Exercises in Algebraic Topology

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No 3 Due: Tuesday, 4th of May 2010

9 Prove the 9-Lemma (aka 3×3-Lemma): If

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f'"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "g'"] & C_2 \arrow[r] \arrow[d, "h'"] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "\alpha_3"] \arrow[d] & B_3 \arrow[r, "\beta_3"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram with exact columns, and the two bottom rows are exact, then the top row is an exact sequence. (You could prove that by a diagram chase, but you don't have to. Try to be as lazy as possible.)

(3 points)

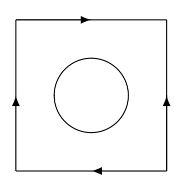
10 Let $X = \mathbb{S}^2$ and $A = \{p_1, \ldots, p_m\}$ where the p_i are m pairwise different points on $\mathbb{S}^2$. Calculate $H_n(X, A)$ in terms of $H_n(X)$ for all n.

(3 points)

11 Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. What can you say about $H_0(X, A)$ and $H_1(X, A)$?

(3 points)

What is H_1 of the Klein bottle? Use that from the picture



and the Seifert-van Kampen Theorem we can deduce that the fundamental group of the Klein bottle is isomorphic to $\langle a, b | aba^{-1}b \rangle$, so just calculate the abelianization of that group.

(1 point)

12 Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence. Prove that $B \cong A \oplus C$ if and only if there is a homomorphism $s \colon C \to B$ with $g \circ s = \mathrm{id}_C$. (Use that $A \cong \ker(g)$ and $C \cong \mathrm{im}(g)$.)

(2 points)