EXAMPLES OF ÉTALE EXTENSIONS OF GREEN FUNCTORS

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Abstract. We provide new examples of étale extensions of Green functors by transferring classical examples of étale extensions to the equivariant setting. Our examples are Tambara functors, and we prove Green étaleness for them, which implies Tambara étaleness. We show that every C_2-Galois extensions of fields gives rise to an étale extension of C_2-Green functors. Here we associate the constant Tambara functor to the base field and the fix-Tambara functor to the extension. We also prove that all C_n-Kummer extensions give rise to étale extensions for arbitrary finite n. Étale extensions of fields induce étale extension of G-Green functors for any finite group G by passing to the corresponding constant G-Tambara functors.

1. Introduction

In the world of commutative rings a finitely generated commutative k-algebra A is étale if it is flat and unramified, and the latter property means that the module of Kähler differentials $\Omega^1_{A|k}$ is trivial. Typical examples include localizations and Galois extensions. Flat extensions with vanishing Kähler differentials are often called formally étale. The module of Kähler differentials can be identified with the first Hochschild homology group of A over k and also with I/I^2, where I is the kernel of the multiplication map $A \otimes_k A \to A$.

In the equivariant setting, this notion turns out to be more involved: Mike Hill defined a module of Kähler differentials for maps of G-Tambara functors $\underline{R} \to \underline{T}$ which we will denote by $\Omega^{1,G}_{\underline{T}|\underline{R}}$ [Hil17, Definition 5.4] and he defined $\underline{R} \to \underline{T}$ to be formally étale, if $\underline{T}$ is flat as an $\underline{R}$-module and if $\Omega^{1,G}_{\underline{T}|\underline{R}} = 0$ [Hil17, Definition 5.9]. Flatness is defined analogously to the non-equivariant context: $\underline{T}$ is flat over $\underline{R}$ if the Mackey box product with $\underline{T}$ over $\underline{R}$, $\underline{T}\Box_{\underline{R}}(-)$, preserves exactness. However, flatness is rarer in the equivariant context than in the non-equivariant one, see [HMQ23]. The definition of the Kähler differentials is also slightly different: the basic object is still the kernel of the multiplication map

$$\underline{I} = \ker(\mathsf{mult}\colon \underline{T}\Box_{\underline{R}}\underline{T} \to \underline{T})$$

but instead of quotienting out by $\underline{I}^2$ one also kills the image of norm maps that are induced by 2-surjective maps of finite G-sets. For a nice explicit description of $\Omega^{1,G}_{\underline{T}|\underline{R}}$ see [Lee19, p. 38]. In all our examples in this paper, we actually prove that $\underline{I}^2 = \underline{I}$, so we only use the corresponding underlying Green functor structures of the Tambara functors involved. We still determine the full Tambara structures of the relevant objects.

Examples of étale extensions are rare; it is known that for a multiplicative subset in $\underline{R}$, the map $\underline{R} \to \underline{R}[N^{-1}]$ is étale [Hil17, Proposition 5.13].

The purpose of this note is to provide new families of examples of étale extensions of Tambara functors. It grew out of our attempt to understand the notion of étaleness in the equivariant context and to transfer classical examples of étale extensions to the setting of Tambara functors. We show that C_2-Galois extensions of fields $K \subset L$ give rise to étale extensions of C_2-Tambara functors. Here we associate the constant Tambara functor, $\underline{K}^c$, to the base field and the fix-Tambara functor, $\underline{L}^{\mathsf{fix}}$ to the extension. We also prove that all C_n-Kummer extensions give rise to Tambara étale extensions for arbitrary finite n.

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Étale extensions of fields $K \subset L$ induce étale extension of G-Tambara functors for any finite group G. Here we consider the extension of the corresponding constant G-Tambara functors, $\underline{K}^c \to \underline{L}^c$.

In addition to sending a commutative G-ring R to $\underline{R}^c$ or $\underline{R}^{\mathsf{fix}}$, there is a third way of importing commutative rings with a group action into the world of Tambara functors: There is a left adjoint functor to the forgetful functor from G-Tambara functors to G-rings. We investigate its properties in upcoming work [LRZ].

It would be interesting to have an example of a Tambara étale extension which is not Green étale. We did not succeed in finding one.

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2. Preliminaries on Mackey functors and Tambara functors

The box product is a symmetric monoidal product in the category of Mackey functors. In this paper we use an explicit formula for box products of C_p-Mackey functors due to [Lew88].

$$(\underline{M}\Box\underline{N})(C_p/e) = \underline{M}(C_p/e) \otimes \underline{N}(C_p/e),$$

$$(\underline{M}\Box\underline{N})(C_p/C_p) = \left(\underline{M}(C_p/C_p) \otimes \underline{N}(C_p/C_p) \oplus \big(\underline{M}(C_p/e) \otimes \underline{N}(C_p/e)\big)_{\mathrm{Weyl}}\right)/\mathrm{FR}.$$

Here, the Weyl group, which is C_p, acts diagonally on $\underline{M}(C_p/e) \otimes \underline{N}(C_p/e)$. The elements in the orbit $\big(\underline{M}(C_p/e) \otimes \underline{N}(C_p/e)\big)_{\mathrm{Weyl}}$ are formal transfers and we will write their equivalence classes in $[-]$ to distinguish from elements in the first part $\underline{M}(C_p/C_p) \otimes \underline{N}(C_p/C_p)$. The relation FR, short for Frobenius reciprocity, forces

$$x \otimes \mathsf{tr}(y) = [\mathsf{res}(x) \otimes y] \text{ and } \mathsf{tr}(x) \otimes y = [x \otimes \mathsf{res}(y)].$$

See [Maz13, 1.2.1] for an explicit description of the box product structure for the groups C_{p^n}. For any commutative ring R and prime p, we denote by $\underline{R}^c$ the constant C_p-Tambara functor with $\underline{R}^c(C_p/e) = \underline{R}^c(C_p/C_p) = R$ and $\mathsf{norm}(a) = a^p$, $\mathsf{tr}(a) = p \cdot a$ and $\mathsf{res}(a) = a$ for all $a \in R$. The action of $C_p = \langle \tau \rangle$ on $\underline{R}^c$ is trivial.

For a commutative C_p-ring T we denote by $\underline{T}^{\mathsf{fix}}$ the C_p-Tambara functor

$$\underline{T}^{\mathsf{fix}}(C_p/e) = T, \qquad \underline{T}^{\mathsf{fix}}(C_p/C_p) = T^{C_p},$$

Here, the restriction map res is the inclusion map $T^{C_p} \hookrightarrow T$, $\mathsf{tr}(a) = a + \tau(a) + \cdots + \tau^{p-1}(a)$, and $\mathsf{norm}(a) = a\tau(a)\cdots\tau^{p-1}(a)$.

Note that the functor $(-)^{\mathsf{fix}}$ from the category of commutative rings with C_p-action to the category of C_p-Tambara functors is right adjoint to the evaluation functor at the free orbit C_p/e. In particular, for every C_p-Galois extension $K \subset L$ the inclusion map $K = \underline{K}^c(C_p/e) \to L$ is a C_p-map so it gives an adjoint map

$$\underline{K}^c \to \underline{L}^{\mathsf{fix}}. \tag{2.1}$$

3. The case of C_2-Galois extensions of fields

Let $K \subset L$ be a C_2-Galois extension of fields. Note that, depending on the characteristic of K, these can take one of two forms: If the characteristic of K is 2, then a C_2-extension is an Artin-Schreier extension [Lan02, Theorem 6.4]. If the characteristic is prime to 2, then $K \subset L$ is a Kummer extension. See [Bir10, p. 89] for background. We will use the specific forms of such extensions in our arguments in both cases.

For any C_2-Galois extension $K \subset L$, we have a map of C_2-Tambara functors $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ adjoint as in (2.1) to the inclusion $K \to L$ which is a C_2 map. In this section, we prove:

Theorem 3.1. *For a C_2-Galois extension $K \subset L$, the map $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ is C_2-Tambara étale.*

The argument depends on the characteristic of the ground field K and has two cases.

3.1. **Characteristic 2: Artin-Schreier Extensions.** In this section, we will assume that K is a field of characteristic 2 and L is a C_2-Galois extension of K. Then $L = K(\alpha)$ where $\alpha^2 + \alpha + a = 0$ for some $a \in K^\times$.

Lemma 3.2. *If $K \subset L$ is a C_2-Galois extension and K is a field of characteristic 2 with $L = K(\alpha)$ as above. Then the Tambara functor $\underline{L}^{\mathsf{fix}} \Box \underline{L}^{\mathsf{fix}}$ is given by the diagram*

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\Box\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu\alpha], \lambda, \mu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\Box\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

The restriction on $K \otimes_{\mathbb{Z}} K$ is the inclusion map, and

$$\mathsf{res}[\lambda\alpha \otimes \mu\alpha] = \lambda\alpha \otimes \mu + \lambda \otimes \mu\alpha + \lambda \otimes \mu. \tag{3.1}$$

The transfer is given for $\lambda, \mu \in K$ by

$$\mathsf{tr}(\lambda \otimes \mu) = 0, \quad \mathsf{tr}(\lambda \otimes \mu\alpha) = \lambda \otimes \mu = \mathsf{tr}(\lambda\alpha \otimes \mu), \quad \mathsf{tr}(\lambda\alpha \otimes \mu\alpha) = [\lambda\alpha \otimes \mu\alpha].$$

The norm on pure tensors is calculated as the tensor product of the norms on generators with the help of the diagonal action; the norm of a sum can be calculated by $\mathsf{norm}(x+y) = \mathsf{norm}(x) + \mathsf{norm}(y) + \mathsf{tr}(x \cdot \tau y)$ *(see* [Maz13, Example 1.4.1] *or, for the more general case of cyclic p-groups,* [HM19, Corollary 2.6]*,* [Maz13, section 1.4.1]*).*

Proof. The box product $\underline{L}^{\mathsf{fix}} \Box \underline{L}^{\mathsf{fix}}$ is given by the following diagram:

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\Box\underline{L}^{\mathsf{fix}}(C_2/C_2): & (K \otimes_{\mathbb{Z}} K \oplus (L \otimes_{\mathbb{Z}} L)_{\mathrm{Weyl}})/\mathrm{FR} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\Box\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

As the tensor product is over the integers we have to consider elements $\lambda + \mu\alpha$ with $\lambda, \mu \in K$. Since α's minimal polynomial over K is $\alpha(\alpha+1) = a$, $\tau(\alpha) = \alpha + 1$ so the Weyl-action identifies $[\lambda \otimes \mu\alpha]$ with

$$[\lambda \otimes \mu(\alpha+1)] = [\lambda \otimes \mu\alpha + \lambda \otimes \mu].$$

Hence in the Weyl quotient all elements $[\lambda \otimes \mu]$ are trivial and we also identify $[\lambda \otimes \mu\alpha]$ with $[\lambda \otimes \mu(\alpha+1)]$. The dual result holds for $[\lambda\alpha \otimes \mu]$. For $[\lambda\alpha \otimes \mu\alpha]$ we obtain

$$[\lambda\alpha \otimes \mu\alpha] \sim [\lambda(\alpha+1) \otimes \mu(\alpha+1)]$$

and this yields $[\lambda\alpha \otimes \mu] \sim [\lambda \otimes \mu\alpha]$. So we get Weyl equivalence classes $[\lambda\alpha \otimes \mu\alpha]$ and $[\lambda \otimes \mu\alpha] = [\lambda\alpha \otimes \mu]$.

The Frobenius reciprocity relations identify $[\lambda \otimes \mu\alpha]$ with

$$\lambda \otimes \mathsf{tr}(\mu\alpha) = \lambda \otimes \mu(\alpha + (\alpha+1)) = \lambda \otimes \mu$$

and $[\lambda\alpha \otimes \mu]$ with $\lambda \otimes \mu$. So we are left with

$$\begin{tikzcd}[column sep=small]
\underline{L}^{\mathsf{fix}} \Box \underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu\alpha], \lambda, \mu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}} \Box \underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} L, \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

as claimed in the lemma. The restriction on $K \otimes_{\mathbb{Z}} K$ is the inclusion map and to prove Equation (3.1), we observe that

$$\begin{aligned}
\mathsf{res}[\lambda\alpha \otimes \mu\alpha] =& \mathsf{res}(\mathsf{tr}(\lambda\alpha \otimes \mu\alpha)) = \mathsf{tr}(\lambda\alpha \otimes \mu\alpha) \\
=& \lambda\alpha \otimes \mu\alpha + \lambda(\alpha + 1) \otimes \mu(\alpha + 1) \\
=& \lambda\alpha \otimes \mu + \lambda \otimes \mu\alpha + \lambda \otimes \mu.
\end{aligned}$$

The transfer is given by $\mathsf{tr}(a \otimes b) = [a \otimes b]$, so $\mathsf{tr}(\lambda \otimes \mu) = [\lambda \otimes \mu] = 0$,

$$\mathsf{tr}(\lambda \otimes \mu\alpha) = [\lambda \otimes \mu\alpha] \sim \lambda \otimes \mu \sim [\lambda\alpha \otimes \mu] = \mathsf{tr}(\lambda\alpha \otimes \mu) \tag{3.2}$$

and $\mathsf{tr}(\lambda\alpha \otimes \mu\alpha) = [\lambda\alpha \otimes \mu\alpha]$. □

Lemma 3.3. *If $K \subset L$ is a C_2-Galois extension and K is a field of characteristic 2 so $L = K(\alpha)$ where $\alpha^2 + \alpha + a = 0$ for some $a \in K^\times$, then the Tambara functor $\underline{L}^{\mathsf{fix}} \Box \underline{K}^c \Box \underline{L}^{\mathsf{fix}}$ is given by the diagram*

$$\begin{tikzcd}[column sep=small]
\underline{L}^{\mathsf{fix}} \Box \underline{K}^c \Box \underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu \otimes \nu\alpha], \lambda, \mu, \nu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}} \Box \underline{K}^c \Box \underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} L, \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

where the restriction on $K \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} K$ is the inclusion map and for all $\lambda, \mu, \nu \in K$

$$\mathsf{res}[\lambda\alpha \otimes \mu \otimes \nu\alpha] = \lambda\alpha \otimes \mu \otimes \nu + \lambda \otimes \mu \otimes \nu\alpha + \lambda \otimes \mu \otimes \nu,$$

$$\begin{aligned}
\mathsf{tr}(\lambda\alpha \otimes \mu \otimes \nu\alpha) &= [\lambda\alpha \otimes \mu \otimes \nu\alpha], \\
\mathsf{tr}(\lambda \otimes \mu \otimes \nu) &= 0, \\
\mathsf{tr}(\lambda\alpha \otimes \mu \otimes \nu) &= \mathsf{tr}(\lambda \otimes \mu \otimes \nu\alpha) = \lambda \otimes \mu \otimes \nu.
\end{aligned}$$

Norms are calculated on pure tensors coordinatewise and extended to sums as in the previous lemma.

Proof. The definition of $\underline{L}^{\mathsf{fix}} \Box \underline{K}^c$ gives the diagram

$$\begin{tikzcd}[column sep=small]
\underline{L}^{\mathsf{fix}} \Box \underline{K}^c(C_2/C_2): & (K \otimes_{\mathbb{Z}} K \oplus (L \otimes_{\mathbb{Z}} K)_{\mathrm{Weyl}})/\mathrm{FR} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}} \Box \underline{K}^c(C_2/e): & L \otimes_{\mathbb{Z}} K. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

The Weyl action sends a generator $\lambda\alpha \otimes \mu$ to $\lambda(\alpha + 1) \otimes \mu = \lambda\alpha \otimes \mu + \lambda \otimes \mu$. Hence $[\lambda \otimes \mu]$ is trivial for all $\lambda, \mu \in K$ and the class $[\lambda\alpha \otimes \mu]$ is equal to $[\lambda(\alpha + 1) \otimes \mu]$.

The only new relation in the Frobenius reciprocity identifies $\mathsf{tr}(\lambda_1 + \mu\alpha) \otimes \lambda_2$ with

$$[\lambda_1 \otimes \lambda_2] + [\mu\alpha \otimes \lambda_2] = [\mu\alpha \otimes \lambda_2].$$

As

$$\mathsf{tr}(\lambda_1 + \mu\alpha) \otimes \lambda_2 = \mu \otimes \lambda_2$$

this identifies $[\mu\alpha \otimes \lambda_2]$ with $\mu \otimes \lambda_2$ and hence we just get

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\Box\underline{K}^c(C_2/C_2): & K \otimes_{\mathbb{Z}} K \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\Box\underline{K}^c(C_2/e): & L \otimes_{\mathbb{Z}} K \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

where the restriction map is the inclusion, $\mathsf{tr}(\lambda\alpha \otimes \mu) = [\lambda\alpha \otimes \mu] = \lambda \otimes \mu$, and $\mathsf{tr}(\lambda \otimes \mu) = 0$. The norm sends a generator $\lambda \otimes \mu$ to $\lambda^2 \otimes \mu^2$ and $\mu\alpha \otimes \nu$ to $\mu^2 a \otimes \nu^2$.

The three-fold box product $\underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}$ is

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/C_2): & (K \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} K \oplus (L \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} L)_{\mathrm{Weyl}})/\mathrm{FR} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

In $L \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} L$, as the Weyl action on $\lambda \otimes \mu \otimes \nu\alpha$ gives $\lambda \otimes \mu \otimes \nu\alpha + \lambda \otimes \mu \otimes \nu$, we see that pure scalar terms $\lambda \otimes \mu \otimes \nu$ are identified to zero. That is, $[\lambda \otimes \mu \otimes \nu] = 0$. Modulo those, the action on $\lambda\alpha \otimes \mu \otimes \nu\alpha$ yields

$$\lambda(\alpha+1) \otimes \mu \otimes \nu(\alpha+1) = \lambda\alpha \otimes \mu \otimes \nu\alpha + \lambda\alpha \otimes \mu \otimes \nu + \lambda \otimes \mu \otimes \nu\alpha$$

and thus $[\lambda\alpha \otimes \mu \otimes \nu] = [\lambda \otimes \mu \otimes \nu\alpha]$. Again, the only new relation given by Frobenius reciprocity identifies $[\lambda\alpha \otimes \mu \otimes \nu]$ and $[\lambda \otimes \mu \otimes \nu\alpha]$ with $\lambda \otimes \mu \otimes \nu$. Thus we get

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu \otimes \nu\alpha], \lambda, \mu, \nu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

The restriction on $K \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} K$ is the inclusion map and

$$\begin{aligned}
\mathsf{res}[\lambda\alpha \otimes \mu \otimes \nu\alpha] &= \mathsf{res}(\mathsf{tr}(\lambda\alpha \otimes \mu \otimes \nu\alpha)) = \mathsf{tr}(\lambda\alpha \otimes \mu \otimes \nu\alpha) \\
&= \lambda\alpha \otimes \mu \otimes \nu + \lambda \otimes \mu \otimes \nu\alpha + \lambda \otimes \mu \otimes \nu.
\end{aligned}$$

For the transfer, we obtain the formulas as given in the statement of the lemma. □

Proposition 3.4. *Let K be a field of characteristic 2 and L be a C_2-Galois extension of K, so $L = K(\alpha)$ where $\alpha^2 + \alpha + a = 0$ for some $a \in K^\times$. Then the relative box product $\underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}$ is given by*

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_K K \oplus \{[\lambda\alpha \otimes \alpha], \lambda \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_K L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

with structure maps as specified in the proof below. Moreover, we have:

1. *The ideal $\underline{I}$ at the fixed level, $\underline{I}(C_2/C_2)$, is spanned by $1 \otimes 1 + [\alpha \otimes \alpha]$.*
2. *The C_2-Tambara Kähler differentials $\Omega^{1,C_2}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c}$ vanish.*

Proof. We use the results of Lemma 3.2 and Lemma 3.3 to construct the coequalizer $\underline{L}^{\mathsf{fix}}\square_{\underline{K}^c}\underline{L}^{\mathsf{fix}}$. In the coequalizer, the K-action on both copies of L and on K is identified and hence we obtain:

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\square_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/C_2): & K\otimes_K K \oplus \{[\lambda\alpha\otimes\alpha], \lambda\in K\} \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\square_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/e): & L\otimes_K L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

For the rest of this proof, unadorned tensor products of elements are over K. The restriction on $K\otimes_K K$ is the inclusion map and

$$\begin{aligned} \mathsf{res}[\lambda\alpha\otimes\alpha] =& \lambda(\alpha\otimes 1 + 1\otimes\alpha + 1\otimes 1), \\ \mathsf{tr}(\lambda\cdot\alpha\otimes\alpha) =& [\lambda\cdot\alpha\otimes\alpha], \\ \mathsf{tr}(\lambda\cdot 1\otimes 1) =& 0, \\ \mathsf{tr}(\lambda\cdot\alpha\otimes 1) =& \mathsf{tr}(\lambda\cdot 1\otimes\alpha) = \lambda, \\ \mathsf{norm}(\lambda\cdot\alpha\otimes\alpha) =& \lambda^2 a^2 \\ \mathsf{norm}(\lambda\cdot 1\otimes 1) =& \lambda^2, \\ \mathsf{norm}(\lambda\cdot\alpha\otimes 1) =& \mathsf{norm}(\lambda\cdot 1\otimes\alpha) = \lambda^2 a. \end{aligned}$$

The multiplication $\mathsf{mult}\colon \underline{L}^{\mathsf{fix}}\square_{\underline{K}^c}\underline{L}^{\mathsf{fix}} \to \underline{L}^{\mathsf{fix}}$ is a morphism of Tambara functors. Hence on $K\otimes_K K$ and $L\otimes_K L$ it is given by the multiplication on K and L. As

$$\begin{aligned} \mathsf{res}(\mathsf{mult}[\lambda\alpha\otimes\alpha]) &= \mathsf{mult}\circ\mathsf{res}[\lambda\alpha\otimes\alpha] \\ &= \mathsf{mult}(\lambda(\alpha\otimes 1 + 1\otimes\alpha + 1\otimes 1)) && \text{by Equation (3.1)} \\ &= \lambda(2\alpha+1) = \lambda = \mathsf{res}(\lambda), \end{aligned}$$

and the fact that restriction is injective in $\underline{L}^{\mathsf{fix}}$, we get that $\mathsf{mult}[\lambda\alpha\otimes\alpha] = \lambda$. Therefore $1\otimes 1 + [\alpha\otimes\alpha] \in \underline{I}(C_2/C_2)$, the kernel of the fixed level of the multiplication map. By examination, $\underline{I}(C_2/C_2)$ is spanned by $1\otimes 1 + [\alpha\otimes\alpha]$.
The element $1\otimes 1 + [\alpha\otimes\alpha]$ is actually idempotent. For the following calculation we use the Frobenius reciprocity formula for Green functors from [Maz13, p. 19].

$$\begin{aligned} (1\otimes 1 + [\alpha\otimes\alpha])(1\otimes 1 + [\alpha\otimes\alpha]) &= 1\otimes 1 + 2[\alpha\otimes\alpha] + [\alpha\otimes\alpha][\alpha\otimes\alpha] \\ &= 1\otimes 1 + [(\alpha\otimes\alpha)\mathsf{res}\,\mathsf{tr}(\alpha\otimes\alpha)] \\ &= 1\otimes 1 + [(\alpha\otimes\alpha)\mathsf{res}(\alpha\otimes 1 + 1\otimes\alpha + 1\otimes 1)] \\ &= 1\otimes 1 + [\alpha^2\otimes\alpha + \alpha\otimes\alpha^2 + \alpha\otimes\alpha] \\ &= 1\otimes 1 + [(\alpha+a)\otimes\alpha + \alpha\otimes(\alpha+a) + \alpha\otimes\alpha] \\ &= 1\otimes 1 + a[\alpha\otimes 1 + 1\otimes\alpha] + [\alpha\otimes\alpha] \\ \text{by Equation (3.2)} &= 1\otimes 1 + 2a(1\otimes 1) + [\alpha\otimes\alpha] \\ &= 1\otimes 1 + [\alpha\otimes\alpha]. \end{aligned}$$

This shows that $\underline{I}/\underline{I}^2(C_2/C_2) = 0$. As $\underline{I}(C_2/e)$ is just the kernel of the multiplication map $L\otimes_K L \to L$ and as $K\subset L$ is étale, we know as well that $\underline{I}/\underline{I}^2(C_2/e) = 0$. As

$$\Omega^{1,C_2}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c} = \underline{I}/\underline{I}^{>1}$$

and as $\underline{I}/\underline{I}^2$ maps surjectively onto $\underline{I}/\underline{I}^{>1}$, we obtain that $\Omega^{1,C_2}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c} = 0$. $\square$

3.2. **Characteristic $\neq 2$: Kummer Extensions.** In this section, we will assume that K is a field of characteristic different than 2 and L is a C_2-Galois extension of K. Then $L = K(\alpha)$ where $\alpha^2 = a \in K^\times$.

Lemma 3.5. *If $K \subset L$ is a C_2-Galois extension and K is a field of characteristic different than 2, then $L = K(\alpha)$ as above. The Tambara functor $\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}$ is given by the diagram*

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu\alpha], \lambda, \mu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

Restriction induces the inclusion on $K \otimes_{\mathbb{Z}} K$ and sends $[\lambda\alpha \otimes \mu\alpha]$ to $2\lambda\alpha \otimes \mu\alpha$. Transfer sends $\lambda \otimes \mu$ to $2\lambda \otimes \mu$ and annihilates terms of the form $\lambda \otimes \mu\alpha$ and $\lambda\alpha \otimes \mu$. It sends $\lambda\alpha \otimes \mu\alpha$ to $[\lambda\alpha \otimes \mu\alpha]$. As $\mathsf{norm}(\alpha) = -\alpha^2 = -a$, we get that the norm map sends $\lambda\alpha \otimes \mu$ to $-a\lambda^2 \otimes \mu^2$, $\lambda \otimes \mu\alpha$ to $-\lambda^2 \otimes a\mu^2$, and $\lambda\alpha \otimes \mu\alpha$ to $a\lambda^2 \otimes a\mu^2$.

Proof. Since α's minimal polynomial over K is $\alpha^2 - a = 0$, the Galois action sends $\tau(\alpha)$ to $-\alpha$. The proof is analogous to the characteristic 2 case.

For $\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}$ we get the following diagram:

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}(C_2/C_2): & (K \otimes_{\mathbb{Z}} K \oplus (L \otimes_{\mathbb{Z}} L)_{\mathrm{Weyl}})/\mathrm{FR} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

The Weyl relation identifies $[\lambda\alpha \otimes \mu\alpha]$ with itself and kills $[\lambda\alpha \otimes \mu]$ and $[\lambda \otimes \mu\alpha]$. Frobenius reciprocity confirms $[\lambda \otimes \mu\alpha] = \lambda \otimes \mathsf{tr}(\mu\alpha) = 0$ and also $[\lambda\alpha \otimes \mu] = 0$. On scalars we obtain

$$[\lambda \otimes \mu] = \lambda \otimes \mathsf{tr}(\mu) = \lambda \otimes 2\mu = 2\lambda \otimes \mu. \tag{3.3}$$

Elements like $[\lambda\alpha \otimes \mu\alpha]$ survive unharmed. So we have

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu\alpha], \lambda, \mu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\square\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

Restriction sends $[\lambda\alpha \otimes \mu\alpha]$ to $2\lambda\alpha \otimes \mu\alpha$. The transfer and norm calculations follow directly. □

Lemma 3.6. *If $K \subset L$ is a C_2-Galois extension and K is a field of characteristic different than 2, so $L = K(\alpha)$ where $\alpha^2 = a \in K^\times$, then the Tambara functor $\underline{L}^{\mathsf{fix}}\square\underline{K}^c\square\underline{L}^{\mathsf{fix}}$ is given by the diagram*

$$\begin{tikzcd}
\underline{L}^{\mathsf{fix}}\square\underline{K}^c\square\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} K \oplus \{[\lambda\alpha \otimes \mu \otimes \nu\alpha], \lambda, \mu, \nu \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\
\underline{L}^{\mathsf{fix}}\square\underline{K}^c\square\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_{\mathbb{Z}} K \otimes_{\mathbb{Z}} L, \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"']
\end{tikzcd}$$

where restriction is inclusion on the left part and $\mathsf{res}[\lambda\alpha \otimes \mu \otimes \nu\alpha] = 2\lambda\alpha \otimes \mu \otimes \nu\alpha$. The transfer sends $\lambda\alpha \otimes \mu \otimes \nu\alpha$ to $[\lambda\alpha \otimes \mu \otimes \nu\alpha]$, $\lambda \otimes \mu \otimes \nu$ to $2\lambda \otimes \mu \otimes \nu$, and the remaining terms to zero. The norm is applied componentwise as before.

Proof. We start with $\underline{L}^{\mathsf{fix}}\Box\underline{K}^c$, given by

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\Box\underline{K}^c(C_2/C_2): & (K\otimes_{\mathbb{Z}}K\oplus(L\otimes_{\mathbb{Z}}K)_{\mathrm{Weyl}})/\mathrm{FR} \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\Box\underline{K}^c(C_2/e): & L\otimes_{\mathbb{Z}}K. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

Again, the Weyl relation kills $[\lambda\alpha\otimes\mu]$ and Frobenius reciprocity identifies $[\lambda\otimes\mu]$ with $\lambda\otimes\mathsf{tr}(\mu)=2\lambda\otimes\mu$. So we are left with

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\Box\underline{K}^c(C_2/C_2): & K\otimes_{\mathbb{Z}}K \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\Box\underline{K}^c(C_2/e): & L\otimes_{\mathbb{Z}}K \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

with restriction, transfers, and norms analogous to previous calculations. For $\underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}$ we have

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/C_2): & (K\otimes_{\mathbb{Z}}K\otimes_{\mathbb{Z}}K\oplus(L\otimes_{\mathbb{Z}}K\otimes_{\mathbb{Z}}L)_{\mathrm{Weyl}})/\mathrm{FR} \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/e): & L\otimes_{\mathbb{Z}}K\otimes_{\mathbb{Z}}L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

We get that $[\lambda\alpha\otimes\mu\otimes\nu\alpha]$ survives, whereas $[\lambda\alpha\otimes\mu\otimes\nu]=0=[\lambda\otimes\mu\otimes\nu\alpha]$. Again, pure scalars $[\lambda\otimes\mu\otimes\nu]$ are identified with $2\lambda\otimes\mu\otimes\nu$ via Frobenius reciprocity, so this three-fold box product is

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/C_2): & K\otimes_{\mathbb{Z}}K\otimes_{\mathbb{Z}}K\oplus\{[\lambda\alpha\otimes\mu\otimes\nu\alpha],\lambda,\mu,\nu\in K\} \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\Box\underline{K}^c\Box\underline{L}^{\mathsf{fix}}(C_2/e): & L\otimes_{\mathbb{Z}}K\otimes_{\mathbb{Z}}L \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

and tr sends $\lambda\alpha\otimes\mu\otimes\nu\alpha$ to $[\lambda\alpha\otimes\mu\otimes\nu\alpha]$ and $\lambda\otimes\mu\otimes\nu$ to $2\lambda\otimes\mu\otimes\nu$. It sends the remaining terms to zero. Restriction is inclusion on the left part and $\mathsf{res}[\lambda\alpha\otimes\mu\otimes\nu\alpha]=2\lambda\alpha\otimes\mu\otimes\nu\alpha$. The norm is applied componentwise.

$\square$

Proposition 3.7. *Let K be a field of characteristic $\neq 2$ and L be a C_2-Galois extension of K, so $L=K(\alpha)$ with $\alpha^2=a\in K^\times$. Then the relative box product $\underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}$ is*

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/C_2): & K\otimes_K K\oplus\{[\lambda\alpha\otimes\alpha],\lambda\in K\} \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/e): & L\otimes_K L, \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

with structure maps as specified in the proof below. Moreover, we have:

1. *The ideal $\underline{I}$ at the fixed level, $\underline{I}(C_2/C_2)$, is spanned by $2a-[\alpha\otimes\alpha]$.*
2. *The C_2-Tambara Kähler differentials $\Omega^{1,C_2}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c}$ vanish.*

Proof. We use the results of Lemma 3.5 and Lemma 3.6 to calculate the coequalizer $\underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}$, to obtain

$$\begin{tikzcd} \underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/C_2): & K \otimes_K K \oplus \{[\lambda\alpha\otimes\alpha], \lambda \in K\} \arrow[d, bend right, "\mathsf{res}"'] \\ \underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}}(C_2/e): & L \otimes_K L. \arrow[u, bend right, "\mathsf{tr}"'] \arrow[u, bend right=60, "\mathsf{norm}"'] \end{tikzcd}$$

The restriction on $K \otimes_K K$ is the inclusion map and

$$\begin{aligned} \mathsf{res}[\lambda\alpha\otimes\alpha] &= 2\lambda\alpha\otimes\alpha, \\ \mathsf{tr}(\lambda\cdot\alpha\otimes\alpha) &= [\lambda\cdot\alpha\otimes\alpha], \\ \mathsf{tr}(\lambda\cdot 1\otimes 1) &= 2\lambda, \\ \mathsf{tr}(\lambda\cdot\alpha\otimes 1) &= \mathsf{tr}(\lambda\cdot 1\otimes\alpha) = 0, \\ \mathsf{norm}(\lambda\cdot\alpha\otimes\alpha) &= \lambda^2a^2 \\ \mathsf{norm}(\lambda\cdot 1\otimes 1) &= \lambda^2, \\ \mathsf{norm}(\lambda\cdot\alpha\otimes 1) &= \mathsf{norm}(\lambda\cdot 1\otimes\alpha) = -\lambda^2a. \end{aligned}$$

The multiplication map $\mathsf{mult}\colon \underline{L}^{\mathsf{fix}}\Box_{\underline{K}^c}\underline{L}^{\mathsf{fix}} \to \underline{L}^{\mathsf{fix}}$ induces the ordinary multiplication on L at the C_2/e-level. As mult is a map of Tambara functors, we obtain that mult induces the multiplication on $K \otimes_K K$. As $\mathsf{res}[\alpha\otimes\alpha] = 2\alpha\otimes\alpha$ and as this is sent to $2\alpha^2 = 2a$ under the multiplication map, we know that $\mathsf{mult}[\alpha\otimes\alpha] = 2a$. Therefore $2a - [\alpha\otimes\alpha]$ generates $\underline{I}(C_2/C_2)$. As $\underline{I}(C_2/e)$ is the kernel of the multiplication map $L\otimes_K L \to L$ and as L is étale, we know that $\underline{I}/\underline{I}^2(C_2/e) = 0$, and hence $\Omega^{1,C_2}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c}(C_2/e) = 0$.
Observe that

$$(2a - [\alpha\otimes\alpha])^2 = 4a^2 - 4a[\alpha\otimes\alpha] + [\alpha\otimes\alpha]^2 = 4a^2 - 4a[\alpha\otimes\alpha] + 4a^2.$$

Here, we use

$$[\alpha\otimes\alpha]^2 = [\alpha\otimes\alpha\cdot\mathsf{res}\,\mathsf{tr}(\alpha\otimes\alpha)] = [\alpha^2\otimes\alpha^2 + (-\alpha^2)\otimes(-\alpha^2)] = 2[a\otimes a] = 4a^2 \text{ by Equation (3.3).}$$

As the characteristic is not 2 and as a is invertible in K, dividing by $4a$ yields that $2a - [\alpha\otimes\alpha]$ is in $\underline{I}^2(C_2/C_2)$. Hence $\Omega^{1,C_2}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c}(C_2/C_2) = 0$. □

3.3. **Projectivity.** To prove étaleness, we still need projectivity. Again, the proof depends on the characteristic of K.

Lemma 3.8. *(Artin-Schreier case) For any C_2-Galois extension $K \subset L$ where the characteristic of K is 2, the C_2-Tambara functor $\underline{L}^{\mathsf{fix}}$ is projective over $\underline{K}^c$.*

Proof. The characteristic of K is 2, so we know that $L = K(\alpha)$ with $\alpha^2 + \alpha + a = 0$ for some $a \in K$ and $\tau(\alpha) = \alpha+1$. Assume that $\pi\colon \underline{M} \to \underline{Q}$ is an epimorphism of C_2-Mackey functors that are $\underline{K}^c$-modules. In particular, the values on the orbits are K-vector spaces. Let $\zeta\colon \underline{L}^{\mathsf{fix}} \to \underline{Q}$

be a morphism of $\underline{K}^c$-modules. Consider the diagram

$$\begin{tikzcd}
K \arrow[drr, bend left, "\zeta(C_2/C_2)"] & & \\
& \underline{M}(C_2/C_2) \arrow[r, "\pi(C_2/C_2)"] \arrow[d, shift right, "\mathsf{res}"'] & \underline{Q}(C_2/C_2) \arrow[d, shift right, "\mathsf{res}"'] \\
& \underline{M}(C_2/e) \arrow[r, "\pi(C_2/e)"] \arrow[u, shift right, "\mathsf{tr}"'] & \underline{Q}(C_2/e) \arrow[u, shift right, "\mathsf{tr}"'] \\
L \arrow[urr, bend right, "\zeta(C_2/e)"'] & &
\end{tikzcd}$$

We define $\xi\colon \underline{L}^{\mathsf{fix}} \to \underline{M}$ as follows: We set $\tilde{m} := \zeta(C_2/e)(\alpha) \in \underline{Q}(C_2/e)$ and choose a preimage m of $\tilde{m}$ under $\pi(C_2/e)$. We define $\xi(C_2/e)(\alpha) := m$. Then $1 = \alpha + \tau\alpha$ is sent to

$$\xi(C_2/e)(\alpha + \tau\alpha) = m + \tau m.$$

We define $\xi(C_2/C_2)(1) := \mathsf{tr}(m)$. Then

$$\mathsf{res}(\xi(C_2/C_2)(1)) = \mathsf{res}\,\mathsf{tr}(m) = m + \tau m = \xi(C_2/e)(1) = \xi(C_2/e)(\mathsf{res}(1))$$

and by construction ξ commutes with tr. $\square$

Lemma 3.9. *(Kummer case) For any C_2-Galois extension $K \subset L$ where the characteristic of K is not 2, the C_2-Tambara functor $\underline{L}^{\mathsf{fix}}$ is projective over $\underline{K}^c$.*

Proof. Since the characteristic of K is not equal to 2, then $\mathsf{tr}(1) = 2$ is invertible. We have $L = K(\alpha)$ with $\alpha^2 = a \in K$ and $\tau(\alpha) = -\alpha$. In particular, the transfer on α is zero.

We claim that any $\underline{K}^c$-module $\underline{M}$ can be canonically decomposed as $\underline{M} = \underline{M}^+ \oplus \underline{M}^-$ in the following way. As $2 \in K^\times$ we can split all C_2-representations into the $\pm$-eigenspaces of the τ-action via the usual trick of writing $x = \frac{x+\tau x}{2} + \frac{x-\tau x}{2}$. So we can write $\underline{M}(C_2/e) = \underline{M}^+(C_2/e) \oplus \underline{M}^-(C_2/e)$. For $x \in \underline{M}(C_2/e)$, we have $\mathsf{tr}(x) = \mathsf{tr}(\tau x) = -\mathsf{tr}(x)$ and hence $\mathsf{tr}(x) = 0$. Define $\underline{M}^+(C_2/C_2) = \underline{M}(C_2/C_2)$ and $\underline{M}^-(C_2/C_2) = 0$. Then $\underline{M}^+$ and $\underline{M}^-$ are sub-C_2-Mackey-functors of $\underline{M}$ and $\underline{K}^c$-modules.

Our lifting diagram looks as follows:

$$\begin{tikzcd}
K = K \oplus 0 \arrow[drr, bend left, "\zeta(C_2/C_2)"] & & \\
& \underline{M}^+(C_2/C_2) \oplus \underline{M}^-(C_2/C_2) \arrow[r, "\pi(C_2/C_2)"] \arrow[d, shift right, "\mathsf{res}"'] & \underline{Q}^+(C_2/C_2) \oplus \underline{Q}^-(C_2/C_2) \arrow[d, shift right, "\mathsf{res}"'] \\
& \underline{M}^+(C_2/e) \oplus \underline{M}^-(C_2/e) \arrow[r, "\pi(C_2/e)"] \arrow[u, shift right, "\mathsf{tr}"'] & \underline{Q}^+(C_2/e) \oplus \underline{Q}^-(C_2/e) \arrow[u, shift right, "\mathsf{tr}"'] \\
K \oplus \{\lambda\alpha, \lambda \in K\} \arrow[urr, bend right, "\zeta(C_2/e)"'] & &
\end{tikzcd}$$

It suffices to show that both $(\underline{L}^{\mathsf{fix}})^+$ and $(\underline{L}^{\mathsf{fix}})^-$ are projective. For the positive part, $(\underline{L}^{\mathsf{fix}})^+ \cong \underline{K}^c$ is a free $\underline{K}^c$-module, so in particular projective. For the negative part, the only nontrivial module is at the C_2/e-level, and as $(\underline{L}^{\mathsf{fix}})^-(C_2/e) = K$ is projective, we can solve the lifting problem. $\square$

Proof of Theorem 3.1. In Propositions 3.4 and 3.7, we proved that $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ has vanishing Kähler differentials. In Lemma 3.8 and Lemma 3.9, we proved that $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ is projective. As projective $\underline{K}^c$-modules are flat (see [Lew99, Corollary 6.4] or [Lee19, Prop 2.2.13]), we know that $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ is flat. □

4. Tambara étaleness of C_n-Kummer extensions

By the above discussion we already know that C_2-Kummer extensions are Tambara étale. We generalize this to arbitrary finite cyclic groups: Let $K \subset L$ be a C_n-Kummer extension [Bir10, p. 89], hence n is invertible in K and the polynomial $X^n - 1$ splits in K. We denote by ζ_n a primitive nth root of unity. We can assume that L is of the form $L = K(\alpha)$ with $\alpha^n = a \in K^\times$, and that the generator σ of the Galois group C_n acts on α via multiplication by ζ_n.

We start with a result that helps to prove flatness. The proof generalizes the one of Lemma 3.9.

Lemma 4.1. *Let K be a field containing a primitive nth root of unity $\zeta = \zeta_n$ with n invertible in K. Then any $\underline{K}^c$-module $\underline{M}$ can be decomposed uniquely and functorially as $\bigoplus_{i=0}^{n-1} \underline{M}^{\zeta^i}$, characterized by the following properties that for any subgroup $C_m \leqslant C_n$,*

1. *the generator of C_n/C_m acts on $\underline{M}^{\zeta^i}(C_n/C_m)$ as multiplication by ζ_n^i;*
2. *if $m \nmid i$, then $\underline{M}^{\zeta^i}(C_n/C_m) = 0$;*
3. *if $m \mid i$, then both $\mathsf{res}_e^{C_m}$ and $\mathsf{tr}_e^{C_m}$ in $\underline{M}^{\zeta^i}$ are isomorphisms.*

Proof. (1) Any C_n-module M can be decomposed into the ζ_n^i-eigenspaces $\bigoplus_{i=0}^{n-1} M^{\zeta_n^i}$ using the identity

$$x = \sum_{i=0}^{n-1} \frac{\sum_{j=0}^{n-1} \zeta_n^{-ij} \sigma^j(x)}{n}.$$

We consider $\underline{M}(C_n/C_m)$ with the C_n-action via $C_n \to C_n/C_m = W_{C_n}(C_m)$. In this way, the restriction maps and the transfer maps are C_n-equivariant. Setting $\underline{M}^{\zeta^i}(C_n/C_m) = \underline{M}(C_n/C_m)^{\zeta_n^i}$, we obtain sub-Mackey functors $\underline{M}^{\zeta^i}$ of $\underline{M}$.

(2) As $\underline{M}(C_n/C_m)$ is C_m-fixed, we obtain $\underline{M}^{\zeta^i}(C_n/C_m) = 0$ if ζ_n^i is not of the form of $\zeta_{n/m}^j$ for some j, or equivalently, if $m \nmid i$.

(3) For a $\underline{K}^c$-Mackey functor, $\mathsf{tr}_e^{C_m} \circ \mathsf{res}_e^{C_m} = m$ (as noticed by [Zen, Remark 2.13]). This holds because by Frobenius reciprocity, $\mathsf{tr}_e^{C_m}(\mathsf{res}_e^{C_m}(x)) = \mathsf{tr}_e^{C_m}(1) \cdot x = m \cdot x$. For $m \mid i$ we get that

$$\mathsf{res}_e^{C_m}(\mathsf{tr}_e^{C_m}(x)) = (1 + (\zeta_n^i)^{n/m} + (\zeta_n^i)^{2n/m} + \cdots + (\zeta_n^i)^{(m-1)n/m}) \cdot x = m \cdot x.$$

As n is invertible in K, so is m. This shows that tr and res are inverses of each other up to multiplication by a unit. □

Example 4.2. For a C_n-Kummer extension $K \subset L = K(\alpha)$ with $\alpha^n = a \in K^\times$, for $0 \leqslant i < n$,

$$(\underline{L}^{\mathsf{fix}})^{\zeta^i}(C_n/C_m) = \begin{cases} K\{\alpha^i\} & m \mid i \\ 0 & m \nmid i. \end{cases}$$

For $d \mid m \mid i$, $\mathsf{res}_{C_d}^{C_m}(\alpha^i) = \alpha^i$; for $d \mid i$ and $d \mid m$,

$$\mathsf{tr}_{C_d}^{C_m}(\alpha^i) = \begin{cases} \frac{m}{d}\alpha^i & m \mid i \\ 0 & m \nmid i. \end{cases}$$

Corollary 4.3. *Let K be a field containing a primitive nth root of unity ζ_n where n is invertible in K. Then:*

(1) *Evaluation at the C_n/e-level gives an equivalence between the category of $\underline{K}^c$-modules in C_n-Mackey functors and the category of $K[C_n]$-modules.*
(2) *All $\underline{K}^c$-modules are projective.*

Proof. (1) By Lemma 4.1, there are natural isomorphisms of $\underline{K}^c$-modules $\underline{M} \cong (\underline{M}(C_n/e))^{\mathsf{fix}}$ for all $\underline{M}$. Lemma 4.1 implies that this isomorphism is defined to be the identity at the C_n/e-level and propagates to the other levels as the relevant non-trivial structure maps are all isomorphisms.

(2) Let $\underline{M}$ be a $\underline{K}^c$-module. By the previous part, in order to show that it is projective, it suffices to show that $\underline{M}(C_n/e)$ is a projective $K[C_n]$-module. But from the assumption on K, all $K[C_n]$-modules split as sums of one-dimensional ones, which are projective. □

Theorem 4.4. *Let $K \subset L$ be a C_n-Kummer extension. Then $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ is C_n-Tambara étale.*

Proof. Recall that $L = K(\alpha)$, $\alpha^n = a \in K^\times$, $n \in K^\times$ and that the generator σ of C_n acts on α via multiplication by ζ_n, an nth root of unity. We show that all elements in $\underline{I} := \ker(\mathsf{mult}\colon \underline{L}^{\mathsf{fix}} \Box_{\underline{K}^c} \underline{L}^{\mathsf{fix}} \to \underline{L}^{\mathsf{fix}})$ are also in $\underline{I}^2$. Note that when n is not prime, there are orbits other than the trivial one and the free one, and there are also transfer elements from there.

In the following we use the formulas for the levels of a box product from [Maz13, Definition 1.2.1]. For a subgroup $C_m \leqslant C_n$ we have $L^{C_m} = K(\alpha^m)$, and

$$(\underline{L}^{\mathsf{fix}} \Box_{\underline{K}^c} \underline{L}^{\mathsf{fix}})(C_n/C_m) = K(\alpha^m) \otimes_K K(\alpha^m) \oplus ((\bigoplus_{d|m} K(\alpha^d) \otimes_K K(\alpha^d))_{\mathrm{Weyl}})/\mathrm{FR}$$

We will first determine $\underline{I}(C_n/C_m)$. Let $[x]^m_d$ denote the transfer class of x from level C_n/C_d to C_n/C_m in the box product and let $C_1 := e$.

- The kernel of the multiplication map $I_m := \ker(\mathsf{mult}\colon K(\alpha^m) \otimes_K K(\alpha^m) \to K(\alpha^m))$ is contained in $\underline{I}(C_n/C_m)$.
- Let $q := m/d$. Note that $L^{C_d} \otimes_K L^{C_d}$ has a C_m/C_d-action, and the generator acts by
$$\sigma^{n/m}(\alpha^d) = (\zeta_n^{n/m})^d \cdot \alpha^d = \zeta_{m/d} \cdot \alpha^d = \zeta_q \cdot \alpha^d.$$
  - If x is not fixed in $L^{C_d} \otimes_K L^{C_d}$, then
  $$\mathsf{res}^{C_m}_{C_d}([x]^m_d) = (1 + \zeta_q + \zeta_q^2 + \cdots + \zeta_q^{q-1})x = 0.$$
  Any element in the kernel of the restriction map to C_n/e is automatically in $\underline{I}(C_n/C_m)$ because the restriction map in $\underline{L}^{\mathsf{fix}}$ is injective.
  - If x is fixed in $L^{C_d} \otimes_K L^{C_d}$, then $\mathsf{res}([x]^m_d) = q \cdot x \neq 0$. This occurs if x is a K-linear combination of elements of the form $\alpha^{id} \otimes \alpha^{jd}$ for $q \mid (i + j)$. In these cases
  $$q \otimes \alpha^{(i+j)d} - [\alpha^{id} \otimes \alpha^{jd}]^m_d \in \underline{I}(C_n/C_m).$$
  Note that this makes sense because $m = qd$ divides $(i + j)d$.

Therefore, $\underline{I}(C_n/C_m)$ contains I_m and $\ker(\mathsf{res})$ and its other generators are of the form

$$q \otimes \alpha^{(i+j)d} - [\alpha^{id} \otimes \alpha^{jd}]^m_d \text{ for } m = qd \text{ and } q \mid (i + j). \tag{4.1}$$

We need to prove that $\Omega^{1,C_n}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c}$ is trivial when evaluated on all C_n-sets C_n/C_m for all subgroups $C_m \leqslant C_n$. We prove, as before, that $\underline{I}(C_n/C_m) = \underline{I}^2(C_n/C_m)$.

First, as $K \subset K(\alpha^m)$ is a $C_{n/m}$-Kummer extension and has vanishing Kähler differentials, we obtain that $I_m = I_m^2 \subset \underline{I}^2(C_n/C_m)$.

Next, we show that if $z \in \underline{I}(C_n/C_m)$ restricts to $0 \in \underline{I}(C_n/e)$, then z is in $\underline{I}^2$. This is true because the element $m \otimes \alpha^n - [\alpha \otimes \alpha^{n-1}]^m_1 = ma - [\alpha \otimes \alpha^{n-1}]^m_1$ is in $\underline{I}(C_n/C_m)$, and

$$z \cdot (ma - [\alpha \otimes \alpha^{n-1}]^m_1) = ma \cdot z - z[\alpha \otimes \alpha^{n-1}]^m_1 = ma \cdot z - [\mathsf{res}(z) \cdot \alpha \otimes \alpha^{n-1}]^m_1 = ma \cdot z$$

agrees with z up to a unit.

We are left with elements of the form $x_{i,i+j} := q \otimes \alpha^{(i+j)d} - [\alpha^{id} \otimes \alpha^{jd}]_d^m$ as in (4.1). Here, $q \mid (i+j)$ and $1 < i, j < n/d$, but we can allow all $i, j \geqslant 0$, because of the relations

$$\begin{aligned} x_{0,t} = x_{t,t} &= 0 && \text{by Frobenius reciprocity} \\ x_{i+n/d,t} &= a x_{i,t} && \text{when } i + n/d \leqslant t \\ x_{i+n/d,t+n/d} &= a x_{i,t}. \end{aligned}$$

Consider the element $1 \otimes \alpha^m - \alpha^m \otimes 1$ in I_m. Then

$$\begin{aligned} &(1 \otimes \alpha^m - \alpha^m \otimes 1) \cdot x_{i,i+j} \\ &= (1 \otimes \alpha^{qd} - \alpha^{qd} \otimes 1) \cdot (q \otimes \alpha^{(i+j)d} - [\alpha^{id} \otimes \alpha^{jd}]_d^m) \\ &= q \otimes \alpha^{(i+j+q)d} - [\alpha^{id} \otimes \alpha^{(j+q)d}]_d^m - q\alpha^{qd} \otimes \alpha^{(i+j)d} - [\alpha^{(i+q)d} \otimes \alpha^{jd}]_d^m \\ &= x_{i,i+j+q} - x_{i+q,i+j+q} + q \otimes \alpha^{(i+j+q)d} - q\alpha^{qd} \otimes \alpha^{(i+j)d}. \end{aligned}$$

As $q \otimes \alpha^{(i+j+q)d} - q\alpha^{qd} \otimes \alpha^{(i+j)d}$ is in I_m and thus in I_m^2, we get

$$x_{i,i+j+q} \equiv x_{i+q,i+j+q} \bmod \underline{I}^2. \tag{4.2}$$

A similar computation shows $x_{i,t} \cdot x_{i',t'} = x_{i,t+t'} + x_{i',t+t'} - x_{i+i',t+t'}$, so

$$x_{i,t} \equiv i \cdot x_{1,t} \bmod \underline{I}^2. \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$x_{0,t} \equiv x_{q,t} \equiv q \cdot x_{1,t} \bmod \underline{I}^2.$$

Now, $x_{0,t} = 0$ and q, being a factor of n, is invertible. This proves that $x_{1,t}$ and thus $x_{i,t}$ is in $\underline{I}^2$. As m, d are arbitrary in the proof, we have shown $\Omega^{1,C_n}_{\underline{L}^{\mathsf{fix}}|\underline{K}^c} = 0$.

Projectivity of $\underline{L}^{\mathsf{fix}}$ over $\underline{K}^c$ can be shown again via the decomposition into eigenspaces, as shown in Corollary 4.3.

□

Remark 4.5. In all our examples the Kähler differentials vanish because the generators of $\underline{I}$ can be shown to be elements of $\underline{I}^2$. So all our examples are étale extensions of the underlying Green functors.

We could however in the C_3-case for instance kill $[\alpha^2 \otimes \alpha] - [\alpha \otimes \alpha^2]$ with a norm, as

$$\mathsf{norm}(1 \otimes \alpha - \alpha \otimes 1) = [\alpha^2 \otimes \alpha] - [\alpha \otimes \alpha^2].$$

For the C_2-Galois examples $1 \otimes 1 + [\alpha \otimes \alpha]$ can also be written as $\mathsf{norm}(1 \otimes \alpha + \alpha \otimes 1)$ in characteristic 2, so it has two reasons to die. Similarly, in the C_2-Kummer case $2a - [\alpha \otimes \alpha] = \mathsf{norm}(1 \otimes \alpha - \alpha \otimes 1)$.

One could of course hope that for any finite G-Galois extension $K \subset L$, the extension $\underline{K}^c \to \underline{L}^{\mathsf{fix}}$ is G-Tambara étale, but such a claim would need a more conceptual proof than chasing generators of $\underline{I}$ to their death.

5. Constant Tambara functors on étale extensions

We assume that K is a field of arbitrary characteristic and that $K \subset L$ is an étale extension. In particular, the underlying K-vector space of L is finite dimensional. The aim of this section is to prove that $\underline{K}^c \to \underline{L}^c$ is a G-Tambara extension for every finite group G. To this end we provide a proof of a probably well-known identification of box-products of constant Tambara functors:

Lemma 5.1. *Assume that A and B are commutative rings. Then $\underline{A}^c \Box \underline{B}^c \cong \underline{(A \otimes B)}^c$ as G-Tambara functors for every finite group G.*

Proof. We use that $\underline{A}^c \Box \underline{B}^c$ is the coproduct of $\underline{A}^c$ and $\underline{B}^c$ in the category of G-Tambara functors [Str, Lemma 9.8].

Let $\underline{R}$ be an arbitrary G-Tambara functor and let $\varphi\colon \underline{A}^c \to \underline{R}$ and $\psi\colon \underline{B}^c \to \underline{R}$ be morphisms of G-Tambara functors. The unique induced morphism $\xi\colon \underline{A}^c \Box \underline{B}^c \to \underline{R}$ is determined by Weyl equivariant ring morphisms

$$\xi(G/H)\colon A \otimes B = \underline{A}^c(G/H) \otimes \underline{B}^c(G/H) \to \underline{R}(G/H)$$

for all $H < G$ that satisfy compatibility constraints with respect to restriction, tr and norm coming from the definition of the box product as a Day convolution product. See [Maz13, Remark 1.2.3 and p. 41] for an explicit list of requirements.

As the tensor product is the coproduct in the category of commutative rings, $\xi(G/H)$ is uniquely determined by ring maps $\xi_A(G/H)\colon A \to \underline{R}(G/H)$ and $\xi_B(G/H)\colon B \to \underline{R}(G/H)$.

As the action on the constant Tambara functors is trivial, the image of the $\xi(G/H)$ is contained in the Weyl fixed points. As restriction on $\underline{A}^c$ and $\underline{B}^c$ is the identity map, we get that for $H < K$, $\mathsf{res}^K_H \circ \xi(G/H) = \xi(G/K)$. The compatibility with tr demands that multiplication by the index in one tensor factor first and then applying ξ is the same as first applying ξ and then applying tr in $\underline{R}$. Similarly, applying first the norm in both factors and then applying ξ has to agree with first applying ξ and then applying the norm in $\underline{R}$. These are exactly the requirements for obtaining a morphism of G-Tambara functors from $\underline{(A \otimes B)}^c$ to $\underline{R}$. □

Corollary 5.2. *For every étale extension $K \subset L$ where K is a field, the extension $\underline{K}^c \to \underline{L}^c$ is G-Tambara étale for every finite group G.*

Proof. As $\underline{K}^c$-modules in G-Mackey functors $\underline{L}^c$ splits into a direct sum of $\underline{K}^c$s and hence is free, thus flat.

The coequalizer $\underline{L}^c \Box_{\underline{K}^c} \underline{L}^c$ of $\underline{L}^c \Box \underline{K}^c \Box \underline{L}^c \rightrightarrows \underline{L}^c \Box \underline{L}^c$ is isomorphic to the coequalizer of $\underline{(L \otimes K \otimes L)}^c \rightrightarrows \underline{(L \otimes L)}^c$ and this is nothing but $\underline{(L \otimes_K L)}^c$. The multiplication map

$$\underline{L}^c \Box_{\underline{K}^c} \underline{L}^c \to \underline{L}^c$$

is induced by the multiplication map on L over K and hence its kernel is $\underline{I}^c$ where I denotes the kernel of the multiplication map on L over K. As $K \subset L$ is étale, $I/I^2 = 0$, and hence $\Omega^{1,G}_{\underline{L}^c|\underline{K}^c} = 0$. □

Remark 5.3. This result can be generalized to étale extensions of rings $R \to A$ as long as $\underline{A}$ is a flat $\underline{R}$-module.

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