^2$ where 5 is the label of the right most top edge of $t_{2,0}$. The shuffles involved in the computation of d_0^2 are the (3,2)-shuffles. For such a (3,2)-shuffle τ the associated sign is given by $\epsilon(\tau; t_{2,0}, t_{2,1})$ where $t_{2,0} = [t_{3,0}, t_{3,1}, t_{3,2}]$ and $t_{2,1} = [t_{3,3}, t_{3,4}]$.

The differential ∂_3 is given by $\partial_3 = (-1)^{3+d(\{a_1, a_2\})} d_0^3 + (-1)^{4+d(\{a_1+a_2+a_3\})} d_1^3 + (-1)^{7+d(\{a_1, \dots, a_6\})} d_3^3 + (-1)^{11+d(\{a_1, \dots, a_8\})} d_5^3 + (-1)^{12+d(\{a_1, \dots, a_{10}\})} d_6^3 + (-1)^{13+d(\{a_1, \dots, a_{11}\})} d_7^3$.

Proof of Lemmas 3.8 and 3.10 The proof that $d = \sum_j \partial_j$ satisfies $d^2 = 0$ is done by induction on n . Since the expression of d in Definition 3.7 coincides with the one in Definition 3.9 when $d(X) = 0$ it is enough to prove Lemma 3.10.

Let $(t, \phi) : X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]$ be an (X, n) -level tree in Epi_n^X . For $j < n$ and for i such that $f_j(i) = f_j(i + 1)$ (we denote by S_j^i the set of such i 's) and for $\tau \in \text{Sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i + 1)) = \text{Sh}_{i,j}^t$ we denote by $\tau_{i,j}$ the map in Epi_n^X defined in Lemma 3.5



For $j = n$ and i such that $f_n(i) = f_n(i + 1)$ there are no shuffles involved and we denote by $d_{i,n}$ the corresponding map. We have to prove that $\partial = \sum_j \partial_j$, where $\partial_j = \sum_{i|f_j(i)=f_j(i+1)} (-1)^{s_{j,i}} d_i^j$ and d_i^j is defined by relation (3.8), satisfies $\partial^2 = 0$. For $x \in F(t, \phi)$, $\partial(x)$ has the following form

$$\partial(x) = \sum_{j < n, i \in S_j^i, \tau \in \text{Sh}_{i,j}^t} \pm F(\tau_{i,j})(x) + \sum_{i \in S_n^i} \pm F(d_{i,n})(x), \tag{3.10}$$

where the signs are described in Definition 3.9.

Let T_n^X be the free k -module generated by the trees in Epi_n^X and for a generator $t \in T_n^X$ consider the map

$$\partial(t) = \sum_{j < n, i \in S_j^t, \tau \in \text{Sh}_{i,j}^t} \pm \tau_{i,j}(t) + \sum_{i \in S_n^t} \pm d_{i,n}(t).$$

Then $\partial^2(t) = 0$ in T_n^X implies that $\partial^2(x) = 0$ for all $x \in F(t, \phi)$.

In order to prove that $\partial^2(t) = 0$ for any $t \in \text{Epi}_n^X$ we use the construction of the iterated bar construction given by Eilenberg and Mac Lane in [5, sections 7–9]. We differ from their convention by using the left instead of the right action of the symmetric group. If (A, ∂) is a differential graded commutative algebra then BA is a differential graded commutative algebra with a differential that is the sum of a residual boundary

$$\partial_r([a_1, \dots, a_k]) = \sum_{i=1}^k (-1)^{i+d(a_1)+\dots+d(a_{i-1})} [a_1, \dots, \partial a_i, \dots, a_k]$$

and a simplicial boundary

$$\partial_s([a_1, \dots, a_k]) = \sum_{i=1}^{k-1} (-1)^{i+d(a_1)+\dots+d(a_i)} [a_1, \dots, a_i \cdot a_{i+1}, \dots, a_k].$$

The graded commutative product of $a = [a_1, \dots, a_k]$ and $b = [a_{k+1}, \dots, a_{k+l}]$ is given by the shuffle product

$$[a_1, \dots, a_k] * [a_{k+1}, \dots, a_{k+l}] = \sum_{\sigma \in \text{Sh}(k,l)} \varepsilon(\sigma; a, b) [a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(k+l)}]$$

where $\varepsilon(\sigma; a, b)$ picks up a factor $(-1)^{d(a_i)+1)d(a_j)+1}$ whenever $\sigma(i) > \sigma(j)$ but $i < j$.

Assume that $n = 1$. To an ordered finite set X of graded elements one can associate a graded algebra $A = \bigoplus_{I \subset X} k[e_I]$ with $d(e_I) = d(I)$ and with the multiplication given by

$$e_I e_J = \begin{cases} \varepsilon(I; J) e_{I \sqcup J} & \text{if } I \cap J = \emptyset, \\ 0 & \text{if not.} \end{cases}$$

The algebra A is graded commutative thanks to relation (3.7). Since the algebra A has no differential the boundary in BA reduces to the simplicial boundary which is

$$\partial_s([e_{X_0}, \dots, e_{X_k}]) = \sum_{i=0}^{k-1} (-1)^{i+1+d(X_0)+\dots+d(X_i)} [e_{X_0}, \dots, e_{X_i} \cdot e_{X_{i+1}}, \dots, e_{X_k}].$$

If we assume furthermore that the family (X_0, \dots, X_k) forms a partition of X then $\partial_s([e_{X_0}, \dots, e_{X_k}])$ is a sum of elements satisfying the same property. As a consequence the free k -module generated by the trees in Epi_1^X is a subcomplex of BA and $\partial_1 = \partial_s$ is a differential, since $s_{1,i}$ is precisely $i + 1 + \sum_{j=0}^i d(X_j)$.

Assume $n > 1$. We denote by T_n^X the free k -module generated by the (X, n) -level trees. We denote by T_n the free k -module generated by the n -level trees whose top vertices are labelled by subsets of X . The labels X_a and X_b of two different vertices a and b may have a non-empty intersection. An (X, n) -level tree $t = [t_{1,0}, \dots, t_{1,r_1}]$ is considered as an element of BT_{n-1} . The k -module T_{n-1} is endowed with the shuffle product since $T_{n-1} = BT_{n-2}$. More precisely, for $a = [t_1, \dots, t_p]$ and $b = [t_{p+1}, \dots, t_{p+q}]$ two $(n - 1)$ -level trees where t_i is an

$(n - 2)$ -level tree for every i , one has $a * b = \sum_{\sigma \in \text{Sh}_{p,q}} \epsilon(\sigma; a, b)[t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(p+q)}]$. From Eilenberg and Mac Lane [5, 7.1] one gets that $(BT_{n-1}, \partial_s + \partial_r)$ is a complex of k -modules.

Let t be an (X, n) -level tree. One has $t = [t_{1,0}, \dots, t_{1,r_1}]$ where each $t_{1,i}$ is labelled by $X_i \subset X$ and the family $(X_i)_{0 \leq i \leq r_1}$ is a partition of X . Hence T_n^X is a k -submodule of BT_{n-1} . The simplicial boundary preserves T_n^X for $t_{1,i} * t_{1,i+1} \in T_{n-1}^{X_i \cup X_{i+1}}$. The residual boundary preserves T_n^X by induction on n . Furthermore

$$\begin{aligned} \partial_s(t) &= \sum_{i=0}^{r_1-1} (-1)^{i+1+d(t_{1,0})+\dots+d(t_{1,i})} [t_{1,0}, \dots, t_{1,i} * t_{1,i+1}, \dots, t_{1,r_1}] \\ &= \sum_{i=0}^{r_1-1} (-1)^{s_{1,i}} \sum_{\tau \in \text{Sh}_{t_{1,i}^r}} \epsilon(\tau; t_{1,i}, t_{1,i+1}) \tau_{i,1}(t) = \partial_1(t). \end{aligned}$$

Similarly, by induction on n we prove that the sum $\partial_2 + \dots + \partial_n$ corresponds to the residual boundary. As a consequence T_n^X is a sub-complex of BT_{n-1} and one gets that $\partial^2 = 0$. \square

As an example, we will determine the zeroth E_n -homology of an Epi_n -functor F . In total degree zero there is just one summand, namely $F([0] \xrightarrow{\text{id}_{[0]}} \dots \xrightarrow{\text{id}_{[0]}} [0])$. The modules $C_{(0,1,0,\dots,0)}^{E_n}(F), \dots, C_{(0,\dots,0,1)}^{E_n}(F)$ are all trivial, so the only boundary term that can occur is caused by the unique map

$$C_{(1,0,\dots,0)}^{E_n}(F) \longrightarrow C_{(0,\dots,0)}^{E_n}(F).$$

Therefore

$$H_0^{E_n}(F) \cong F([0] \xrightarrow{\text{id}_{[0]}} \dots \xrightarrow{\text{id}_{[0]}} [0]) / \text{image} \left(F([1] \xrightarrow{d_0} [0] \xrightarrow{\text{id}_{[0]}} \dots \xrightarrow{\text{id}_{[0]}} [0]) \right). \tag{3.11}$$

We can view an Epi_n -module F as an Epi_k -module for all $k \leq n$ via the functors ι_n^k .

Proposition 3.12 *For every Epi_n -module F there is a map of chain complexes $\text{Tot}(C_*^{E_k}(F \circ \iota_n^k)) \longrightarrow \text{Tot}(C_*^{E_n}(F))$ and therefore a map of graded k -modules*

$$H_*^{E_k}(F \circ \iota_n^k) \longrightarrow H_*^{E_n}(F).$$

Proof There is a natural identification of the module $C_{(r_k, \dots, r_1)}^{E_k}(F \circ \iota_n^k)$ with the module $C_{(r_k, \dots, r_1, 0, \dots, 0)}^{E_n}(F)$ and this includes $\text{Tot}(C_*^{E_k}(F \circ \iota_n^k))$ as a subcomplex into $\text{Tot}(C_*^{E_n}(F))$. \square

3.1 Relationship to higher order Hochschild homology

For a non-unital commutative k -algebra \bar{A} we define $\mathcal{L}^n(\bar{A}): \text{Epi}_n \rightarrow k\text{-mod}$ as

$$\mathcal{L}^n(\bar{A}) \left([r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1] \right) = \bar{A}^{\otimes(r_n+1)}.$$

A morphism

$$\begin{array}{ccccccc}
 [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \cdots & \xrightarrow{f_2} & [r_1] \\
 \downarrow \sigma_n & & \downarrow \sigma_{n-1} & & & & \downarrow \sigma_1 \\
 [r'_n] & \xrightarrow{f'_n} & [r'_{n-1}] & \xrightarrow{f'_{n-1}} & \cdots & \xrightarrow{f'_2} & [r'_1]
 \end{array}$$

induces a map $\bar{A}^{\otimes(r_n+1)} \rightarrow \bar{A}^{\otimes(r'_n+1)}$ via

$$a_0 \otimes \cdots \otimes a_{r_n} \mapsto (\sigma_n)_*(a_0 \otimes \cdots \otimes a_{r_n}) = b_0 \otimes \cdots \otimes b_{r'_n}$$

with $b_i = \prod_{\sigma_n(j)=i} a_j$. The E_n -homology of the functor $\mathcal{L}^n(\bar{A})$ coincides with the homology of the n -fold bar construction of the non-unital algebra \bar{A} , hence with the E_n -homology of \bar{A} . The total complex has been described in [6, Appendix] and it coincides with ours.

There is a correspondence between augmented commutative k -algebras and non-unital k -algebras that sends an augmented k -algebra A to its augmentation ideal \bar{A} . Under this correspondence, the $(m + n)$ th homology group of the n -fold bar construction $B^n(A)$ is isomorphic to the m th homology group of the n -fold iterated bar construction of \bar{A} , $B^n(\bar{A})$. As the chain complex $B(A)$ is the chain complex for the Hochschild homology of A with coefficients in k (compare [5, (7.5)]), we can express $B(A)$ as $A \tilde{\otimes} \mathbb{S}^1$. Here, \mathbb{S}^1 is the simplicial model of the 1-sphere, which has $n + 1$ elements in simplicial degree n and $(A \tilde{\otimes} \mathbb{S}^1)_n = k \otimes A^{\otimes n}$. Therefore

$$B^n(A) \cong (\dots(A \tilde{\otimes} \mathbb{S}^1)\dots) \tilde{\otimes} \mathbb{S}^1 \cong A \tilde{\otimes} ((\mathbb{S}^1)^{\wedge n}) \cong A \tilde{\otimes} \mathbb{S}^n$$

which gives rise to higher order Hochschild homology of order n of A with coefficients in k , $HH_*^{[n]}(A; k)$, in the sense of Pirashvili [9]. Thus, $HH_{*+n}^{[n]}(A; k) \cong H_*^{E_n}(\bar{A})$.

By Proposition 3.12 there is a sequence of maps

$$HH_{*+1}(A; k) \cong H_*^{\text{bar}}(\bar{A}) = H_*^{E_1}(\bar{A}) \rightarrow H_*^{E_2}(\bar{A}) \rightarrow H_*^{E_3}(\bar{A}) \rightarrow \dots \tag{3.12}$$

and the map from $H_*^{E_1}(\bar{A})$ to the higher E_n -homology groups is given on the chain level by the inclusion of $C_m^{\text{bar}}(\bar{A})$ into $C_{(m,0,\dots,0)}^{E_n}(\bar{A})$.

Suspension induces maps

$$\begin{array}{ccc}
 HH_\ell(A; k) = \pi_\ell \mathcal{L}(A; k)(\mathbb{S}^1) & \longrightarrow & HH_{\ell+1}^{[2]}(A; k) = \pi_{\ell+1} \mathcal{L}(A; k)(\mathbb{S}^2) \longrightarrow \dots \\
 & \searrow & \downarrow \\
 & & H\Gamma_{\ell-1}(A; k) \cong \pi_\ell^s(\mathcal{L}(A; k)).
 \end{array}$$

For the last isomorphism see [10]. Fresse proves a comparison [6, 9.6] between Gamma homology of A and E_∞ -homology of \bar{A} . Using the isomorphisms above this sequence gives rise to a sequence of maps involving graded k -modules that are isomorphic to the ones in (3.12).

The explicit form of the suspension maps is described in [5, (7.9)]: an element $a \in \bar{A}$ is sent to $[a]$ in the bar construction. The iterations of this map correspond precisely to the maps $u_n^{n-1}: B^{n-1}(\bar{A}) \rightarrow B^n(\bar{A})$. Therefore we actually have an isomorphism of sequences, i.e., the suspension maps $HH_{\ell+n}^{[n]}(A; k) \rightarrow HH_{\ell+n+1}^{[n+1]}(A; k)$ are related to the natural maps $H_\ell^{E_n}(\bar{A}) \rightarrow H_\ell^{E_{n+1}}(\bar{A})$ via the isomorphisms $HH_{*+n}^{[n]}(A; k) \cong H_*^{E_n}(\bar{A})$.

We have $\mathcal{L}^n(\bar{A})([1] \xrightarrow{\text{id}_{[0]}} \dots \xrightarrow{\text{id}_{[0]}} [0]) = \bar{A}^{\otimes 2}$ and hence for all $n \geq 1$ the zeroth E_n -homology group is

$$H_0^{E_n}(\bar{A}) \cong \bar{A}/\bar{A} \cdot \bar{A}.$$

As a concrete example, we calculate the E_n -homology of a polynomial algebra in one and in two variables. Let Γ denote the skeleton of the category of finite pointed sets with objects $[n] = \{0, \dots, n\}$ for $n \geq 0$ with 0 as basepoint. Pirashvili’s definition of n th order Hochschild homology is given for Γ -modules, i.e., functors from Γ to k -modules. In particular, we consider $\mathcal{L}(A; k): \Gamma \rightarrow k\text{-mod}$, for an augmented commutative algebra A with $\mathcal{L}(A; k)[n] = A^{\otimes n}$ where maps in Γ induce multiplication in A , augmentation to k or insertion of units. The functor \mathcal{L} for a polynomial algebra with coefficients in k evaluated on a simplicial model of the n -sphere is the symmetric algebra functor evaluated on the n -sphere and thus arguing as in [13, section 4] we obtain

$$\begin{aligned} H_*^{E_n}(\overline{k[x]}) &\cong HH_{*+n}^{[n]}(k[x]; k) = H_{*+n}(\mathcal{L}(k[x]; k)(\mathbb{S}^n)) \cong H_{*+n}(\text{Sym} \circ L(\mathbb{S}^n)) \\ &\cong H_{*+n}(SP(\mathbb{S}^n); k). \end{aligned}$$

Here, SP stands for the infinite symmetric product and L is the Γ -module that sends $[n]$ to the free k -module generated by the set $\{1, \dots, n\}$. Note that in this case $k[x]$ is augmented over k via the augmentation ε_0 that sends x to zero. The augmentation affects the $k[x]$ -module structure of k , but in [13, 4.1] it is shown that the resulting homotopy groups are independent of the module structure.

Evaluated on an n -sphere, the functor SP yields an Eilenberg–MacLane space of type (\mathbb{Z}, n) and hence the above is isomorphic to $H_{*+n}(K(\mathbb{Z}, n); k)$.

Consider the Γ -module $\mathcal{L}(k[x, y]; k)$. To $[n]$ it associates $k \otimes k[x, y]^{\otimes n} \cong k[x]^{\otimes n} \otimes k[y]^{\otimes n} \cong \mathcal{L}(k[x]; k)[n] \otimes \mathcal{L}(k[y]; k)[n]$. A morphism of finite pointed sets $f: [n] \rightarrow [m]$ sends $\lambda \otimes a_1 \otimes \dots \otimes a_n$ (with $\lambda \in k$ and a_i in $k[x, y]$) to $\mu \otimes b_1 \otimes \dots \otimes b_m$ where $b_i = \prod_{f(j)=i} a_j$ and $\mu = \lambda \cdot \prod_{f(j)=0, j \neq 0} \varepsilon(a_j)$. Therefore the above isomorphism of $\mathcal{L}(k[x]; k)[n] \otimes \mathcal{L}(k[y]; k)[n]$ and $\mathcal{L}(k[x, y]; k)[n]$ induces an isomorphism of Γ -modules between $\mathcal{L}(k[x, y]; k)(\mathbb{S}^n)$ and the pointwise tensor product $\mathcal{L}(k[x]; k)(\mathbb{S}^n) \otimes \mathcal{L}(k[y]; k)(\mathbb{S}^n)$. Furthermore, we get

$$\begin{aligned} \pi_*((\text{Sym} \circ L(\mathbb{S}^n)) \otimes (\text{Sym} \circ L(\mathbb{S}^n))) &\cong \pi_*(\text{Sym} \circ (L(\mathbb{S}^n) \oplus L(\mathbb{S}^n))) \\ &\cong \pi_*(\text{Sym} \circ (L(\mathbb{S}^n \vee \mathbb{S}^n))) \cong H_*(SP(\mathbb{S}^n \vee \mathbb{S}^n); k) \\ &\cong H_*(K(\mathbb{Z} \times \mathbb{Z}, n); k). \end{aligned}$$

Therefore, we obtain that

$$H_*^{E_n}(\overline{k[x, y]}) \cong H_{*+n}^{[n]}(k[x, y]) \cong H_{*+n}(K(\mathbb{Z} \times \mathbb{Z}, n); k).$$

We close this part with the description of a spectral sequence. Our description of E_2 -homology leads to the following result.

Proposition 3.13 *If \bar{A} and $H_*^{\text{bar}}(\bar{A})$ are k -flat, then there is a spectral sequence*

$$E_{p,q}^1 = \bigoplus_{\ell_0 + \dots + \ell_q = p-q} H_{\ell_0}^{\text{bar}}(\bar{A}) \otimes \dots \otimes H_{\ell_q}^{\text{bar}}(\bar{A}) \Rightarrow H_{p+q}^{E_2}(\bar{A})$$

where the d_1 -differential is induced by the shuffle differential.

Proof The double complex for E_2 -homology looks as follows:

$$\begin{array}{ccccccc}
 & & & & \vdots & & \\
 & & & & \downarrow & & \\
 & & & & \bar{A}^{\otimes 3} & \longleftarrow & \dots \\
 & & & & \downarrow & & \\
 & & & \bar{A}^{\otimes 2} & \longleftarrow & \bar{A}^{\otimes 3} & \longleftarrow \dots \\
 & & & \downarrow & & \downarrow & \\
 \bar{A} & \longleftarrow & \bar{A}^{\otimes 2} & \longleftarrow & \bar{A}^{\otimes 3} & \longleftarrow & \dots
 \end{array}$$

The horizontal maps are induced by the b' -differential whereas the vertical maps are induced by the shuffle maps. The horizontal homology of the bottom row is precisely $H_*^{\text{bar}}(\bar{A})$. We can interpret the second row as the total complex associated to the following double complex:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 \downarrow \text{id} \otimes b' & & \downarrow \text{id} \otimes b' & & \downarrow \text{id} \otimes b' & & \\
 \bar{A} \otimes \bar{A}^{\otimes 3} & \xleftarrow{b' \otimes \text{id}} & \bar{A}^{\otimes 2} \otimes \bar{A}^{\otimes 3} & \xleftarrow{b' \otimes \text{id}} & \bar{A}^{\otimes 3} \otimes \bar{A}^{\otimes 3} & \xleftarrow{b' \otimes \text{id}} & \dots \\
 \downarrow \text{id} \otimes b' & & \downarrow \text{id} \otimes b' & & \downarrow \text{id} \otimes b' & & \\
 \bar{A} \otimes \bar{A}^{\otimes 2} & \xleftarrow{b' \otimes \text{id}} & \bar{A}^{\otimes 2} \otimes \bar{A}^{\otimes 2} & \xleftarrow{b' \otimes \text{id}} & \bar{A}^{\otimes 3} \otimes \bar{A}^{\otimes 2} & \xleftarrow{b' \otimes \text{id}} & \dots \\
 \downarrow \text{id} \otimes b' & & \downarrow \text{id} \otimes b' & & \downarrow \text{id} \otimes b' & & \\
 \bar{A} \otimes \bar{A} & \xleftarrow{b' \otimes \text{id}} & \bar{A}^{\otimes 2} \otimes \bar{A} & \xleftarrow{b' \otimes \text{id}} & \bar{A}^{\otimes 3} \otimes \bar{A} & \xleftarrow{b' \otimes \text{id}} & \dots
 \end{array}$$

Therefore the horizontal homology groups of the second row are the homology of the tensor product of the $C^{\text{bar}}(\bar{A})$ -complex with itself. Our flatness assumptions guarantee that we obtain $H_*^{\text{bar}}(\bar{A})^{\otimes 2}$ as homology. An induction then finishes the proof. \square

4 Tor interpretation of E_n -homology

The purpose of this section is to prove the following theorem.

Theorem 4.1 *For any Epi_n -module F*

$$H_p^{E_n}(F) \cong \text{Tor}_p^{\text{Epi}_n}(b_n^{\text{epi}}, F), \quad \text{for all } p \geq 0$$

where

$$b_n^{\text{epi}}(t) \cong \begin{cases} k & \text{for } t = [0] \xrightarrow{\text{id}_{[0]}} \dots \xrightarrow{\text{id}_{[0]}} [0], \\ 0 & \text{for } t \neq [0] \xrightarrow{\text{id}_{[0]}} \dots \xrightarrow{\text{id}_{[0]}} [0]. \end{cases}$$

As will be explained in the proof below it is enough to prove that for every $t \in \text{Epi}_n$, the representable functor $\text{Epi}_n^t = k[\text{Epi}_n(t, -)]$ is acyclic with respect to E_n -homology. In order

to prove this we proceed by induction on n . The case $n = 1$ has been treated in Sect. 2. We consider the case $n = 2$ from Proposition 4.4 to Corollary 4.9 and this case is the core of the proof. The induction process is explained in Proposition 4.10. Unfortunately, the signs involved in the differential of the complex $C_*^{E_n}(\text{Epi}_n^t)$ are not compatible with an induction process as in Proposition 4.10 and this is the reason why we introduced the category Epi_n^X and the homology $H_*^{E_n, X}$ of an Epi_n^X -module in Definition 3.9. We explain in the proof below how we use information on $\text{Epi}_n^{X^t, \phi}$ in order to get information on Epi_n^t . Here X is a finite ordered set of graded elements and $\text{Epi}_n^{X^t, \phi}$ is the representable functor $k[\text{Epi}_n^X[(t, \phi), -]]$. Consequently propositions and Lemmas 4.4 to 4.9 are expressed in terms of the representable functors $\text{Epi}_n^{X^t, \phi}$ as well as the representable functors Epi_n^t , whereas Proposition 4.10 gives results on $\text{Epi}_n^{X^t, \phi}$ only.

Proof As in the proof of Proposition 2.3, we have to show that $H_*^{E_n}(-)$ maps short exact sequences of Epi_n -modules to long exact sequences, that $H_*^{E_n}(-)$ vanishes on projectives in positive degrees and that $H_0^{E_n}(F)$ and $b_n^{\text{epi}} \otimes_{\text{Epi}_n} F$ agree for all Epi_n -modules F . The homology $H_*^{E_n}(-)$ is the homology of a total complex $C_*^{E_n}(-)$ sending short exact sequences as in (2.2) to short exact sequences of chain complexes and therefore the first claim is true. Note that the left Epi_n -module b_n^{epi} is the cokernel of the map between contravariant representables

$$(d_0)_* : \text{Epi}_{n, [1]} \rightarrow [0] \rightarrow \dots \rightarrow [0] \rightarrow \text{Epi}_{n, [0]} \rightarrow [0] \rightarrow \dots \rightarrow [0].$$

This remark together with the computation of $H_0^{E_n}(F)$ in relation (3.11) implies the last claim, similarly to the proof of Proposition 2.3.

In order to show that $H_*^{E_n}(P)$ is trivial in positive degrees for any projective Epi_n -module P it suffices to show that the representables Epi_n^t are acyclic for any planar tree $t = [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_2} [r_1]$.

Let t be such an n -level tree, let X be a finite ordered set and let $\phi : X \rightarrow [r_n]$ be a fixed surjection. Assume that every element in X has degree 0. Then we claim that the complexes $C_*^{E_n}(\text{Epi}_n^t)$ and $C_*^{E_n, X}(\text{Epi}_n^{X^t, \phi})$ are isomorphic. One has

$$k[\text{Epi}_n(t, t')] \cong \bigoplus_{\phi' : X \rightarrow [r'_n]} k[\text{Epi}_n^X((t, \phi), (t', \phi'))],$$

because any morphism of n -trees $\sigma : t \rightarrow t'$ determines a component $\phi' = \sigma_n \circ \phi$. This defines an injective map $k[\text{Epi}_n(t, t')] \rightarrow \bigoplus_{\phi' : X \rightarrow [r'_n]} k[\text{Epi}_n^X((t, \phi), (t', \phi'))]$. As every morphism from (t, ϕ) to (t', ϕ') is a morphism of n -trees $\sigma : t \rightarrow t'$ with $\sigma_n \circ \phi = \phi'$, the map is surjective. By relations (3.6) and (3.9) one has

$$C_*^{E_n}(\text{Epi}_n^t) = \bigoplus_{t' \in \text{Epi}_n} \text{Epi}_n(t, t') = \bigoplus_{(t', \phi') \in \text{Epi}_n^X} \text{Epi}_n^X((t, \phi), (t', \phi')) = C_*^{E_n, X}(\text{Epi}_n^{X^t, \phi}) \tag{4.1}$$

and as every element of X has degree zero, the differentials ∂_j coincide for all j .

In Proposition 4.10, we will prove by induction that for an (X, n) -level tree $(t, \phi) : X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} \dots \xrightarrow{f_{j+1}} [r_j] \xrightarrow{f_j} \dots \xrightarrow{f_2} [r_1]$ with $X = \{x_0 < \dots < x_{r_n}\}$ and $\phi(x_i) = i$, the representable $\text{Epi}_n^{X^t, \phi}$ is acyclic. In particular, if every element in X has degree zero, then this implies that Epi_n^t is acyclic for any n -level tree t .

The case $n = 1$ has been proved in Proposition 2.3 in the ungraded case and in Remark 2.4 in the graded case. For $n = 2$ we study the bicomplex $C_{(*,*)}^{E_2, X}(\text{Epi}_2^{X^t, \phi})$. In Proposition 4.4 we give the k -module structure of the homology with respect to the differential ∂_2 and give its generators in Propositions 4.7 and 4.8. Corollaries 4.6 and 4.9 state the result for $n = 2$. For the general case, one uses induction on n and Proposition 4.10. As a consequence $H_*^{E_n}(\text{Epi}_n^t) = 0$ for all $* \geq 0$ if $t \neq [0] \rightarrow [0] \dots \rightarrow [0]$ and in that case

$$H_*^{E_n}(\text{Epi}_n^{[0] \rightarrow [0] \dots \rightarrow [0]}) = \begin{cases} 0 & \text{for } * > 0, \\ k & \text{for } * = 0. \end{cases}$$

□

In the following we need some technical tools from the homology of small categories, as in Mitchell [12, section 17]. We review the standard resolution of a small category in the graded context. References on the more general context of differential graded categories can be found also in [8].

Definition 4.2 Let \mathcal{C} be a small category that is a graded k -linear category, i.e., for every pair of objects $a, b \in \mathcal{C}$ the set of morphisms $\mathcal{C}(a, b)$ is a graded k -module and the structure maps of the category $\mathcal{C}(b, c) \otimes \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, c)$ are morphisms of graded k -modules. We denote by $\mathcal{C} \otimes \mathcal{D}$ the tensor product of two small graded k -categories, which is given by the product on objects and the tensor product on morphisms. The standard resolution of \mathcal{C} is the simplicial bifunctor from $\mathcal{C}^{op} \otimes \mathcal{C}$ to the category of graded k -modules defined by

$$S_n(\mathcal{C}) = \bigoplus_{p_1, \dots, p_{n+1}} \mathcal{C}(-, p_1) \otimes \mathcal{C}(p_1, p_2) \otimes \dots \otimes \mathcal{C}(p_n, p_{n+1}) \otimes \mathcal{C}(p_{n+1}, -)$$

where the face maps are

$$d_i(\alpha_0 \otimes \dots \otimes \alpha_{n+1}) = (-1)^{d(\alpha_0) + \dots + d(\alpha_i)} \alpha_0 \otimes \dots \otimes \alpha_{i+1} \alpha_i \otimes \dots \otimes \alpha_{n+1}$$

and the degeneracy maps are

$$s_j(\alpha_0 \otimes \dots \otimes \alpha_{n+1}) = (-1)^{d(\alpha_0) + \dots + d(\alpha_j)} \alpha_0 \otimes \dots \otimes \alpha_j \otimes \text{id} \otimes \alpha_{j+1} \otimes \dots \otimes \alpha_{n+1}.$$

A covariant (contravariant) functor from the category \mathcal{C} to the category of graded k -modules is called a left (right) graded \mathcal{C} -module. For any left graded \mathcal{C} -module L and any right graded \mathcal{C} module R , we define

$$H_*(R; L) = H_*(L \otimes_{\mathcal{C}^{op}} S_*(\mathcal{C}) \otimes_{\mathcal{C}^{op}} R) = \text{Tor}_*^{\mathcal{C}}(R; L).$$

Note that one can also use the normalized standard resolution, combined with the Yoneda lemma to compute this homology

$$L \otimes_{\mathcal{C}^{op}} N_n(\mathcal{C}) \otimes_{\mathcal{C}^{op}} R = \bigoplus_{p_1, \dots, p_{n+1}} L(p_1) \otimes \tilde{\mathcal{C}}(p_1, p_2) \otimes \dots \otimes \tilde{\mathcal{C}}(p_n, p_{n+1}) \otimes R(p_{n+1}), \tag{4.2}$$

where $\tilde{\mathcal{C}}(p_1, p_2) = \mathcal{C}(p_1, p_2)$ if $p_1 \neq p_2$ and $\tilde{\mathcal{C}}(p_1, p_2)$ is the cokernel of the map $k \rightarrow \mathcal{C}(p_1, p_1)$ if $p_1 = p_2$.

Let $a > 0$ be an integer and let Y be a graded ordered set and $\pi = \pi^1 \sqcup \dots \sqcup \pi^a$ be a partition of Y . We denote by $[a]^\pi$ the following graded category: objects in $[a]^\pi$ are the elements i for $0 \leq i \leq a$ and morphisms in $[a]^\pi$ are the graded k -modules given by

$$[a]^\pi(i, j) = \begin{cases} k \text{ in degree } d(\pi^{i+1} \cup \dots \cup \pi^j) & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases}$$

where by convention $d(\emptyset) = 0$.

The composition of $\alpha \in [a]^\pi(j, k)$ and $\beta \in [a]^\pi(i, j)$, $i \leq j \leq k$ is given by

$$\alpha \circ \beta = \epsilon \left(\pi^{i+1} \cup \dots \cup \pi^j; \pi^{j+1} \cup \dots \cup \pi^k \right) \alpha \beta.$$

The category $[a]^\pi$ is a poset category with a minimal element 0 and a maximal element a . We denote by L_0 the left graded $[a]^\pi$ -module which assigns k in degree 0 to 0 and 0 to $0 < i \leq a$ and by R_a the right graded $[a]^\pi$ -module which assigns k in degree 0 to a and 0 to $0 \leq i < a$.

Lemma 4.3 *For a tensor product $[a_0]^{\pi_0} \otimes \dots \otimes [a_{r_1}]^{\pi_{r_1}}$ of categories we obtain*

$$\text{Tor}_n^{[a_0]^{\pi_0} \otimes \dots \otimes [a_{r_1}]^{\pi_{r_1}}} \left(R_{a_0} \otimes \dots \otimes R_{a_{r_1}}; L_0 \otimes \dots \otimes L_0 \right) \cong \begin{cases} k & \text{if } n = r_1 + 1 \text{ and } \forall j, a_j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof The Künneth formula (see e.g. [17, section 3.6]) gives

$$\begin{aligned} & \text{Tor}_n^{[a_0]^{\pi_0} \otimes \dots \otimes [a_{r_1}]^{\pi_{r_1}}} \left(R_{a_0} \otimes \dots \otimes R_{a_{r_1}}; L_0 \otimes \dots \otimes L_0 \right) \\ & \cong \bigoplus_{n_0 + \dots + n_{r_1} = n} \text{Tor}_{n_0}^{[a_0]^{\pi_0}}(R_{a_0}; L_0) \otimes \dots \otimes \text{Tor}_{n_{r_1}}^{[a_{r_1}]^{\pi_{r_1}}}(R_{a_{r_1}}; L_0). \end{aligned}$$

Consequently it is enough to compute $\text{Tor}_n^{[a]^\pi}(R_a; L_0)$. Relation (4.2) gives the complex computing this homology:

$$\begin{aligned} C_n([a]^\pi) &= L_0 \otimes_{\mathcal{C}^{\text{op}}} N_n(\mathcal{C}) \otimes_{\mathcal{C}^{\text{op}}} R_a \\ &= \begin{cases} 0 & \text{if } n = 0, \\ k & \text{if } n = 1, \\ \bigoplus_{0 < p_2 < \dots < p_n < a} [a]^\pi(0, p_2) \otimes \dots \otimes [a]^\pi(p_n, a), & \text{if } n > 1, \end{cases} \end{aligned}$$

with the differential given by $d = \sum_{i=1}^{n-1} (-1)^i d_i$.

If $a = 1$, then $C_n([a]^\pi) = 0$ for $n > 1$ and the result follows.

Assume $a > 1$, and let $\xi_{i,j}$ be the generator of $[a]^\pi(i, j)$ in degree $d(\pi^{i+1} \cup \dots \cup \pi^j)$.

The homotopy

$$h(\xi_{0p_2} \otimes \dots \otimes \xi_{p_n a}) = \begin{cases} 0 & \text{if } p_2 = 1, \\ (-1)^{d(\pi^1)} \epsilon(\pi^1; \pi^2 \cup \dots \cup \pi^{p_2}) \xi_{01} \otimes \xi_{1p_2} \otimes \dots \otimes \xi_{p_n a} & \text{if } p_2 > 1, \end{cases}$$

proves that the complex is acyclic.

The E_n -homology of an Epi_n -module F (resp. the (E_n, X) -homology of an Epi_n^X -module F) can be computed in different ways, since it is the homology of the total complex associated to an n -complex. The notation $H_*(F, \partial_i)$ stands for the homology of the complex $C_*^{E_n}(F)$ (resp. $C_*^{E_n, X}(F)$) with respect to the differential ∂_i . The complex $(C_*^{E_n}(F), \partial_i)$ splits into subcomplexes

$$C_{(s_n, s_{n-1}, \dots, s_{i+1}, *, s_{i-1}, \dots, s_1)}^{E_n}(F) = \bigoplus_{t=[s_n] \xrightarrow{g_n} \dots \xrightarrow{g_{i+2}} [s_{i+1}] \xrightarrow{g_{i+1}} [*] \xrightarrow{g_i} [s_{i-1}] \xrightarrow{g_{i-1}} \dots \xrightarrow{g_2} [s_1]} F(t), \tag{4.3}$$

whose homology is denoted by $H_{(s_n, s_{n-1}, \dots, s_{i+1}, *, s_{i-1}, \dots, s_1)}(F, \partial_i)$. There is an analogous splitting for the complex $(C_*^{E_n, X}(F), \partial_i)$.

In the proposition below we compute $H_*(F, \partial_2)$ for the representable functors for $n = 2$, give its generators in Propositions 4.7 and 4.8 and then prove that the representable functors are acyclic for $n = 2$ in Corollaries 4.6 and 4.9 depending on the form of the 2-level tree t , whether it is a fork tree or not.

Proposition 4.4 *Let $(t, \phi) = X \xrightarrow{\phi} [r_2] \xrightarrow{f} [r_1]$ be an $(X, 2)$ -level tree in Epi_2^X .*

$$\begin{aligned}
 H_{(*,s)}(\text{Epi}_2^{X^t, \phi}, \partial_2) &= 0 && \text{if } r_2 \neq r_1, \\
 H_{(*,s)}(\text{Epi}_2^{X^t, \phi}, \partial_2) &\cong \begin{cases} 0 & \text{for } * \neq r_2, \\ k^{\oplus |\Delta^{\text{epi}}([r_2], [s])|} & \text{for } s \leq * = r_2, \end{cases} && \text{if } r_2 = r_1.
 \end{aligned}$$

Proof Let F denote the covariant functor $\text{Epi}_2^{X^t, \phi}$.

Assume $s = 0$. We first prove that the chain complex $\partial_2 : C_{(*,0)}^{E_2, X}(F) \rightarrow C_{(*-1,0)}^{E_2, X}(F)$ is the normalized chain complex associated to a small category as in Definition 4.2.

The chain complex $(C_{(*,0)}^{E_2, X}(F), \partial_2)$ has the following form, for $0 < u \leq r_2$

$$\bigoplus_{\psi} k \left[\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [u] \rightarrow [0]) \right] \xrightarrow{\sum_{i=0}^{u-1} (-1)^{s_2, i} d_i^2} \bigoplus_{\psi} k \left[\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [u-1] \rightarrow [0]) \right].$$

Let (A_0, \dots, A_{r_1}) be the sequence of preimages of f , and a_i the number of elements in A_i . For a fixed ψ , the set $\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [u] \rightarrow [0])$ is either empty or has only one element uniquely determined by a surjective map $\sigma : [r_2] \rightarrow [u]$: since ϕ is surjective, the requirement $\psi = \sigma \phi$ uniquely determines the surjection σ if it exists. In that case, σ determines an element in $\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [u] \rightarrow [0])$ if it is order-preserving on the fibres of f .

The map σ can be described by the sequence of its preimages (S_0, \dots, S_u) with the condition (C_S) : if $a < b \in A_i$ then $i_a \leq i_b$ where i_α is the unique index for which $\alpha \in S_{i_\alpha}$. One has

$$\begin{aligned}
 d(S_0, \dots, S_u) &= \sum_{i=0}^{u-1} (-1)^{i+d(\phi^{-1}(S_0 \cup \dots \cup S_i))} \varepsilon(\phi^{-1}(S_i); \phi^{-1}(S_{i+1})) \\
 &\quad \times (S_0, \dots, S_i \cup S_{i+1}, \dots, S_u).
 \end{aligned}$$

Let \mathcal{C} be the tensor product of the categories $[a_j]^{\pi_j}$, $0 \leq j \leq r_1$ where the partitions π_j 's will be defined later. The category \mathcal{C} is a poset category and the order is given by the product order. Any object of the category \mathcal{C} can be written as $p = (p^0, \dots, p^{r_1})$. We denote by 0 the minimal element $(0, \dots, 0)$, by α the maximal element (a_0, \dots, a_{r_1}) and by L_0 and R_α the corresponding left and right graded \mathcal{C} -module. For $p \leq q$ the element ξ_{pq} denotes the unit of $k = \mathcal{C}(p, q)$. The reduced complex (4.2) computing $\text{Tor}_{u+1}^{\mathcal{C}}(R_\alpha; L_0)$ is given by

$$C_{u+1}(\mathcal{C}) = L_0 \otimes_{\mathcal{C}^{\text{op}}} N_{u+1}(\mathcal{C}) \otimes_{\mathcal{C}^{\text{op}}} R_\alpha = \bigoplus_{0 < p_1 < \dots < p_u < \alpha} k \xi_{0p_1} \otimes \xi_{p_1 p_2} \cdots \otimes \xi_{p_u \alpha},$$

with the differential $d = \sum_{i=1}^u (-1)^i d_i$ given in Definition 4.2. The sequence (S_0, \dots, S_u) is in 1-to-1 correspondence with the element $\xi_{0p_1} \otimes \xi_{p_1 p_2} \cdots \otimes \xi_{p_u \alpha}$ where the j th coordinate of p_i, p_i^j , is given by the number of elements in $A_j \cap (\cup_{k \leq i-1} S_k)$. Conversely, let $p_0 = 0 < p_1 < \dots < p_u < p_{u+1} = \alpha$ be a sequence of objects in $\otimes [a_j]^{\pi_j}$. The set A_i is $\{\sum_{k=0}^{i-1} a_k + l, 0 \leq l \leq a_i - 1\}$. Let $S_i \subset [r_2]$ be the set defined by $S_i \cap A_j = \{\sum_{k=0}^{j-1} a_k + l, p_i^j \leq l \leq p_{i+1}^j - 1\}$. The sequence (S_0, \dots, S_u) satisfies the condition (C_S) . The partition π_j is the partition of $(f\phi)^{-1}(j)$ given by $\pi_j^k = \phi^{-1}(k)$ whenever $f(k) = j$.

As a consequence, the two complexes, $(C_{u+1}(C), -d)$ and $(C_{(u,0)}^{E_2,X}(F), \partial_2)$, coincide. By Lemma 4.3 the homology $H_{(u,s)}(\text{Epi}_2^{X^t,\phi}, \partial_2)$ is 0 but for the case where $u + 1 = r_1 + 1$ and $a_j = 1$ for all j . The latter condition is equivalent to $r_2 = r_1$ and $f = \text{id}$. This concludes the case $s = 0$.

Assume $s > 0$.

The complex $(C_{(*,s)}^{E_2,X}(F), \partial_2)$ splits into subcomplexes

$$C_{(*,s)}^{E_2,X}(F) = \bigoplus_{\sigma \in \Delta^{\text{epi}}([r_1],[s])} C_{(*,s)}(F_\sigma) = \bigoplus_{\sigma \in \Delta^{\text{epi}}([r_1],[s])} \bigoplus_{g \in \Delta^{\text{epi}}([*],[s]), \psi} F_\sigma \left(X \xrightarrow{\psi} [*] \xrightarrow{g} [s] \right)$$

where $F_\sigma(X \xrightarrow{\psi} [u] \xrightarrow{g} [s]) \subset \text{Epi}_2^{X^t,\phi}(X \xrightarrow{\psi} [u] \xrightarrow{g} [s])$ is the free k -module generated by morphisms of the form

$$\begin{CD} X @>\phi>> [r_2] @>f>> [r_1] \\ @V\text{id}VV @V\tau VV @V\sigma VV \\ X @>\psi>> [u] @>g>> [s]. \end{CD} \tag{4.4}$$

Let (A_0, \dots, A_s) denote the sequence of preimages of σf and (B_0, \dots, B_s) that of g . The latter has to satisfy the condition $|B_i| \leq |A_i|, 0 \leq i \leq s$. Note that $g \in \Delta^{\text{epi}}([u], [s])$ is also uniquely determined by the sequence (b_0, \dots, b_s) of the cardinalities of its preimages. The differential $\partial_2: C_{(u,s)}(F_\sigma) \rightarrow C_{(u-1,s)}(F_\sigma)$ has the following form:

$$\begin{aligned} \partial_2 \left(\begin{CD} X @>\phi>> [r_2] @>f>> [r_1] \\ @V\text{id}VV @V\tau VV @V\sigma VV \\ X @>\tau\phi>> [u] @>g>> [s] \end{CD} \right) &= \sum_{i|g(i)=g(i+1)} (-1)^{s_2,i} \epsilon((\tau\phi)^{-1}(i); (\tau\phi)^{-1}(i+1)) \begin{CD} X @>\phi>> [r_2] @>f>> [r_1] \\ @V\text{id}VV @Vd_i\tau VV @V\sigma VV \\ X @>d_i\tau\phi>> [u-1] @>g|_{i=i+1}>> [s] \end{CD} \\ &= \sum_{j=0}^s \left(\sum_{i \in B_j | g(i)=g(i+1)} (-1)^{s_2,i} \epsilon((\tau\phi)^{-1}(i); (\tau\phi)^{-1}(i+1)) \begin{CD} X @>\phi>> [r_2] @>f>> [r_1] \\ @V\text{id}VV @Vd_i\tau VV @V\sigma VV \\ X @>d_i\tau\phi>> [u-1] @>g|_{i=i+1}>> [s] \end{CD} \right). \end{aligned}$$

Define D_j by restricting the sum over indices i such that $g(i) = g(i + 1)$ to the sum over indices $i \in B_j$ such that $g(i) = g(i + 1)$. One has

$$D_j: C_{(u,s)}(F_\sigma) = \bigoplus_{b_0+\dots+b_s=u+1} C_{((b_0,\dots,b_j,\dots,b_s),s)}(F_\sigma) \rightarrow \bigoplus_{b_0+\dots+b_s=u+1} C_{((b_0,\dots,b_{j-1},\dots,b_s),s)}(F_\sigma)$$

and $\partial_2 = D_0 + \dots + D_s$. We claim that the D_j are anti-commuting differentials:

Let i be in B_j and ℓ be in B_k . For $j < k$ it follows that $i + 1 < \ell$ for $g(i) = g(i + 1) = j < g(\ell) = k$ therefore we have the relation $d_i d_\ell = d_{\ell-1} d_i$. Furthermore the ϵ -signs involved do not depend on the way we compose: the resulting sign is $\epsilon((\tau\phi)^{-1}(i); (\tau\phi)^{-1}(i+1))\epsilon((\tau\phi)^{-1}(\ell); (\tau\phi)^{-1}(\ell+1))$ for $d_i^{-1}(\ell-1) = \ell$ and $d_i^{-1}(i) = i$. In order to calculate the effect of $d_{\ell-1} d_i$ we have to determine $s_{2,\ell-1}$ after the application of d_i . Let \tilde{S}_j denote the preimage $(d_i \circ \tau \circ \phi)^{-1}(j)$ and S_j the preimage $(\tau \circ \phi)^{-1}(j)$ for $j \in [u]$. Then

$$d(\tilde{S}_j) = \begin{cases} d(S_j), & j < i, \\ d(S_i) + d(S_{i+1}), & j = i, \\ d(S_{j+1}), & j > i. \end{cases}$$

Thus $s_{2,\ell-1}$ is $\ell - 1 + k + 2 + \sum_{j=0}^{\ell-1} d(\tilde{S}_j) = \ell + k + 1 + \sum_{j=0}^{\ell} d(S_j)$ whereas $s_{2,\ell} = \ell + k + 2 + \sum_{j=0}^{\ell} d(S_j)$. A similar argument shows that the D_j are differentials.

The complex $(C_{(u,s)}(F_\sigma), D_s)$ splits into subcomplexes $(C_{((b_0, \dots, b_{s-1}), *)}(F_\sigma), D_s)$ for fixed $b_i \leq a_i = |A_i|$, $i < s$. With the notation of Definition 3.9, the tree (t, ϕ) can be written as $t = [t_{1,0}, \dots, t_{1,r_1}]$, with $t_{1,i}$ being an $(X_{1,i}, 1)$ -level tree. Let p be the first integer such that $\sigma(p) = s$. Let $X_{s-1} = \cup_{0 \leq i \leq p-1} X_{1,i}$ and $\tilde{X} = \cup_{p \leq i \leq r_1} X_{1,i}$. Denote by t_{s-1} the $(X_{s-1}, 2)$ -level tree $t_{s-1} = [t_{1,0}, \dots, t_{1,p-1}]$ and by \tilde{t} the $(\tilde{X}, 2)$ -level tree $\tilde{t} = [t_{1,p}, \dots, t_{1,r_1}]$. See Example 4.5 for an example. Let σ_{s-1} (resp. ϕ_{s-1}) be the map obtained from σ (resp. ϕ) by restriction $\sigma_{s-1}: \sigma^{-1}([s-1]) \xrightarrow{\sigma} [s-1]$. Let $u_{s-1} = (\sum_{i < s} b_i) - 1$. The subcomplex $(C_{((b_0, \dots, b_{s-1}), *)}(F_\sigma), D_s)$ can be expressed as

$$\bigoplus_{\psi; \gamma \in (\text{Epi}_2^{X_{s-1} t_{s-1} \phi_{s-1}})_{\sigma_{s-1}}(X_{s-1} \xrightarrow{\psi} [u_{s-1}] \xrightarrow{g! [u_{s-1}]} [s-1])} (C_{(*,0)}^{E_2, \tilde{X}}(\text{Epi}_2^{\tilde{X}, \tilde{\phi}}), (-1)^{b_0 + \dots + b_{s-1} + s + d(X_{s-1})} \partial_2).$$

If $f \neq \text{id}$, then there exists $j \in [s]$ such that the restriction of f on $(\sigma \circ f)^{-1}(j) \rightarrow \sigma^{-1}(j)$ is different from the identity. Without loss of generality we can assume that $j = s$, hence \tilde{t} is a non-fork tree and the homology of the complex is 0. If $f = \text{id}$, then we deduce from the case $s = 0$ that the complex $(C_{(*,0)}^{E_2, \tilde{X}}(\text{Epi}_2^{\tilde{X}, \tilde{\phi}}, \partial_2)$ has only top homology of rank one; consequently when $t: [r_2] \rightarrow [r_2]$ is the fork tree,

$$(H_*(C_{(*,s)}((\text{Epi}_2^{X,t,\phi})_\sigma), D_s), D_1 + \dots + D_{s-1}) \cong (C_{(*,s-1)}((\text{Epi}_2^{X_{s-1} t_{s-1} \phi_{s-1}})_{\sigma_{s-1}}), \partial_2).$$

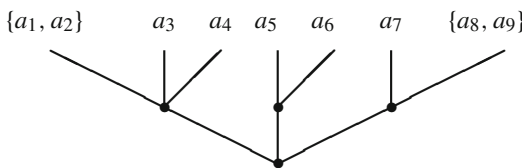
We then have an inductive process to compute the homology of the total complex $(C_{(*,s)}(F_\sigma), \partial_2)$. Consequently, for a fixed $\sigma: [r_2] \rightarrow [s]$

$$H_{(*,s)}(F_\sigma, \partial_2) = 0 \quad \text{if } r_2 \neq r_1,$$

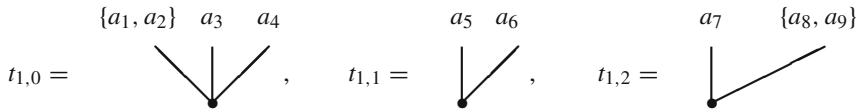
$$H_{(*,s)}(F_\sigma, \partial_2) \cong \begin{cases} 0 & \text{for } * \neq r_2 \\ k & \text{for } s \leq * = r_2 \end{cases} \quad \text{if } r_2 = r_1.$$

Since each $\sigma \in \Delta^{\text{epi}}([r_2], [s])$ contributes to one summand in $H_{(r_2,s)}(F, \partial_2)$, this proves the claim. The computation of the generators for $s > 0$ is given in Proposition 4.8. \square

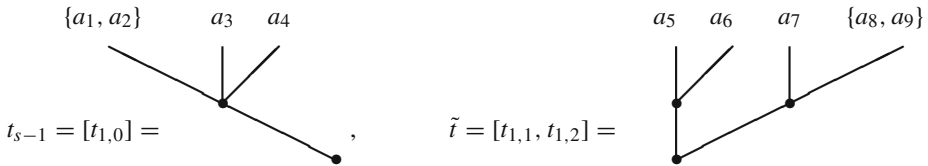
Example 4.5 Let $(t, \phi) = X \xrightarrow{\phi} [6] \xrightarrow{f} [2]$ be the following tree



where $(t, \phi) = [t_{1,0}, t_{1,1}, t_{1,2}]$ with



and $X_{1,0} = \{a_1, a_2, a_3, a_4\}$, $X_{1,1} = \{a_5, a_6\}$ and $X_{1,2} = \{a_7, a_8, a_9\}$. Let $\sigma : [2] \rightarrow [1]$ be the map assigning 0 to 0 and 1 to 1 and 2. One has $s = 1, p = 1$, so that $X_{s-1} = \{a_1, \dots, a_4\}$ and $\tilde{X} = \{a_5, \dots, a_9\}$. Moreover



Corollary 4.6 For any non-fork tree $(t, \phi) = X \xrightarrow{\phi} [r_2] \xrightarrow{f} [r_1], r_2 \neq r_1, \text{Epi}_2^{X^t, \phi}$ is acyclic. For any non-fork tree $t = [r_2] \xrightarrow{f} [r_1], r_2 \neq r_1, \text{Epi}_2^t$ is acyclic.

Proof The first assertion is a direct consequence of the first equation of Proposition 4.4. The second one is a direct consequence of relation (4.1). \square

Proposition 4.7 Let $(t, \phi) : X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]$ be a fork tree and let $X_i = \phi^{-1}(i)$. Then the top homology $H_{(r,0)}(\text{Epi}_2^{X^t, \phi}, \partial_2)$ is freely generated by $c_{r,X} := \sum_{\sigma \in \Sigma_{r+1}} \text{sgn}(\sigma; X)\sigma$, where the sign $\text{sgn}(\sigma; X)$ picks up a factor $(-1)^{(d(X_i)+1)(d(X_j)+1)}$ whenever $\sigma(i) > \sigma(j)$ but $i < j$.

In particular, for a fork tree $t : [r] \xrightarrow{\text{id}} [r]$, the top homology $H_{(r,0)}(\text{Epi}_2^t, \partial_2)$ is freely generated by $c_r := \sum_{\sigma \in \Sigma_{r+1}} \text{sgn}(\sigma)\sigma$.

Proof The second assertion is a consequence of the first one using relation (4.1). The computation of the top homology amounts to determining the kernel of the map

$$\bigoplus_{\psi} k \left[\text{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]; X \xrightarrow{\psi} [r] \rightarrow [0]) \right] \xrightarrow{\partial_2} \bigoplus_{\psi} k \left[\text{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]; X \xrightarrow{\psi} [r-1] \rightarrow [0]) \right].$$

The set $\text{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]; X \xrightarrow{\psi} [r] \rightarrow [0])$ is either empty or has only one element uniquely determined by the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\text{id}} & [r] \\ \text{id} \downarrow & & \downarrow \tau & & \downarrow \\ X & \xrightarrow{\psi = \tau \phi} & [r] & \longrightarrow & [0]. \end{array}$$

Hence, the surjection ψ determines a bijection τ and this induces a permutation of the set $\{X_0, \dots, X_r\}$. We denote such an element by $\tau \cdot X := (X_{\tau^{-1}(0)}, \dots, X_{\tau^{-1}(r)})$. As a consequence the computation of the top homology amounts to determining the kernel of the map

$$\partial_2 : k[\Sigma_{r+1}] \longrightarrow \bigoplus_{\psi} k \left[\text{Epi}_2^X(X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]; X \xrightarrow{\psi} [r-1] \rightarrow [0]) \right]$$

where

$$\partial_2(\tau \cdot X) = \sum_{i=0}^{r-1} (-1)^{i+d(X_{\tau^{-1}(0)})+\dots+d(X_{\tau^{-1}(i)})} \epsilon(X_{\tau^{-1}(i)}; X_{\tau^{-1}(i+1)}) \times (X_{\tau^{-1}(0)}, \dots, X_{\tau^{-1}(i)} \cup X_{\tau^{-1}(i+1)}, \dots, X_{\tau^{-1}(r)}).$$

Therefore, if $x = \sum_{\tau \in \Sigma_{r+1}} \lambda_\tau \tau \cdot X$ is in the kernel of ∂_2 , then for all transpositions $(i, i + 1)$ and all τ one has $\lambda_{(i,i+1)\tau} = (-1)^{1+d(X_{\tau^{-1}(i)})+d(X_{\tau^{-1}(i+1)})+d(X_{\tau^{-1}(i)})d(X_{\tau^{-1}(i+1)})} \lambda_\tau$. Since the transpositions generate the symmetric group one has $\lambda_\tau = \text{sgn}(\tau; X) \lambda_{\text{id}}$ and $x = \lambda_{\text{id}} c_{r, X}$. \square

For $s > 0$, the computation of the top homology of $(C_{*,s}^{E_2, X}(\text{Epi}_2^{X^t, \phi}), \partial_2)$ amounts to calculating the kernel of the map ∂_2

$$\bigoplus_{\psi, g \in \Delta^{\text{epi}}([r], [s])} k \left[\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [r] \xrightarrow{g} [s]) \right] \rightarrow \bigoplus_{\psi, h \in \Delta^{\text{epi}}([r-1], [s])} k \left[\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [r-1] \xrightarrow{h} [s]) \right].$$

We know from Proposition 4.4 that it is free of rank equal to the cardinality of $\Delta^{\text{epi}}([r], [s])$. As before, the set $\text{Epi}_2^X((t, \phi); X \xrightarrow{\psi} [r] \xrightarrow{g} [s])$ is either empty or has only one element determined by the commuting diagram

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\text{id}} & [r] \\ \downarrow \text{id} & & \downarrow \tau & & \downarrow g' \\ X & \xrightarrow{\psi = \tau \phi} & [r] & \xrightarrow{g} & [s]. \end{array}$$

An element g in $\Delta^{\text{epi}}([r], [s])$ is uniquely determined by the sequence (x_0, \dots, x_s) of the cardinalities of its preimages. Furthermore, any map in $\text{Epi}_2([r] \xrightarrow{\text{id}} [r]; [r] \xrightarrow{g} [s])$ is given by $g' : [r] \rightarrow [s]$ in Δ^{epi} and $\tau : [r] \rightarrow [r]$ in Σ_{r+1} such that $g' = g\tau$. This implies that $g' = g$ and $\tau \in \Sigma_{x_0} \times \dots \times \Sigma_{x_s}$. If there is such a τ satisfying $\psi = \tau\phi$ then the set is non-empty and τ is unique. Let $X_{(x_i)} = (g\phi)^{-1}(i)$. Then $X_{(x_i)}$ is a subset of X and there is a natural partition of it given by $X_{(x_i)} = \sqcup_{j \in g^{-1}(\{i\})} X_j$.

Let $c_{(x_0, \dots, x_s); X}$ be the element

$$c_{(x_0, \dots, x_s); X} = \left(\sum_{\sigma^0 \in \Sigma_{x_0}} \text{sgn}(\sigma^0; X_{(x_0)}) \sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \text{sgn}(\sigma^s; X_{(x_s)}) \sigma^s \right) \in k[\Sigma_{x_0} \times \dots \times \Sigma_{x_s}].$$

If every element of X has degree zero, we denote $(\sum_{\sigma^0 \in \Sigma_{x_0}} \text{sgn}(\sigma^0) \sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \text{sgn}(\sigma^s) \sigma^s)$ by $c_{(x_0, \dots, x_s)}$.

Proposition 4.8 *Let $(t, \phi): X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]$ be a fork tree. The top homology $H_{(r,s)}(\text{Epi}_2^{X^t, \phi}, \partial_2)$ is freely generated by the elements $c_{(x_0, \dots, x_s); X} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \text{sgn}(\sigma^0; X_{(x_0)}) \sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \text{sgn}(\sigma^s; X_{(x_s)}) \sigma^s)$, for $g = (x_0, \dots, x_s) \in \Delta^{\text{epi}}([r], [s])$, $X_{(x_k)} = (g\phi)^{-1}(\{k\})$.*

Let $t : [r] \xrightarrow{\text{id}} [r]$ be a fork tree. The top homology $H_{(r,s)}(\text{Epi}_2^t, \partial_2)$ is freely generated by the elements $c_{(x_0, \dots, x_s)} = (\sum_{\sigma^0 \in \Sigma_{x_0}} \text{sgn}(\sigma^0) \sigma^0, \dots, \sum_{\sigma^s \in \Sigma_{x_s}} \text{sgn}(\sigma^s) \sigma^s)$, for $(x_0, \dots, x_s) \in \Delta^{\text{epi}}([r], [s])$.

Proof As in the proof of Proposition 4.7 we compute the kernel of ∂_2 which decomposes into the sum of anti-commuting differentials $\partial_2 = D_0 + \dots + D_s$, as in the proof of Proposition 4.4. As a consequence $\ker(\partial_2) = \cap_i \ker(D_i)$, which gives the result. \square

Corollary 4.9 For any fork tree $(t, \phi) = X \xrightarrow{\phi} [r] \xrightarrow{\text{id}} [r]$, $\text{Epi}_2^{X^t, \phi}$ is acyclic. In particular, Epi_2^t is acyclic for any fork tree $t = [r] \xrightarrow{\text{id}} [r]$.

Proof It remains to compute the homology of the complex $((H_{(r,*)}(C^{E_2, X}(\text{Epi}_2^{X^t, \phi}), \partial_2), \partial_1)$ and prove that it vanishes for all $*$ if $r > 0$. From Propositions 4.7 and 4.8 one has

$$H_{(r,s)}(C^{E_2, X}(\text{Epi}_2^{X^t, \phi}), \partial_2) = \bigoplus_{(x_0, \dots, x_s) \in \Delta^{\text{epi}}([r], [s])} k c_{(x_0, \dots, x_s); X}.$$

To compute $\partial_1(c_{(x_0, \dots, x_s); X})$ it is enough to compute $\partial_1(\text{id}_{\Sigma_0 \times \dots \times \Sigma_s})$ in $C_{(r,s)}^{E_2, X}(\text{Epi}_2^{X^t, \phi})$. We apply relations (3.8) and (3.9):

$$\begin{aligned} & \partial_1 \left(\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\text{id}} & [r] \\ \downarrow \text{id} & & \downarrow \text{id} & & \downarrow (x_0, \dots, x_s) \\ X & \xrightarrow{\phi} & [r] & \xrightarrow{(x_0, \dots, x_s)} & [s] \end{array} \right) \\ &= \sum_{i=0}^{s-1} (-1)^{i+1+x_0+d(X_{x_0})+\dots+x_i+d(X_{x_i})} \left(\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\text{id}} & [r] \\ \downarrow \text{id} & & \downarrow \text{id} & & \downarrow d_i(x_0, \dots, x_s) \\ X & \xrightarrow{\phi} & [r] & \xrightarrow{d_i(x_0, \dots, x_s)} & [s-1] \end{array} \right) \\ & \pm \sum_{\xi} \left(\begin{array}{ccccc} X & \xrightarrow{\phi} & [r] & \xrightarrow{\text{id}} & [r] \\ \downarrow \text{id} & & \downarrow \xi & & \downarrow d_i(x_0, \dots, x_s) \\ X & \xrightarrow{\xi \phi} & [r] & \xrightarrow{d_i(x_0, \dots, x_s)} & [s-1] \end{array} \right), \end{aligned}$$

with ξ running over the $(X_{x_i}, X_{x_{i+1}})$ -shuffles with $\xi \neq \text{id}$. Thus,

$$\partial_1(c_{(x_0, \dots, x_s); X}) = \sum_{i=0}^{s-1} (-1)^{i+1+x_0+d(X_{x_0})+\dots+x_i+d(X_{x_i})} c_{(x_0, \dots, x_i+x_{i+1}, \dots, x_s); X}$$

and the complex $(H_{(r,*)}(C^{E_2, X}(\text{Epi}_2^{X^t, \phi}), \partial_2), \partial_1)$ agrees with the graded version of the complex $C_*^{\text{bar}}((\Delta^{\text{epi}})^r)$ of Remark 2.4. Therefore it is acyclic, with

$$H_0(C_*^{\text{bar}}(\Delta^{\text{epi}})^r) = \begin{cases} 0 & \text{if } r > 0, \\ k & \text{if } r = 0. \end{cases}$$

As a consequence the spectral sequence associated to the bicomplex $(C_{(*,*)}^{E_2, X}(\text{Epi}_2^{X^t, \phi}), \partial_1 + \partial_2)$ collapses at the E^2 -stage and one gets $H_p^{E_2, X}(\text{Epi}_2^{X^t, \phi}) = 0$ for all $p > 0$. \square

Proposition 4.10 *Let $X = \{x_0 < \dots < x_{r_n}\}$ be an ordered set of graded elements. Let $\phi : X \rightarrow [r_n]$ be the map sending x_i to i . Let $(t, \phi) = X \xrightarrow{\phi} [r_n] \xrightarrow{f_n} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1]$ be an (X, n) -level tree and let \bar{t} be its $(n - 1)$ -truncation $X[1] \xrightarrow{f_n\phi} [r_{n-1}] \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} [r_1]$, where $X[1]$ is the ordered set obtained from X by increasing the degree of its elements by 1, then*

$$H_{(*, s_{n-1}, \dots, s_1)}(\text{Epi}_n^{X^t, \phi}, \partial_n) = 0 \quad \text{if } r_n \neq r_{n-1},$$

$$H_{(*, s_{n-1}, \dots, s_1)}(\text{Epi}_n^{X^t, \phi}, \partial_n) \cong \begin{cases} 0 & \text{for } * \neq r_n \\ C_{(s_{n-1}, \dots, s_1)}^{E_{n-1}, X[1]}(\text{Epi}_{n-1}^{X[1]^{\bar{t}}, \phi}) & \text{for } s_{n-1} \leq * = r_n \end{cases} \quad \text{if } r_n = r_{n-1}.$$

Furthermore when $f_n = \text{id}$, the $(n - 1)$ -complex structure induced on $H_{(r_n, s_{n-1}, \dots, s_1)}(\text{Epi}_n^{X^t, \phi}, \partial_n)$ by the n -complex structure of $C_{(*, \dots, *)}^{E_n, X}(\text{Epi}_n^{X^t, \phi})$ coincides with the one on $C_{(s_{n-1}, \dots, s_1)}^{E_{n-1}, X[1]}(\text{Epi}_{n-1}^{X[1]^{\bar{t}}, \phi})$.

Proof Recall from Definition 3.9 that

$$\begin{aligned} \partial_n & \left(\begin{array}{ccccccc} X & \xrightarrow{\phi} & [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \dots & \xrightarrow{f_2} & [r_1] \\ \downarrow \text{id} & & \downarrow \sigma_n & & \downarrow \sigma_{n-1} & & & & \downarrow \sigma_1 \\ X & \xrightarrow{\sigma_n \phi} & [s_n] & \xrightarrow{g_n} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \dots & \xrightarrow{g_2} & [s_1] \end{array} \right) \\ &= \sum_{i | g_n(i) = g_n(i+1)} (-1)^{s_{n,i}} \in ((\sigma_n \phi)^{-1}(i); (\sigma_n \phi)^{-1}(i+1)) \\ & \quad \times \begin{array}{ccccccc} X & \xrightarrow{\phi} & [r_n] & \xrightarrow{f_n} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \dots & \xrightarrow{f_2} & [r_1] \\ \downarrow \text{id} & & \downarrow d_i \sigma_n & & \downarrow \sigma_{n-1} & & & & \downarrow \sigma_1 \\ X & \xrightarrow{d_i \sigma_n \phi} & [s_n - 1] & \xrightarrow{g_n |_{i=i+1}} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \dots & \xrightarrow{g_2} & [s_1]. \end{array} \end{aligned}$$

The same proof as in Proposition 4.4 provides the computation of the homology of the complex with respect to the differential ∂_n : if t is not a fork tree, then the homology of the complex vanishes, and if t is the fork tree $f_n = \text{id}_{[r_{n-1}]}$, then its homology groups are concentrated in top degree r_n . Let us describe all the bijections τ of $[r_{n-1}]$ such that the following diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{\phi} & [r_{n-1}] & \xrightarrow{\text{id}} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \dots & \xrightarrow{f_2} & [r_1] \\ \downarrow \text{id} & & \downarrow \tau & & \downarrow \sigma_{n-1} & & & & \downarrow \sigma_1 \\ X & \xrightarrow{\tau \phi} & [r_{n-1}] & \xrightarrow{g_n} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \dots & \xrightarrow{g_2} & [s_1]. \end{array}$$

Let $(x_0, \dots, x_{s_{n-1}})$ be the sequence of cardinalities of the preimages of σ_{n-1} , which also determines g_n . There exists a bijection ξ of $[r_{n-1}]$ such that $\sigma_{n-1} = g_n \xi$. If ξ, ξ' are bijections of $[r_{n-1}]$ both satisfying the previous equality then $\xi(\xi')^{-1} \in \Sigma_{x_0} \times \dots \times \Sigma_{x_{s_{n-1}}}$. Any element τ that makes the diagram commute is of the form $\alpha \xi$ for $\alpha \in \Sigma_{x_0} \times \dots \times \Sigma_{x_{s_{n-1}}}$. As in Proposition 4.8, the element $\text{sgn}(\xi; X)(c_{(x_0, \dots, x_{s_{n-1}}); X}) \xi$ does not depend on the choice of ξ and it is a generator of $H_{(r_n, s_{n-1}, \dots, s_1)}(\text{Epi}_n^{X^t, \phi}, \partial_n)$. This gives the desired isomorphism of k -modules between this homology group and $C_{(s_{n-1}, \dots, s_1)}^{E_{n-1}, X[1]}(\text{Epi}_{n-1}^{X[1]^{\bar{t}}, \phi})$.

A direct inspection of the signs in Definition 3.9 shows that the induced differential ∂_i coincides with the one on $C_{(s_{n-1}, \dots, s_1)}^{E_{n-1}, X[1]}(\text{Epi}_{n-1}^{X[1]^i, \phi})$ for $1 \leq i \leq n-2$. The case $i = n-1$ is similar to the proof of Corollary 4.9: let us choose a bijection $\xi: [r_{n-1}] \rightarrow [r_{n-1}]$ with the property that $i < j$ and $g_n \xi(i) = g_n \xi(j)$ imply $\xi(i) < \xi(j)$. We take $\text{sgn}(\xi; X)(c_{(x_0, \dots, x_{s_{n-1}})}; X)\xi$ as the corresponding generator.

On the one hand, one has $\partial_{n-1}(\xi, \sigma_{n-1}, \dots, \sigma_1) = \sum_{j|g_{n-1}(j)=g_{n-1}(j+1)} (-1)^{s_{n-1}.j} \epsilon_j \alpha_j$ where α_j is a sum of trees which is a generator of the form $\text{sgn}(\xi; X)c_{(x_0, \dots, x_j+x_{j+1}, \dots, x_{s_{n-1}})}; X\xi$ and ϵ_j is a sign to be determined. If $Y_j = \{y_1 < \dots < y_{x_j}\}$ and $Y_{j+1} = \{z_1 < \dots < z_{x_{j+1}}\}$ are the sets $(g_{n-1}g_n \xi \phi)^{-1}(j)$ and $(g_{n-1}g_n \xi \phi)^{-1}(j+1)$ respectively then the sign ϵ_j is precisely $\epsilon(Y_j[1]; Y_{j+1}[1])$. On the other hand, computing

$$\partial_{n-1} \left(\begin{array}{ccccccc} X[1] & \xrightarrow{\phi} & [r_{n-1}] & \xrightarrow{f_{n-1}} & \dots & \xrightarrow{f_2} & [r_1] \\ \downarrow \text{id} & & \downarrow \sigma_{n-1} & & & & \downarrow \sigma_1 \\ X[1] & \xrightarrow{\sigma_{n-1}\phi} & [s_{n-1}] & \xrightarrow{g_{n-1}} & \dots & \xrightarrow{g_2} & [s_1] \end{array} \right)$$

gives the same signs for $Y_j[1] = (g_{n-1}\sigma_{n-1}\phi)^{-1}(j) = (g_{n-1}g_n \xi \phi)^{-1}(j)$. □

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