Exercises in Algebraic Topology

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No 10 Due: Tuesday, 29th of June 2010

37 Consider a product of spheres $\mathbb{S}^n \times \mathbb{S}^n$ and a map $f\colon \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n$. Then f has bidegree (d_1, d_2), if the map $\mathbb{S}^n \ni x \mapsto f(x,1)$ has degree d_1 and the map $\mathbb{S}^n \ni x \mapsto f(1,x)$ has degree d_2.

Show that for spheres of even dimension $n \geqslant 2$ only maps of bidegree $(0, d_2)$ or $(d_1, 0)$ can occur. (Use generators $H_n(\mathbb{S}^n \times \mathbb{S}^n)$ and $H^n(\mathbb{S}^n \times \mathbb{S}^n)$ and the extra structure on H_* and H^*.)

(4 points)

38 Describe the local orientations for spheres $\mathbb{S}^n$, $n \geqslant 2$ as explicit as possible. (How are these related to $\mu_n \in H_n(\mathbb{S}^n)$?)

Describe a manifold structure on the torus T with only two charts. Draw a picture of that.

(2 points)

39 Let V be a finite-dimensional real vector space. Prove that an orientation of V in the sense of linear algebra is the same as an orientation of V as a manifold. (What has this to do with $\pi_0(GL_n(\mathbb{R}))$?)

(3 points)

40 Let M be a manifold of dimension $m \geqslant 2$ and let $\tilde{M}$ be the set

$$\tilde{M} := \{(x, o_x) | x \in M, o_x \in H_m(M, M \backslash x) \text{ a generator}\}.$$

We define a map $p\colon \tilde{M} \to M$ as $p(x, o_x) = x$. This map is surjective and the *fibre* $f^{-1}(x)$ consists of two points for all $x \in M$.

Find a topology on $\tilde{M}$ such that $\tilde{M}$ is again an m-manifold and such that the map p is continuous. Show that $\tilde{M}$ is orientable. The map $p\colon \tilde{M} \to M$ is called the *orientation cover of* M.

Show that $x \mapsto o_x$ is an orientation of M if and only if the map $s\colon M \to \tilde{M}$, $x \mapsto (x, o_x)$ is continuous.

What are the orientation covers of $\mathbb{R}P^n$, of the Klein bottle and of the torus?

(3 points)