Exercises in Algebraic Topology

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No 4 Due: Tuesday, 11th of May 2010

13 Let F_g be again the closed orientable compact connected surface of genus g. Calculate $H_n(F_g)$ for all n and g. This is possible with induction over g, if you calculate the homology groups of $T_{1,1}$, where $T_{1,1}$ is a torus where you've cut out a small disc. Use these building blocks to construct F_g and then use the Mayer-Vietoris sequence to calculate the homology groups.

(5 points)

14 Right and wrong generalizations of the 5-lemma. You get one point for each correct answer, so three points in total. (Just give the answer, don't tell us why!)

a) Let

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

be a commutative diagram with exact rows. Assume that f_2 and f_4 are monomorphisms and f_1 is an epimorphism. Is it true that in that case f_3 is always a monomorphism?

b) Take the same diagram and assume that f_2 and f_4 are monomorphisms and f_1 is an epimorphism. Is then f_3 always surjective?

c) If f_2 and f_4 are epimorphisms and f_5 is a monomorphism, is then f_3 always injective?

(3 points)

15 Show that for every natural number $N > 0$ there exists a path-connected space X with $H_n(X) = \mathbb{Z}/N\mathbb{Z}$ and $H_k(X) = 0$ for all $0 \neq k \neq n$. Such a space is called *Moore space* and is often denoted by $M(\mathbb{Z}/N\mathbb{Z}, n)$. (Try to write $M(\mathbb{Z}/N\mathbb{Z}, n)$ as a mapping cone $C_f = CX \cup_f Y$ for some suitable $f\colon X \to Y$. Start with the case $n = 1$ and then use suspension.)

(3 points)

16 Use **15** to show that for any finitely generated abelian group A, there is a Moore space $M(A, n)$, *i.e.*, a path-connected space with $H_k(M(A, n)) = 0$ for all $0 \neq k \neq n$ and $H_n(M(A, n)) \cong A$.

(1 point)