Exercises in Algebraic Topology

Prof. Dr. Birgit Richter

Summer term 2010

No 6 Due: Tuesday, 1st of June 2010

21 Let X be a finite CW complex. The Euler characteristic of X is defined as

$$\chi(X) := \sum_{i \geqslant 0} (-1)^i \mathrm{rk}(H_i(X)).$$

Here, the rank of a finitely generated abelian group A, $\mathrm{rk}(A)$, is the number of its free summands.

Why is this number well-defined? Can you weaken the assumption and still get a well-defined number? What is the relationship between the Euler characteristic and Euler's formula for polyhedra?

What is $\chi(F_g)$ for $g \geqslant 0$?

(3 points)

22 Let X be

$$X = \mathbb{D}^{n+1} \cup_f \mathbb{S}^n := \mathbb{D}^{n+1} \sqcup \mathbb{S}^n / z \sim f(z) \text{ for } z \in \partial \mathbb{D}^{n+1}$$

Thus X is a mapping cone of a self-map $f\colon \mathbb{S}^n \to \mathbb{S}^n$. Calculate the cellular homology of X in dependence of the mapping degree $k = \mathrm{grad}(f)$ of f. (You know the answer already, but here you should write down the cellular chain complex and calculate the cellular boundary maps. It could be useful to consider the case $k = 0$ separately.)

(4 points)

23 For $g \geqslant 2$ let $E_{2g} \subset \mathbb{R}^2$ be a regular $2g$-gon. We call the vertices of E_{2g} $w_1, \ldots, w_{2g}$ and we consider

$$N_g := E_{2g}/\sim$$

where the equivalence relation $\sim$ is generated by

$$(1-t)w_{2j-1} + tw_{2j} \sim (1-t)w_{2j} + tw_{2j+1}, \quad 0 \leqslant t \leqslant 1.$$

Here the indices should be read modulo $2g$. The identification only identifies points on the boundary of E_{2g}. We call N_g the non-orientable closed surface of genus g.

What is N_2?

Calculate the homology groups of N_g using cellular homology. (In order to calculate the boundary maps, it really helps to draw pictures!)

(4 points)

24 Show that $\mathbb{C}P^1$ is homeomorphic to $\mathbb{S}^2$.

(1 point)