Exercises in Algebraic Topology

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No 1 Due: Tuesday, 20th of April 2010

1 Let $\mathbb{D}^n$ be the chain complex with $(\mathbb{D}^n)_n = \mathbb{Z} = (\mathbb{D}^n)_{n-1}$, $(\mathbb{D}^n)_i = 0, i \neq n, n-1$, and whose only non-trivial differential is the identity on the integers. Similarly, let $\mathbb{S}^m$ be the chain complex with

$$(\mathbb{S}^m)_\ell = \begin{cases} \mathbb{Z}, & \ell = m, \\ 0, & \text{otherwise.} \end{cases}$$

What are the homology groups of $\mathbb{S}^m$ and $\mathbb{D}^n$ $(n, m \in \mathbb{Z})$?

(1 point)

Prove that chain maps from $\mathbb{S}^n$ into a chain complex C_* correspond to the n-cycles $Z_n(C)$. Describe what chain maps from $\mathbb{D}^n$ to C_* correspond to. Are there chain maps between $\mathbb{D}^n$ and $\mathbb{S}^m$?

(3 points)

2 Let $(C^i, d^i)_{i\in I}$ be a family of chain complexes. Find a suitable definition of the chain complex $\bigoplus_{i\in I} C^i$ and prove that $H_*(\bigoplus_{i\in I} C^i) \cong \bigoplus_{i\in I} H_*(C^i)$.

(1 point)

Let $(A_n)_{n\in\mathbb{Z}}$ be an arbitrary family of finitely generated abelian groups. Is there a chain complex F_* such that F_n is a free abelian group for all n and $H_n(F_*) \cong A_n$? If so, describe F_*, if not, give a counterexample. (You'll need the classification of finitely generated abelian groups for this exercise. If you don't know that, just look it up for instance on `http://en.wikipedia.org/wiki/Finitely_generated_abelian_group`)

(3 points)

3 Let X and Y be topological spaces. Is every chain map $S_*(X) \to S_*(Y)$ induced by a continuous map $f\colon X \to Y$? (Proof or counterexample)

(1 point)

4 If $f\colon C_* \to C'_*$ is a chain map, then let C_f be the chain complex

$$(C_f)_n = C_{n-1} \oplus C'_n, \quad d_{C_f} = d\colon (C_f)_n \to (C_f)_{n-1}, d(c, c') = (-d^C(c), f(c) + d^{C'}(c')).$$

This is called the mapping cone of the chain map f. (It's related to the mapping cone of topological spaces.)

Prove that C_f is indeed a chain complex and show that the inclusion $C'_* \to C_f$ is a chain map.

Let $\mathrm{cone}(C_*)$ be the mapping cone of the identity of C_*. Show that f is null-homotopic if and only if f extends to a chain map $\mathrm{cone}(C_*) \to C'_*$.

(3 points)