# Exercises in Algebraic Topology

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**No 2** Due: Tuesday, 27th of April 2010

**5** Prove the snake lemma: If

$$\begin{tikzcd} & A' \arrow[r, "\alpha"] \arrow[d, "f'"] & A \arrow[r, "\beta"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "\alpha'"] & B \arrow[r, "\beta'"] & B'' & \end{tikzcd}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$\ker(f') \to \ker(f) \to \ker(f'') \xrightarrow{\delta} \mathrm{coker}(f') \to \mathrm{coker}(f) \to \mathrm{coker}(f'').$$

Here, $\delta$ is a suitable connecting homomorphism; all other maps are induced ones.

(5 points)

**6** The fundamental group of a connected, compact, closed and orientable surface $F_g$ of genus $g \geqslant 1$ is

$$\pi_1(F_g) \cong \langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

Calculate $H_1(F_g)$ via the Hurewicz homomorphism.

(3 points)

**7** Recall, that $i\colon A \hookrightarrow X$ is a *strong deformation retract*, if there a map $H\colon X \times [0,1] \to X$ such that

(1) $H(x, 0) = x$ for all $x \in X$,
(2) $H(x, 1) \in A$ for all $x \in X$,
(3) $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$.

If only the first two properties are satisfied and if $H(a, 1) = a$ for all $a \in A$, then $A$ is called a *deformation retract* of $X$. If there is just an $r\colon X \to A$ with $r \circ i = \mathrm{id}$, then $A$ is called a *retract*.

Back to linear algebra. Use the Gram-Schmidt process to show that $O(n) \subset GL_n(\mathbb{R})$ is a retract. Can you improve this retraction to a (strong) deformation retraction? (In order to simplify things, you could use the Gram-Schmidt process to prove a splitting $GL_n(\mathbb{R}) \cong O(n) \times B_n$ for some suitable $B_n$.)

(3 points)

**8** Let $C_*$ be an arbitrary chain complex. Is it true that the sequence of chain complexes

$$0 \to C_* \xrightarrow{p\cdot} C_* \to C_*/pC_* \to 0$$

is always exact? (Proof or counterexample)

(1 point)