Exercises in Algebraic Topology

Prof. Dr. Birgit Richter

Summer term 2010

No 9 Due: Tuesday, 22nd of June 2010

33 Recall the definition of the Moore space $M(\mathbb{Z}/p\mathbb{Z}, n)$ for $n \geqslant 1$ and some prime p as a mapping cone of a degree p self-map of $\mathbb{S}^n$. Then the quotient of $M(\mathbb{Z}/p\mathbb{Z}, n)$ by its n-skeleton is homeomorphic to $\mathbb{S}^{n+1}$, so there is a map $\pi\colon M(\mathbb{Z}/p\mathbb{Z}, n) \to \mathbb{S}^{n+1}$. Use $\pi \times \mathrm{id}_{M(\mathbb{Z}/p\mathbb{Z},n)}$ so show that the splitting in the Künneth formula is not natural.

(3 points)

34 Compute all possible cap products for the spaces $\mathbb{S}^n$ and $\mathbb{S}^n \times \mathbb{S}^n$.

We defined the cap product as a map

$$\cap\colon H^q(X, A) \otimes H_n(X, A) \to H_{n-q}(X).$$

Argue why the image is not contained in $H_{n-q}(X, A)$ and construct the following variant: Let A, B be open subspaces of X, then the cap product is a map

$$\cap\colon H^q(X, A) \otimes H_n(X, A \cup B) \to H_{n-q}(X, B).$$

(4 points)

35 Calculate $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ and $H^*(\mathbb{R}P^n; \mathbb{Z}/3\mathbb{Z})$ for all n.

(2 points)

36

Let k be a field and let C_* and C'_* be two chain complexes that C_n and C'_n are vector spaces over k and the differentials are k-linear. Prove that

$$H_n(C_* \otimes C'_*) \cong \oplus_{p+q=n} H_p(C_*) \otimes H_q(C'_*).$$

(In particular, this shows that for two topological spaces X and Y there are isomorphisms

$$H_n(X \times Y; k) \cong \oplus_{p+q=n} H_p(X; k) \otimes H_q(Y; k)$$

for all n.)

(3 points)