GALOIS THEORY AND LUBIN-TATE COCHAINS ON CLASSIFYING SPACES

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Abstract. We consider brave new cochain extensions $F(BG_+, R) \to F(EG_+, R)$, where $R$ is either a Lubin-Tate spectrum $E_n$ or the related 2-periodic Morava K-theory $K_n$, and $G$ is a finite group. When $R$ is an Eilenberg-Mac Lane spectrum, in some good cases such an extension is a $G$-Galois extension in the sense of John Rognes, but not always faithful. We prove that for $E_n$ and $K_n$ these extensions are always faithful in the $K_n$-local category. However, for a cyclic $p$-group $C_{p^r}$, the cochain extension $F(BC_{p^r}+, E_n) \to F(EC_{p^r}+, E_n)$ is not a Galois extension because it ramifies. As a consequence, it follows that the $E_n$-theory Eilenberg-Moore spectral sequence for $G$ and $BG$ does not always converge to its expected target.

1. Introduction

In the algebraic Galois theory of commutative rings [6], faithful flatness is a property implied by separability. However, in the topological analogue, the brave new Galois theory of Rognes [19], this is not true. The simplest counterexample, due to Ben Wieland [20], is provided by the $C_2$-Galois extension

\[(1.1) \quad F(BC_2+, H\mathbb{F}_2) \to F(EC_2+, H\mathbb{F}_2) \sim H\mathbb{F}_2\]

which is not faithful. This example relies on the algebraic fact that

\[\pi_*(F(BC_2+, H\mathbb{F}_2)) = H^{-*}(BC_2; \mathbb{F}_2)\]

is a polynomial algebra and so has finite global dimension.

In this note we consider this question for a Lubin-Tate spectrum $E_n$ and the related Morava $K$-theory $K_n$, and show that for any finite group $G$, the extension

\[(1.2) \quad E_n^{BG} = F(BG_+, E_n) \to F(EG_+, E_n) \sim E_n\]

is faithful as an $E_n$-module. We also show that the non-commutative extension

\[(1.3) \quad F(BG_+, K_n) \to F(EG_+, K_n) \sim K_n\]

is faithful and $F(BG_+, K_n)$ is a faithful $E_n$-module. A crucial difference from $F(BG_+, H\mathbb{F}_p)$ is that $K_n(BG_+)$ is always an Artinian algebra over $(K_n)_*$, and so if $K_n(BG_+) \neq K_n^*$ then it has infinite global dimension by Proposition 2.2.

Our approach to this involves introducing an analogue of the algebraic socle series for a module over an Artinian ring, and we show that this behaves well enough to prove our result.

We show in Section 5 that for a cyclic $p$-group $C_{p^r}$, the cochain extension $F(BC_{p^r}+, E_n) \to F(EC_{p^r}+, E_n)$ is ramified and hence it is not a Galois extension. As a consequence it follows that the $E_n$-theory Eilenberg-Moore spectral sequence for such groups does not converge to its expected target, whereas work of Tilman Bauer indicates that this is not the case for Morava $K$-theory.

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Notation, etc. In discussing purely algebraic notions we will often use boldface symbols $A, M, \ldots$ to denote rings, modules, etc, while for topological objects such as $S$-algebras and their modules we will use italic symbols $A, M, \ldots$, thereby hopefully reducing the possibility of confusion between the two settings. For an associative $S$-algebra $A$, we denote by $\mathcal{D}_A$ the derived category of $A$-module spectra defined in [7, chapter III, construction 2.11].

We follow Lam [12, theorem 19.1] in using the phrase local ring to indicate a ring with a unique maximal left ideal (necessarily 2-sided and equal to its Jacobson radical); the quotient of such a ring by its Jacobson radical is a division ring. For non-commutative rings other terminology is often encountered such as scalar local ring.

Brave new Galois extensions. The following definition of a Galois extension is due to John Rognes [19]. Let $A$ be a commutative $S$-algebra and let $B$ be a commutative cofibrant $A$-algebra. Let $G$ be a finite (discrete) group and suppose that there is an action of $G$ on $B$ by commutative $A$-algebra morphisms. Then $B/A$ is a $G$-Galois extension if it satisfies the following two conditions:

- The natural map $A \longrightarrow B^{hG} = F(EG_+, B)^G$ is a weak equivalence of $A$-algebras.
- There is a natural equivalence of $B$-algebras
  \[ \Theta: B \wedge_A B \sim \longrightarrow F(G_+, B) \]
  induced from the action of $G$ on the right hand factor of $B$.

Furthermore, $B/A$ is a faithful $G$-Galois extension if it also satisfies

- $B$ is faithful as an $A$-module, i.e., for any $A$-module $M$, $B \wedge_A M \sim \ast$ implies that $M \sim \ast$.

Examples like (1.1) show that not every Galois extension is faithful.

2. Recollections on modules over Artinian algebras

In this section we review some standard algebraic background material; good sources for this are [1] [12].

Let $D$ be a division ring. A ring $A$ equipped with homomorphisms of rings $\eta: D \longrightarrow A$ and $\varepsilon: A \longrightarrow D$ is an augmented $D$-algebra if the following diagram commutes.

\[
\begin{array}{ccc}
D & \xrightarrow{=} & D \\
\eta \downarrow & & \downarrow \varepsilon \\
A & &
\end{array}
\]

The augmentation $\varepsilon$ splits the unit $\eta$. We will also say that $A$ is an Artinian local $D$-algebra if it is Artinian and local.

If $A$ is an Artinian local augmented $D$-algebra, then the Jacobson radical of $A$ is

\[ J = \text{rad}(A) = \ker \varepsilon. \]

By [12, theorem 4.12], $J$ is nilpotent, say $J^e = 0$ and $J^{e-1} \neq 0$.

Lemma 2.1. Let $A$ be as above and let $M$ be a left $A$-module. If $D \otimes_A M = 0$, then $M = 0$.

Proof. Comparing the two horizontal exact sequences

\[
\begin{array}{c}
J \otimes_A M \longrightarrow A \otimes_A M \longrightarrow D \otimes_A M \longrightarrow 0 \\
0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0
\end{array}
\]

we see that if $D \otimes_A M = 0$ then

\[ M = JM = \ldots = J^e M = 0. \]

\[ \square \]
Let $M$ be a left $A$-module. The socle of $M$ is the submodule
\[ \text{soc}^1 \ M = \text{soc} \ M = \{ x \in M : Jx = 0 \}, \]
which can also be characterized as the sum of all the simple $A$-submodules of $M$. The socle series of $M$ is the increasing sequence of submodules
\[ 0 = \text{soc}^0 \ M \subseteq \text{soc}^1 \ M \subseteq \ldots \subseteq \text{soc}^k \ M \subseteq \text{soc}^{k+1} \ M \subseteq \ldots \subseteq M, \]
where for each $k$ the following is a pullback square
\[
\begin{array}{ccc}
\text{soc}^{k+1} \ M & \longrightarrow & \text{soc}(M/\text{soc}^k \ M) \\
\downarrow & & \downarrow \\
M & \longrightarrow & M/\text{soc}^k \ M
\end{array}
\]
so we have
\[ \text{soc}^k \ M = \{ x \in M : J^k x = 0 \}, \]
and
\[ \text{soc}^e \ M = M. \]
In fact, for small $k$
\[ \text{soc}^k \ M \subseteq \text{soc}^{k+1} \ M, \]
until we reach a value $k = k_0 \leq e$ for which $\text{soc}^{k_0} \ M = M$.

It is also clear that given a homomorphism $\varphi : M \longrightarrow N$ of $A$-modules there are compatible homomorphisms
\[ \text{soc}^k \ M \longrightarrow \text{soc}^k \ N. \]

For details on the socle series see [12], especially Ex. 4.18, and [1, chapter I, section 1].

We end this section with a result that supplies an algebraic backdrop for some of our later work. We give a proof suggested by K. Brown.

**Proposition 2.2.** Let $A$ be a local left-Artinian ring which is not a division ring. Then
\[ \text{proj dim}(A/\text{rad}(A)) = \text{gl dim} \ A = \infty, \]
where $A/\text{rad}(A)$ is the unique simple left $A$-module.

**Proof.** Since $A$ is local, it has only one simple module and therefore
\[ \text{proj dim}(A/\text{rad}(A)) = \text{gl dim} \ A. \]
Also, since $A$ is Artinian it has a left ideal $I$ isomorphic to $A/\text{rad}(A)$. The corresponding exact sequence
\[ (2.1) \quad 0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0 \]
cannot split since $A$ is local and therefore it has no non-trivial idempotents.

If
\[ \text{proj dim}(A/\text{rad}(A)) = \text{gl dim} \ A < \infty, \]
then (2.1) would give
\[ \text{proj dim}(A/\text{rad}(A)) + 1 = \text{proj dim}(A/I) \leq \text{gl dim} \ A = \text{proj dim}(A/\text{rad}(A)), \]
which is impossible. \qed

**Remark 2.3.** We end this section by noting that the above discussion works as well if we assume that $A$ is graded, provided this is suitably interpreted. In our work below we are interested in $\mathbb{Z}$-gradings which are also 2-periodic, i.e., for all $n \in \mathbb{Z}$, $(-)^{n+2} = (-)^n$. This can be interpreted as a $\mathbb{Z}/2$-grading.
3. Socle series in topology

Let $D$ be an $S$-algebra for which $\pi_0 D$ is a non-trivial division ring, $\pi_1 D = 0$, and the graded ring $\pi_* D = D$ has period two. Suppose that $A$ is an $S$-algebra both under and over $D$, giving the following diagram of morphisms of $S$-algebras.

\[
\begin{array}{ccc}
D & \cong & D \\
\downarrow \cong & & \downarrow \cong \\
A & \rightarrow & A \\
\end{array}
\]

We assume that $A = \pi_* A$ is an Artinian local augmented $D$-algebra, so that the augmentation ideal $\ker \varepsilon$ is the Jacobson radical of $A$, $\rad(A)$, and also $\rad(A)^e = 0$ and $\rad(A)^e-1 \neq 0$.

**Remark 3.1.** Let $M$ be a left $A$-module. Then $M = \pi_* M$ is a left $A$-module and its socle $soc M$ is a $D$-module through both the unit $\eta$ and the augmentation $\varepsilon$, and these module structures agree since $\rad(A) = \ker \varepsilon$.

**Theorem 3.2.** There are functors $\soc^k : \mathcal{D}_A \rightarrow \mathcal{D}_A$ for $0 \leq k \leq e$ such that

(a) for each $k$, $\pi_* (soc^k M) = soc^k M$;

(b) there are natural transformations $soc^k M \rightarrow soc^{k+1} M$ giving a commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & \pi_* soc^1 M & \rightarrow & \pi_* soc^2 M & \rightarrow & \cdots & \rightarrow & \pi_* soc^e M & \rightarrow & 0 \\
\cong & & \cong & & \cong & & & & \cong & & \\
0 & \rightarrow & soc^1 M & \rightarrow & soc^2 M & \rightarrow & \cdots & \rightarrow & soc^e M & \rightarrow & 0
\end{array}
\]

which is natural with respect to morphisms of $A$-modules.

**Proof.** As $D$ is a graded division ring, $soc M$ is a $D$-vector space. Since $M$ is a $D$-module via the unit we can find a morphism of $D$-modules

\[
\bigvee_j \Sigma^s(j) D \rightarrow M
\]

to realize an algebraic isomorphism

\[
\bigoplus_j D_{s-s(j)} \cong \soc M \subseteq M.
\]

Now Remark 3.1 implies that the morphism of (3.2) is actually one of $A$-modules. We set $soc M = \bigvee_j \Sigma^s(j) D$.

Now we can repeat this on the cofibre $M/soc M$ of the map $soc M \rightarrow M$, obtaining $soc(M/soc M) \rightarrow M/soc M$. We then define $soc^2 M$ using the right hand pullback square in the diagram

\[
\begin{array}{ccc}
soc M & \rightarrow & soc^2 M \\
\downarrow & & \downarrow \\
M & \rightarrow & M/soc M
\end{array}
\]

from which we see by a standard diagram chase that $\pi_* (soc^2 M) \cong soc^2 M$. Continuing in this way we inductively build the socle tower

\[
* \rightarrow soc^1 M \rightarrow soc^2 M \rightarrow \ldots \rightarrow soc^{e-1} M \rightarrow soc^e M = M,
\]

using pullback squares

\[
\begin{array}{ccc}
soc^{k+1} M & \rightarrow & soc(M/soc^k M) \\
\downarrow & & \downarrow \\
M & \rightarrow & M/soc^k M
\end{array}
\]
for each \( k \). These satisfy
\[
\pi_*(\text{soc}^k M) = \text{soc}^k M. \quad \square
\]

An important consequence of this construction is that there is a minimal \( k_0 \) for which \( \text{soc}^{k_0} M = M \), so since \( \text{soc}^{k_0-1} M \neq M \), using the fibre sequence
\[
(3.3) \quad \text{soc}^{k_0-1} M \to M \to M/\text{soc}^{k_0-1} M,
\]
we obtain \( \pi_*(M/\text{soc}^{k_0-1} M) \neq 0 \).

**Lemma 3.3.** The \( A \)-module \( D \) satisfies \( \pi_*(D \wedge_A D) \neq 0 \).

**Proof.** There is a diagram of left \( D \)-modules induced from \((3.1)\)
\[
\begin{array}{ccc}
D \wedge_D D & = & D \wedge_D D \\
\downarrow & & \downarrow \\
D \wedge_A D & & D \wedge_A D
\end{array}
\]
in which \( D \wedge_D D \cong D \). On applying \( \pi_*(-) \) we see that \( \pi_*(D \wedge_A D) \neq 0 \). \( \square \)

**Theorem 3.4.** Let \( M \) be an \( A \)-module for which \( \pi_* M \neq 0 \). Then \( \pi_*(D \wedge_A M) \neq 0 \), i.e., \( D \) is a faithful \( A \)-module.

**Proof.** Using the socle series we can find a fibration sequence as in \((3.3)\),
\[
(3.4) \quad M' \to M \to M'',
\]
where \( M'' = \pi_* M'' \neq 0 \), \( JM'' = 0 \) and there is a short exact sequence
\[
(3.5) \quad 0 \to \pi_*(M') \to \pi_*(M) \to \pi_*(M'') \to 0.
\]

As remarked in the proof of Theorem \((3.2)\), \( M'' \) is weakly equivalent to a wedge of copies of suspensions of the \( A \)-module \( D \). So \( \pi_*(M'') \) is a direct sum of copies of suspensions of \( \pi_*(D) \), hence by Lemma \((3.3)\), \( \pi_*(M'') \neq 0 \). The fibre sequence \((3.4)\) induces a commutative diagram
\[
\begin{array}{ccc}
0 & \to & \pi_*(D \wedge_D M') \\
\downarrow & & \downarrow \\
\pi_*(D \wedge_A M') & \to & \pi_*(D \wedge_A M)
\end{array}
\begin{array}{ccc}
\to & \pi_*(D \wedge_D M) & \to & \pi_*(D \wedge_D M'') \\
\to & \pi_*(D \wedge_A M) & \to & \pi_*(D \wedge_A M'') \\
\to & \pi_*(D \wedge_D M'') & \to & \pi_*(D \wedge_D M'')
\end{array}
\]
in which a non-zero element \( x \in \pi_*(D \wedge_D M'') \) lifts to \( \pi_*(D \wedge_D M) \) and so is in the image of composition passing through \( \pi_*(D \wedge_A M) \). Therefore \( \pi_*(D \wedge_A M) \neq 0 \). \( \square \)

\[ \text{4. Lubin-Tate cohomology of classifying spaces} \]

We will denote by \( E \) any Lubin-Tate spectrum such as \( E_n \) or \( E_n^{ur} \), and then \( K \) will denote the corresponding version of Morava \( K \)-theory see \([3]\) for details. The spectrum \( E \) is a commutative \( S \)-algebra, while \( K \) is an \( E \)-algebra in the sense of \([7]\). The homotopy groups \( \pi_* E \) and \( \pi_* K \) are 2-periodic and \( \pi_0 E \) is Noetherian; \( \pi_0 K \) is a field, although \( K \) is only homotopy commutative if \( p \) is an odd prime, while when \( p = 2 \) it is not even that. Nevertheless, we will view \( K \) as a kind of ‘topological division ring’.

The following lemma will allows us in certain circumstances to relate modules over \( E^{BG} = F(BG_+, E) \) to modules over \( K^{BG} = F(BG_+, K) \).
Lemma 4.1. For any $E^{BG}$-module $M$, there is isomorphism of $K$-modules

$$K \wedge_{E^{BG}} M \cong (K \wedge E) \wedge_{K \wedge E^{BG}} (K \wedge M).$$

In particular, there is an isomorphism of $K$-modules

$$K \wedge_{E^{BG}} E \cong K \wedge_{K^{BG}} K.$$

Proof. This follows from an obvious generalization of [7, proposition III.3.10]. Since there are isomorphisms of $E$-algebras $K \cong K \wedge E$ and $K^{BG} \cong K \wedge E^{BG}$, for any $E^{BG}$-module $M$,

$$K \wedge_{E^{BG}} M \cong K \wedge (E \wedge_{E^{BG}} M) \cong (K \wedge K) \wedge (E \wedge_{E^{BG}} M) \cong (K \wedge E) \wedge_{K \wedge E^{BG}} (K \wedge M).$$

Remark 4.2. By a standard argument making use of the Becker-Gottlieb transfer [5], after $p$-localization, $\Sigma^{\infty}BG_{+}$ is a retract of $\Sigma^{\infty}BG'_{+}$ where $G'$ is any $p$-Sylow subgroup of $G$. In particular, when $p \nmid |G|$ we have

$$F(BG_{+}, E) \sim E, \quad F(BG_{+}, K) \sim K.$$

Theorem 4.3. Let $G$ be a finite group.

(a) The $K$-cohomology $K^{*}(BG_{+})$ is a finite dimensional $K^{*}$-vector space and the $E$-cohomology $E^{*}(BG_{+})$ is a finitely generated $E^{*}$-module.

(b) If $K^{*}(BG_{+})$ is concentrated in even degrees, then $E^{*}(BG_{+})$ is a free $E^{*}$-module of finite rank and

$$K^{*}(BG_{+}) = K^{*} \otimes_{E^{*}} E^{*}(BG_{+}) = E^{*}(BG_{+})/mE^{*}(BG_{+}).$$

(c) $K^{*}(BG_{+})$ is an augmented Artinian local $K^{*}$-algebra whose maximal ideal is nilpotent. Hence $E^{*}(BG_{+})$ is an augmented pro-Artinian local $E^{*}$-algebra,

$$E^{*}(BG_{+}) = \lim_{r} E^{*}(BG_{+})/m^{r}E^{*}(BG_{+}).$$

Proof. (a) See [8, 9] for example.

(b) See [10, proposition 2.5].

(c) Following Remark 4.2 we can reduce to the case where $G$ is a $p$-group using the transfer associated with a $p$-Sylow subgroup $G' \leq G$. The case of a cyclic $p$-group $C_{p^{r}}$ is well known and

$$K^{*}(BC_{p^{r}}) = K^{*}[y]/(y^{p^{r}}).$$

The case of a general $p$-group $G$ of order $p^{m}$ follows by induction on $m$ since there is always a normal subgroup $N \triangleleft G$ of index $p$ and this permits an argument with the Serre spectral sequence associated with the fibration

$$BN \rightarrow BG \rightarrow BC_{p}$$

as used in [16] to calculate $K^{*}(BG_{+})$ from knowledge of $K^{*}(BN_{+})$ as input.

It is known that $K^{*}(BG_{+})$ need not be concentrated in even degrees [11].

We are interested in the $E$-algebras $E^{BG} = F(BG_{+}, E)$ and $K^{BG} = F(BG_{+}, K)$, each of which is $K$-local. Of course the diagonal $BG \rightarrow BG \times BG$ induces the product on each of these, but only $E^{BG}$ is strictly commutative, while $K^{BG}$ is homotopy commutative when $p \neq 2$ and merely associative when $p = 2$. At the level of homotopy groups, $E^{*}(BG_{+}) = \pi_{*}(E^{BG})$ and $K^{*}(BG_{+}) = \pi_{*}(K^{BG})$ are both graded commutative.

Now we can apply our earlier results to give

Theorem 4.4. For any finite group $G$, $E$ and $K$ are faithful $E^{BG}$-modules in the $K$-local category.
Proof. It suffices to show that $K$ is faithful. By Lemma 4.1 for any $E^{BG}$-module there is an isomorphism

\[ K \wedge_{EBG} M \cong (K \wedge_E E) \wedge_{K \wedge_E B} (K \wedge_E M). \]

The natural morphism of $E$-algebras

\[ K \wedge_E F(BG_+, E) \longrightarrow F(BG_+, K \wedge_E E) \]

is a weak equivalence since $K$ is a finite cell $E$-module, so by [7, theorem III.4.2] it is enough to know that

\[ (K \wedge_E E) \wedge_{K^{BG}} (K \wedge_E M) \cong K \wedge_{K^{BG}} (K \wedge_E M) \sim *. \]

If $M$ is $K$-local and non-trivial, then $K \wedge_{K^{BG}} (K \wedge_E M) \sim *$, because we know from Theorem 3.4 that $K$ is faithful as a $K^{BG}$-module. \hfill \Box

5. Galois theory and $E^{BG}$

In this section we will consider extensions of the form

\[ E^{BG} = F(BG_+, E) \longrightarrow F(EG_+, E) \sim E \]

with $G$ a finite group and consider whether or not they are Galois. Since we know they are faithful, the issue is whether such an extension satisfies the unramified condition that the map

\[ \Theta : F(BG_+, E) \wedge_{EBG} F(BG_+, E) \longrightarrow F(G_+, E) \]

is a weak equivalence, and therefore there is a weak equivalence

\[ (5.1) \quad E \wedge_{EBG} E \sim \prod G. \]

In particular, this condition implies that $\pi_*(E \wedge_{EBG} E)$ is concentrated in even degrees.

We begin by considering the case of cyclic $p$-groups $C_{p^r}$.

Theorem 5.1. For each $r \geq 1$, the extension

\[ E^{BC_{p^r}} = F(BC_{p^r}+, E) \longrightarrow F(EC_{p^r}+, E) \]

is ramified and hence it is not $C_{p^r}$-Galois.

Proof. We recall (see for example [9, lemma 5.1]) that

\[ (E^{BC_{p^r}})_* = E^*[y]/([p^r]y), \]

where $y \in (E^{BC_{p^r}})_0 = E^0(BC_{p^r}+)$ and the $p$-series $[p]y$ has the form

\[ [p]y = y^{p^n} \mod m, \]

so for each $r \geq 1$ the $p^r$-series is inductively defined by

\[ [p^r]y = [p]([p^{r-1}]y) = p^r y + \cdots + y^{p^n} + \cdots \equiv y^{p^{r^n}} \mod m. \]

By the Weierstrass preparation theorem, there is a polynomial

\[ (p^r)y = p^r + \cdots + y^{p^{r^n}} - 1 \equiv y^{p^{r^n}} - 1 \mod m \]

for which

\[ [p^r]y = y(p^r)(1 + y f_r(y)), \]

where $f_r(y) \in E^*[y]$. Then we have

\[ (E^{BC_{p^r}})_* = E^*[y]/(y(p^r)y). \]

The $(E^{BC_{p^r}})_*$-module $E_*$ admits the periodic minimal free resolution

\[ (5.2) \quad 0 \leftarrow E_* \leftarrow (E^{BC_{p^r}})_* \leftarrow (E^{BC_{p^r}})_* \leftarrow (E^{BC_{p^r}})_* \leftarrow \cdots, \]
so $\text{Tor}^{(E_{BC_p'})_*}(E_*, E_*)$ is the homology of the complex

$$0 \leftarrow E_* \otimes_{(E_{BC_p'})_*} (E_{BC_p'})_* \xleftarrow{I \otimes y} E_* \otimes_{(E_{BC_p'})_*} (E_{BC_p'})_* = E_* \otimes_{(E_{BC_p'})_*} (E_{BC_p'})_* \leftarrow \ldots,$$

which is equivalent to

$$(5.3) \quad 0 \leftarrow E_* \leftarrow E_* \leftarrow E_* \leftarrow E_* \leftarrow \ldots.$$

Since $E_*$ is torsion-free, for $s \geq 0$ this gives

$$\text{Tor}^{(E_{BC_p'})_*}(E_*, E_*) = \begin{cases} E_* & \text{if } s = 0, \\ E_* / p^s E_* & \text{if } s \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus in the Künneth spectral sequence

$$E^2_{s,t} = \text{Tor}^{(E_{BC_p'})_*}(E_*, E_*) \Longrightarrow \pi_{s+t}(E \wedge_{E_{BC_p'}} E)$$

there can be no non-trivial differentials since for degree reasons the only possibilities involve $E_*$-module homomorphisms of the form

$$d^{2k-1} : E^2_{2k-1,t} = E_t / p^t E_t \longrightarrow E^2_{t+2k-2} = E_{t+2k-2},$$

with torsion-free target. This shows that the odd degree terms in $\pi_*(E \wedge_{E_{BC_p'}} E)$ are not zero, contradicting the unramified condition (5.1) for a Galois extension.

**Remark 5.2.** If we work rationally, then the Künneth spectral sequence

$$E^2_{s,t}(C_p' ; \mathbb{Q}) = \text{Tor}^{((E_{BC_p'})_Q)_*}(E_* \mathbb{Q}, E_* \mathbb{Q}) \Longrightarrow \pi_{s+t}(E \mathbb{Q} \wedge_{(E_{BC_p'})_Q} E \mathbb{Q})$$

has $E^2_{s,t}(C_p' ; \mathbb{Q}) = 0$ except when $s = 0$, giving

$$\pi_*(E \mathbb{Q} \wedge_{(E_{BC_p'})_Q} E \mathbb{Q}) = E_* \mathbb{Q} \otimes_{(E_{BC_p'})_Q} E_* \mathbb{Q}.$$

This shows that higher filtration terms in the Künneth spectral sequence contribute $p$-torsion.

Now we extend Theorem (5.1) to arbitrary $p$-groups.

**Theorem 5.3.** Let $G$ be a non-trivial $p$-group. Then the extension

$$F(BG_+, E) \longrightarrow F(EG_+, E)$$

is not $G$-Galois. More precisely, this extension is ramified:

$$F(EG_+, E) \wedge_{F(BG_+, E)} F(EG_+, E) \simeq \prod_G F(EG_+, E).$$

**Proof.** Choose a non-trivial epimorphism $G \longrightarrow C_p$; then for some $k \geq 1$ there is a factorization

$$(5.6) \quad G \twoheadrightarrow C_p^k \longrightarrow C_p$$

inducing morphisms between the associated Künneth spectral sequences

$$(5.7) \quad E^*_{s+1}(C_p^k) \longrightarrow E^*_{s+1}(C_p) \longrightarrow E^*_{s+1}(C_p^k).$$

As we saw in the proof of Theorem (5.1) the two outer spectral sequences have trivial differentials. We will analyze the composite morphism $E^*_{s+1}(C_p^k) \longrightarrow E^*_{s+1}(C_p^k)$.

On choosing generators appropriately, the canonical epimorphism $C_p^k \longrightarrow C_p$ induces the $E_*$-algebra monomorphism

$$E_{BC_p'} = E_*(y)/([p]y) \longrightarrow (E_{BC_p'})_* = E_*(y)/([p]y): \quad y \mapsto [p^{k-1}]y,$$
hence the induced map between the two resolutions of the form \[5.2\] is

\[
\begin{array}{cccccccc}
0 & \to & E_* & \to & (E^{B_{C_r}})_* & \to & (E^{B_{C_p}})_* & \to & \cdots \\
\rho_0 & \downarrow & \rho_1 & \downarrow & \rho_2 & \downarrow & \cdots \\
0 & \to & E_* & \to & (E^{B_{C_{pk}}})_* & \to & (E^{B_{C_{pk}}})_* & \to & \cdots \\
\end{array}
\]

where the vertical maps are given by

\[
\rho_{2s}: g(y) \mapsto g([p^{k-1}]y), \quad \rho_{2s-1}: h(y) \mapsto h([p^{k-1}]y)(p^{k-1})y.
\]

Applying \(E_* \otimes (E^{B_{C_{p^r}}})_*(-)\) to the first and second rows with \(r = 1\) and \(k\) respectively, we obtain a map of chain complexes

\[
\begin{array}{cccccccc}
0 & \to & E_* & \to & 0 & \to & E_* & \to & 0 & \cdots \\
\rho_0' & \downarrow & \rho_1' \cdot p^{k-1} & \downarrow & \rho_2' & \downarrow & \cdots \\
0 & \to & E_* & \to & 0 & \to & E_* & \to & 0 & \cdots \\
\end{array}
\]

where

\[
\rho_2' = \text{id}, \quad \rho_2' \cdot 1 = p^{k-1}.\]

Applying this to the odd degree terms given in \[5.4\] we see that the induced map

\[
E_* / pE_* \stackrel{p^{k-1}}{\longrightarrow} E_* / p^k E_*
\]

is always a monomorphism. Therefore in \[5.7\], the first of the induced morphisms

\[
E_{s}^2(C_p) \longrightarrow E_{s}^*(G) \longrightarrow E_{s}^*(C_{pk})
\]

is a monomorphism. There can be no higher differentials killing elements in its image because they map to non-trivial elements of \(E_{s}^2(C_{pk})\) which survive the right hand spectral sequence. This shows that \(E_{s}^*(G)\) contains elements of odd degree, and as in the cyclic group case this is incompatible with the unramified condition. \(\square\)

We can extend this result to the class of \(p\)-nilpotent groups. A finite group \(G\) is \(p\)-nilpotent if one and hence each \(p\)-Sylow subgroup \(P \leq G\) has a normal \(p\)-complement, \(i.e.,\) there is a normal subgroup \(N \triangleleft G\) with \(p \nmid |N|\) and \(G = PN = P \ltimes N\). A convenient summary of the properties of such groups can be found in \cite{14} section 7, see also \cite{18}.

**Corollary 5.4.** If \(G\) is a \(p\)-nilpotent group for which \(p\) divides \(|G|\), then the extension

\[
F(BG_+, E) \longrightarrow F(EG_+, E)
\]

is ramified and so is not \(G\)-Galois.

**Proof.** By a result of Tate \cite{21}, \(G\) being \(p\)-nilpotent is equivalent to the restriction homomorphism giving an isomorphism

\[
\text{res}^G_0: H^*(BG; \mathbb{F}_p) \xrightarrow{\cong} H^*(BP; \mathbb{F}_p),
\]

and in fact it is sufficient that this holds in degree 1. Comparison of the Serre spectral sequences for \(K^*(BG_+\)) and \(K^*(BP_+)\) shows that

\[
K^*(BG_+) \xrightarrow{\cong} K^*(BP_+).
\]

It now follows that

\[
E^*(BG_+) \xrightarrow{\cong} E^*(BP_+),
\]

and the result can be deduced from Theorem \[5.3\] \(\square\)
Remark 5.5. The condition of $G$ being a $p$-nilpotent group should not be confused with the condition that the conjugation action of $G$ on $F_p[G]$ is nilpotent. The latter is used in [19] proposition 5.6.3 to ensure convergence of the Eilenberg-Moore spectral sequence and so to prove that for such groups

$$F(BG_+, H\mathbb{F}_p) \rightarrow F(EG_+, H\mathbb{F}_p)$$

is a $G$-Galois extension. The example of $G = \Sigma_3$, the third symmetric group, for the prime $p = 2$ illustrates this. For each of the Sylow 2-subgroups

$$\{\text{id}, (1, 2), (1, 3), (2, 3)\}$$

has as normal complement

$$N = \{\text{id}, (1, 2, 3), (1, 3, 2)\},$$

therefore $\Sigma_3$ is 2-nilpotent. However, the $\Sigma_3$-module $\mathbb{F}_2[\Sigma_3]$ contains the 2-dimensional non-trivial simple submodule

$$V = \{x(1, 2) + y(1, 3) + z(2, 3) : x + y + z = 0\},$$

so by Jordan-Hölder theory every composition series for $\mathbb{F}_2[\Sigma_3]$ must have this as a composition factor. Hence the action of $\Sigma_3$ on $\mathbb{F}_2[\Sigma_3]$ cannot be nilpotent.

6. Some observations on the Eilenberg-Moore spectral sequence

In [19] section 5.6, it is shown that for a finite $p$-group $G$, the Eilenberg-Moore spectral sequence with

$$(6.1) \quad E_2^{s,t} = \text{Tor}^{H^*(BG_+, \mathbb{F}_p)}_{s,t}(\mathbb{F}_p, \mathbb{F}_p)$$

converges to $\pi_*(F(G_+, H\mathbb{F}_p)) = \pi_*(\prod_C \mathbb{F}_p)$. By comparing it with the Künneth spectral sequence for $\pi_*(H\mathbb{F}_p \wedge F(BG_+, H\mathbb{F}_p))$, it is also shown that

$$F(BG_+, H\mathbb{F}_p) \rightarrow F(EG_+, H\mathbb{F}_p)$$

is a $G$-Galois extension.

Let us consider in detail the case $G = C_p$ for $p$ an odd prime. The case when $p = 2$ is similar. First we write

$$H^*(BC_p) = H^*(BC_{p+}; \mathbb{F}_p) = \mathbb{F}_p[y] \otimes \Lambda(z),$$

where $y \in H^2(BC_p)$ and $z \in H^1(BC_p)$. Then (6.1) becomes

$$E_2^{s} = \Gamma(\sigma z) \otimes \Lambda(\sigma y),$$

where $\sigma y \in E_1^{2,-2}$ and $\sigma z \in E_1^{1,-1}$ are the suspensions of $y$ and $z$, see [17]. Writing $\gamma_r = \gamma_r(\sigma z)$. The first non-trivial differential is

$$d^{p-1}\gamma_p = \sigma y,$$

and we have

$$E_3^{s} = \mathbb{F}_p[\zeta]/(\zeta^{p'}) \otimes \Gamma(\gamma_{p'}) \otimes \Lambda(\gamma_{p'}\sigma y),$$

where $\zeta$ represents the class of $\sigma z$. The remaining differentials are determined by the formulae

$$d^{p' - p^{t-1} - 1}\gamma_{p'} = \gamma_{p^{t-1}}\sigma y$$

in

$$E_3^{s} = \mathbb{F}_p[\zeta]/(\zeta^{p'}) \otimes \Gamma(\gamma_{p'}) \otimes \Lambda(\gamma_{p^{t-1}}\sigma y).$$

Finally we have

$$E_\infty^{s} = \mathbb{F}_p[\zeta]/(\zeta^{p'})$$

which is an avatar of $\prod_C \mathbb{F}_p$. These differentials are forced by the known answer and multiplicativity, and are also related to the discussion of [17] section 6. For Lubin-Tate theory ($E^{BC_{p'}}$) is free over $E$, and the comparison of the Eilenberg-Moore with the Künneth spectral sequence together with our Theorems 5.1 and 5.3 has the following consequence.
Proposition 6.1. For the cyclic $p$-group $C_p$ the $E$-theory Eilenberg-Moore spectral sequence for $BC_p$ with

$$L^T E^2_{s,t} = \text{Tor}^{(E_{BC_p})_*}(E_*, E_s)$$

does not converge to $\pi_*(\prod_{C_p} E)$.

Just as in the $HF_p$ case, we can compare the Morava $K$-theory based Eilenberg-Moore spectral sequence with the Künneth spectral sequence. Work of Bauer [4] on the convergence of the Cotor-version of this Eilenberg-Moore spectral sequence shows that the corresponding spectral sequence converges for $G = C_p$ and odd primes $p$, and therefore

$$K \wedge_{KBC_p} K \sim \prod_{C_p} K.$$

The extension of $S$-algebras $K^{BC_p} \to K^{EC_p}$ can be interpreted as a Galois extension of non-commutative $S$-algebras.

References


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