Symmetry properties of the Dold-Kan correspondence

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1 Introduction

The aim of this paper is to prove that the inverse of the normalization functor in the Dold-Kan correspondence $D : \mathsf{Ch}(\mathsf{Ab}) \to \mathsf{sAb}$ is an E_∞-monoidal functor. This proves that generalized Eilenberg-MacLane spectra on differential graded commutative algebras are E_∞-monoids in the category of $H\mathbb{Z}$-module spectra.

The Dold-Kan correspondence between the category of simplicial abelian groups and the category of non-negative chain complexes of abelian groups is a classical result. It states that the normalization functor N, which sends a simplicial abelian group A_* to its normalized chain complex possesses an adjoint functor D, such that

$$N : \mathsf{sAb} \rightleftarrows \mathsf{Ch}(\mathsf{Ab}) : D$$

is an equivalence of categories (see for instance [13, 8.4]).

It is known that the functor N respects the tensor product of simplicial abelian groups and that N is in fact a lax symmetric monoidal functor: There is a classical natural chain transformation from the tensor product of two normalized chain complexes $N(A_*)$ and $N(B_*)$ to the normalized chain complex of the diagonal tensor product $(A_* \hat{\otimes} B_*)_n = A_n \otimes B_n$ (see for instance [13, 8.5.4]):

$$g : N(A_*) \otimes N(B_*) \longrightarrow N(A_* \hat{\otimes} B_*). \tag{1}$$

As this transformation is given by shuffle maps, it is symmetric, i.e., the following diagram commutes

$$\begin{array}{ccc} N(A_*) \otimes N(B_*) & \xrightarrow{g} & N(A_* \hat{\otimes} B_*) \\ \downarrow t & & \downarrow N(t) \\ N(B_*) \otimes N(A_*) & \xrightarrow{g} & N(B_* \hat{\otimes} A_*). \end{array}$$

Here t denotes the map, which twists tensor factors. In particular the normalization of a commutative monoid in the category of simplicial abelian groups – that is a commutative simplicial ring – is send to a differential graded commutative algebra.

We will prove a slightly weaker result for the inverse of the normalization: The functor D is lax monoidal. For two chain complexes X_* and Y_* the Alexander-Whitney map yields a natural map $D(X_*) \hat{\otimes} D(Y_*) \to D(X_* \otimes Y_*)$ which is associative but *not* symmetric. We will prove that the functor D maps commutative chain algebras to E_∞-monoids in the category sAb, i.e., there is an E_∞-operad which acts on the image $D(R_*)$ of every commutative chain algebra R_*. Using methods of [9], we will prove this fact via showing that D is an "E_∞-monoidal functor" (see Definition 2.3 below).

One important application of this statement is the proof that given a differential graded commutative algebra R_* the generalized Eilenberg-MacLane spectrum $H(R_*)$ (see [10, §1]) is an E_∞-monoid in the category of modules over the Eilenberg-MacLane spectrum of $\mathbb{Z}$ in the category of symmetric spectra or Γ-spaces.

2 E_∞-monoidal functors

The way how we will prove the result that the functor D maps a differential graded commutative algebra to an E_∞-monoid is that we will show that there is an E_∞-operad $\mathcal{O}_D$ acting on the images of D. The aim of this section is to make this statement precise.

We use the standard terminology of operads as it can be found for instance in [6, part I], in particular an operad is always equipped with an action of the symmetric groups Σ_n. Let $(\mathcal{C}, \circledcirc, \mathbb{1}_\mathcal{C})$ be a symmetric monoidal category with $\circledcirc$ denoting the product and $\mathbb{1}_\mathcal{C}$ being the unit of the monoidal structure. Recall that the augmentation ε of an unital operad $\mathcal{O}$ is the map which is induced via the following composition:

$$\mathcal{O}(n) \cong \mathcal{O}(n) \circledcirc \underbrace{\mathcal{O}(0) \circledcirc \cdots \circledcirc \mathcal{O}(0)}_{n} \longrightarrow \mathcal{O}(0).$$

Assume that $\mathcal{C}$ has in addition a notion of weak equivalences, e.g. if $\mathcal{C}$ is a monoidal model category.

Definition 2.1 *Let $\mathcal{C}$ be a category as above. An* E_∞-operad in $\mathcal{C}$ *is a unital operad which is (non-equivariantly) weakly equivalent via its augmentation to the operad* Com *with* $\mathsf{Com}(n) = \mathbb{1}_\mathcal{C}$ *which characterizes commutative monoids in $\mathcal{C}$.*

Note that we do not demand the n-th term of the operad $\mathcal{C}(n)$ to be free over Σ_n. In the examples which we will consider, a replacement by a Σ-free operad can be achieved by taking a product (concerning the monoidal structure in $\mathcal{C}$) with an E_∞-operad $\mathcal{P}$ which is Σ-free. The action of the product on an $\mathcal{C}$-algebra is then given by the augmentation of $\mathcal{P}$ and the action of $\mathcal{C}$. We will apply this definition in the following examples of monoidal model categories:

- The category of simplicial abelian groups sAb is closed symmetric monoidal with the degreewise tensor product $\hat{\otimes}$. The model category structure is described in [8, II.4, Theorem 4].

- The category Ch(Ab) of non-negative chain complexes of abelian groups is closed symmetric monoidal concerning the usual tensor product of chain complexes; its standard model category structure is determined by choosing the weak equivalences as the maps which induce isomorphisms in homology and the fibrations to be the maps which are surjective in positive degrees.

- Let R be a commutative ring and let $H(R)$ denote the Eilenberg-MacLane spectrum of R. In the categories of Gamma spaces and symmetric spectra the category of module spectra over $H(R)$ is a closed symmetric monoidal model category if R is commutative. This fact is shown by Schwede and Shipley in [11, Theorem 4.1]. The model category structure of $H(R)$-modules is cofibrantly generated.

Remark 2.2 *For the category of (co)chain complexes Hinich and Schechtman in [4] called algebras over acyclic operads "May algebras".*

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories with the above properties. We denote the product in the monoidal structure of $\mathcal{C}$ by $\circledcirc$ and the one in $\mathcal{C}'$ by $\odot$. Note that we do not assume our monoidal structure to be strict. Thus the following diagrams should be actually written down with parentheses. But as this would lead to heavy notation, we omit this.

Definition 2.3 *A functor F from $\mathcal{C}$ to $\mathcal{C}'$ is called an E_∞-*monoidal functor*, if there is an E_∞-operad $\mathcal{O}_F$ in $\mathcal{C}'$ and if for every n-tupel $(C_1, \dots, C_n)$ of objects in $\mathcal{C}$ there are natural maps*

$$\gamma : \mathcal{O}_F(n) \odot F(C_1) \odot \cdots \odot F(C_n) \longrightarrow F(C_1 \circledcirc \cdots \circledcirc C_n)$$

with the following properties:

- *The action is unital: Let $\eta : \mathbb{1}_{\mathcal{C}'} \to \mathcal{O}_F(1)$ denote the unit of the operad, with $\mathbb{1}_{\mathcal{C}'}$ being the unit in the monoidal structure of $\mathcal{C}'$. Then the diagram*

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{C}'} \odot F(C) & \xrightarrow{\eta \odot id} & \mathcal{O}_F(1) \odot F(C) \\ & \searrow^{\cong} & \downarrow \gamma \\ & & F(C) \end{array}$$

  is commutative.

- *The action is equivariant: For every $\sigma \in \Sigma_n$, the right action with σ on $\mathcal{O}_F(n)$ interacts nicely with the left action on n-fold products by permuting the entries :*

$$\begin{array}{ccc} \mathcal{O}_F(n) \odot (F(C_1) \odot \cdots \odot F(C_n)) & \xrightarrow{\gamma} & F(C_1 \circledcirc \cdots \circledcirc C_n) \\ \downarrow{\scriptstyle \sigma \odot \sigma} & & \downarrow{\scriptstyle F(\sigma)} \\ \mathcal{O}_F(n) \odot (F(C_{\sigma^{-1}(1)}) \odot \cdots \odot F(C_{\sigma^{-1}(n)})) & \xrightarrow{\gamma} & F(C_{\sigma^{-1}(1)} \circledcirc \cdots \circledcirc C_{\sigma^{-1}(n)}) \end{array}$$

- *The more or less obvious associativity condition is fulfilled (see [9, p.552]).*

The reason for introducing this terminology is the following fact.

Proposition 2.4 *Given an E_∞-monoidal functor F, the image of a commutative monoid R in $\mathcal{C}$ under F is an E_∞-monoid in $\mathcal{C}'$.*

Proof Let us denote the multiplication map of R by μ. We prolong the given operad action on F with this multiplication and obtain

$$\mathcal{O}_F(n) \odot F(R) \odot \cdots \odot F(R) \xrightarrow{\gamma} F(R \circledcirc \cdots \circledcirc R) \xrightarrow{F(\mu)} F(R).$$

As R is commutative, this is a well defined operad action of the E_∞-operad $\mathcal{O}_F$ on $F(R)$.

□

3 Definition of the operad

Let us briefly recall the definition of the normalized chain complex associated to a simplicial abelian group A_*. In degree n the complex $N(A_*)$ is $N(A_*)_n = \bigcap_{i=1}^n \ker d_i$; here the d_i are the face maps arising from the simplicial structure of A_*. The differential on $N(A_*)$ is the remaining zeroth face map d_0.

The aim of this section is to define an E_∞-operad $\mathcal{O}_D$ which acts on the images of the functor D. The natural attempt is to use an enriched analog of the endomorphism operad. To be more precise we define an operad in the category of simplicial abelian groups in the following way. Let Δ denote as usual the category of finite ordinal numbers and order preserving maps and let $\mathbb{Z}[\Delta_k]$ denote the free abelian group on the standard k-simplex $\Delta_k = \mathsf{hom}_\Delta(-, k)$. We will use the definition of D which comes out of this equivalence, namely as D and N are adjoint and as we can apply the Yoneda-lemma to $\mathbb{Z}[\Delta_k]$, we obtain for a chain complex X_*

$$D(X_*)_k \cong \mathsf{hom}_{\mathsf{sAb}}(\mathbb{Z}[\Delta_k], D(X_*)) \cong_{\text{adj.}} \mathsf{hom}_{\mathsf{Ch(Ab)}}(N\mathbb{Z}[\Delta_k], X_*).$$

Hence D is given by $D(-)_k = \mathsf{hom}_{\mathsf{Ch(Ab)}}(N\mathbb{Z}[\Delta_k], -)$.

We construct our operad using the natural transformations of two functors from the products of the category of chain complexes $\mathsf{Ch(Ab)}$ to the category of simplicial abelian groups. For $n > 0$ and n chain complexes $C_1, \ldots, C_n$ define

$$\begin{aligned} D^{\hat{\otimes} n}(C_1, \ldots, C_n) &= D(C_1) \hat{\otimes} \cdots \hat{\otimes} D(C_n) \\ D^{\otimes n}(C_1, \ldots, C_n) &= D(C_1 \otimes \cdots \otimes C_n). \end{aligned}$$

Definition 3.1 *Let $\mathcal{O}_D$ be the operad of simplicial abelian groups whose n-th term in simplicial degree k is*

$$\mathcal{O}_D(n)_k = \mathsf{Nat}_{\mathsf{sAb}^{\mathsf{Ch}^n}}(D^{\hat{\otimes} n} \hat{\otimes} \mathbb{Z}[\Delta_k], D^{\otimes n}).$$

The action of the symmetric groups on the operad is given in the standard way, i.e. if f is a natural transformation in $\mathcal{O}_D(n)_k$ and σ is a permutation in Σ_n, then for an n-tuple of chain complexes $(X_1, \ldots, X_n)$ the transformation $f.\sigma$ is defined via the following diagram:

$$\begin{array}{ccc}
D(X_1)\hat{\otimes} \cdots \hat{\otimes} D(X_n) & \xrightarrow{f.\sigma} & D(X_1 \otimes \cdots \otimes X_n) \\
\downarrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle D(\sigma^{-1})} \\
D(X_{\sigma^{-1}(1)})\hat{\otimes} \cdots \hat{\otimes} D(X_{\sigma^{-1}(n)}) & \xrightarrow{f} & D(X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)})
\end{array}$$

The composition in $\mathcal{O}_D$ is given by the composition of natural transformations. This operad acts naturally on the images of D via the evaluation map. Its zeroth term $\mathcal{O}_D(0)$ consists of the enriched natural transformations from the constant functor which assigns $\underline{\mathbb{Z}}$ to every chain complex to the functor D. Here $\underline{\mathbb{Z}}$ is the simplicial abelian group which is $\mathbb{Z}$ in every simplicial degree and which has the identity of $\mathbb{Z}$ as all face and degeneracy maps. Note that $\mathcal{O}_D(0)$ can be identified with $\underline{\mathbb{Z}}$ because the functor D applied to the chain complex $(\mathbb{Z}, 0)$ which is $\mathbb{Z}$ in degree zero and zero in all other degrees gives the constant simplicial abelian group $\underline{\mathbb{Z}}$.

4 Proof of the E_∞-property

What remains to be shown is that the operad $\mathcal{O}_D$ is weakly equivalent to the operad Com of commutative monoids in sAb which is $\mathsf{Com}(n) = \underline{\mathbb{Z}}$ for all n.

Theorem 4.1 *The operad $\mathcal{O}_D$ is an E_∞-operad.*

Proof For a cosimplicial and simplicial set X_*^* the total space Tot of X_*^* is the simplicial set given in simplicial degree k by the maps of cosimplicial simplicial sets

$$\Delta \times \Delta_k \longrightarrow X_*^*.$$

Let us denote by $\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})$ the functor from $\Delta^{op} \times \Delta$ into abelian groups given by

$$\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})(\ell, m) = \mathsf{Nat}_{\mathsf{Ab}^{\mathsf{Ch}^n}}(D_m^{\hat{\otimes} n}, D_\ell^{\otimes n}).$$

Then our operad is the total simplicial space of the cosimplicial simplicial abelian group $\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})$ (compare [1, X,3.3]):

$$\mathsf{Tot}\left(\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})\right) \cong \mathcal{O}_D(n)$$

and we can use the tower of fibration spectral sequence for the total space which arises from the skeleton filtration of Δ (as described in [1, X,§§6,7] or [3, VIII.1]). The corresponding E^2-term is the following:

$$E_2^{p,q} = \pi^p \pi_q \underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n}) \Rightarrow \pi_{q-p}\mathsf{Tot}\left(\underset{\sim}{\mathsf{nat}}(D^{\hat{\otimes} n}, D^{\otimes n})\right) \cong \pi_{q-p}\mathcal{O}_D(n).$$

With the following slight generalization of the Yoneda lemma we can finish the proof:

Lemma 4.2 *Let G be a functor from the n-fold product of the category of chain complexes $\mathsf{Ch}(\mathsf{Ab})^n$ to the category of abelian groups Ab, which is linear in every component. Then the natural transformations from the n-fold tensor product of representables $\mathsf{hom}_{\mathsf{Ch}}(X_i, -)$ to G is isomorphic to G evaluated on the X_i:*

$$\mathsf{Nat}_{\mathsf{Ab}^{\mathsf{Ch}^n}}(\mathsf{hom}_{\mathsf{Ch}}(X_1, -) \otimes \cdots \otimes \mathsf{hom}_{\mathsf{Ch}}(X_n, -), G) \cong G(X_1, \ldots, X_n).$$

Proof The usual proof of the Yoneda lemma works, but the multilinearity of G leads to the identification with the tensor product of the representable functors. □

In the above situation this n-dimensional version of the Yoneda-lemma identifies $\underset{\sim}{\mathsf{nat}}(D_m^{\hat{\otimes} n}, D_\ell^{\otimes n})$ with $D^{\otimes n}(N\mathbb{Z}\Delta_m)_\ell$ because D is representable. Determining homotopy in simplicial direction in the above E^2-tableau thus reduces to calculate the homotopy groups of $D(N\mathbb{Z}\Delta_m \otimes \cdots \otimes N\mathbb{Z}\Delta_m)_*$. The shuffle-map from (1) is a homotopy equivalence. As D preserves quasi-isomorphisms we obtain that

$$\pi_q D(N\mathbb{Z}\Delta_m \otimes \cdots \otimes N\mathbb{Z}\Delta_m)_* \cong \pi_q D(N(\mathbb{Z}\Delta_m \hat{\otimes} \cdots \hat{\otimes} \mathbb{Z}\Delta_m))_*$$

which is nothing but the homotopy of $\mathbb{Z}[\Delta_m \times \cdots \times \Delta_m]$. Hence the homotopy groups are trivial except in dimension zero where they are $\mathbb{Z}$. The codifferentials in m-direction are induced by maps $\Delta_m \to \Delta_{m+1}$; hence depending on the parity of m the codifferential is the identity of $\mathbb{Z}$ or zero and the cohomotopy is concentrated in degree zero.

□

Remark 4.3 *The augmentation $\varepsilon : \mathcal{O}_D(n) \to \mathcal{O}_D(0) \cong \underline{\mathbb{Z}}$ provides a weak equivalence from $\mathcal{O}_D(n)$ to $\underline{\mathbb{Z}}$. The map ε restricts a natural transformation from $D^{\hat{\otimes} n}$ to $D^{\otimes n}$ to its value on the n-fold tensor product of the simplicial abelian group $\underline{\mathbb{Z}}$ and the calculation in the spectral sequence shows that the terms $N\mathbb{Z}\Delta_m \simeq (\mathbb{Z}, 0)$ determine the homology of $\mathcal{O}_D(n)$.*

5 Symmetries of N and D

Given an equivalence of categories with one functor of the equivalence lax symmetric monoidal, one can dualize the symmetry properties of the involved functors:

Let us note that the lax symmetry of the functor N translates to the fact that D is lax symmetric comonoidal. That means there are natural transformations

$$c_{(X_*,Y_*)} : D(X_* \otimes Y_*) \longrightarrow D(X_*)\hat{\otimes} D(Y_*)$$

for any two chain complexes X_* and Y_* which are symmetric and coherent with respect to coassociativity and counit. The transformation c is given by

$$D(\mathsf{shuffle}) : D(N \circ D(X_*) \otimes N \circ D(Y_*)) \longrightarrow D(N(D(X_*)\hat{\otimes} D(Y_*)))$$

where the latter term is naturally isomorphic to $D(X_*)\hat{\otimes} D(Y_*)$. The symmetry results from the symmetry of N and the naturality of D.

Dually to 2.3 one can define the notion of E_∞-comonoidal functors (see [9, p.553]). Using that D is an E_∞-monoidal functor via the operad $\mathcal{O}_D$ leads to an action of the E_∞-operad $N(\mathcal{O}_D)$ on N. This makes N an E_∞-comonoidal functor: First the lax symmetric transformation given by the shuffle map induces

$$N(\mathcal{O}_D(n)) \otimes N(A^1_*\hat{\otimes} \cdots \hat{\otimes} A^n_*) \longrightarrow N(\mathcal{O}_D(n)\hat{\otimes} DN(A^1_*)\hat{\otimes} \cdots \hat{\otimes} DN(A^n_*)).$$

With the help of the action of $\mathcal{O}_D$ on D we end up in $N(D(N(A^1_*) \otimes \cdots \otimes N(A^n_*))$ and by the equivalence of N and D this is naturally isomorphic to

$$N(D(N(A^1_*) \otimes \cdots \otimes N(A^n_*))) \cong N(A^1_*) \otimes \cdots \otimes N(A^n_*).$$

Thus the image of the E_∞-operad $\mathcal{O}_D$ under N can be used to split the image of a product under N into pieces. Note that we used the lax symmetry of N for this process.

Remark 5.1 *For the functor C : $\mathsf{sAb} \longrightarrow \mathsf{Ch}(\mathsf{Ab})$ which maps a simplicial abelian group to its unnormalized chain complex, it is already known that C is E_∞-comonoidal (see [9, § 7,p.563] and [12]).*

In [2, Satz 1.6] Dold proved the necessary identifications: The homology of the enriched natural transformations from the (unreduced) chain complex on the n-fold tensor product of simplicial abelian groups $C((-)\hat{\otimes} \cdots \hat{\otimes}(-))$ to the tensor product of the complexes $C(-) \otimes \cdots \otimes C(-)$ is concentrated in degree zero where it is $\mathbb{Z}$.

5.1 The Eilenberg-Zilber operad

For the normalization functor N^* of cosimplicial abelian groups, Hinich and Schechtman [4] proved the acyclicity of the corresponding operad, called the Eilenberg-Zilber operad. We will abbreviate this operad with $\mathcal{EZ}$. Let $\mathsf{Ch}^\bullet(\mathsf{Ab})$ denote the category of cochain complexes and let cAb be the category of cosimplicial abelian groups. We consider $N(\mathbb{Z}[\Delta])$ as the cosimplicial object in chain complexes which is $N(\mathbb{Z}[\Delta_k])_n$ in cosimplicial degree k and chain degree n. The normalization N^* of a cosimplicial abelian group $A^* \in \mathsf{cAb}$ is then given by the morphisms of cosimplicial abelian groups from $N(\mathbb{Z}[\Delta])$ to A^*. The differential on the normalized chain complex makes this object a cochain complex. The operad $\mathcal{EZ}$ is defined as the cochain complex of the morphisms from this normalization to its n-fold tensor product

$$\mathcal{EZ}(n) = \mathsf{hom}_{\mathsf{cAb}}(N(\mathbb{Z}[\Delta]), N(\mathbb{Z}[\Delta])^{\hat{\otimes} n}).$$

Here $\hat{\otimes}$ denotes the degreewise tensor product of cosimplicial abelian groups. In degree n this operad gives natural transformations from $N^*(A^*) \otimes \cdots \otimes N^*(A^*)$ to $N^*(A^*)$ for any cosimplicial commutative ring A^*

The natural attempt would be to let an analog of $\mathcal{EZ}$ act as well on the images of $D : \mathsf{Ch}(\mathsf{Ab}) \to \mathsf{sAb}$ using the definition of D as $\mathsf{hom}_{\mathsf{Ch}(\mathsf{Ab})}(N\mathbb{Z}[\Delta], -)$ but to this end one would have to give $\mathcal{EZ}$ a structure of a simplicial abelian group.

6 Generalized Eilenberg-MacLane spectra

The category of symmetric spectra defined in [5] and the category of Γ-spaces (see [7] and [10]) lead to nice symmetric monoidal categories of spectra. In both of these models for the stable homotopy category there is a standard model of an Eilenberg-MacLane spectrum for a given simplicial abelian group. The functor H which assigns the Eilenberg-MacLane spectrum $H(A_*)$ to a simplicial abelian group A_* is a symmetric functor.

The construction of this functor is straightforward in both categories: Let $\mathbb{S}^1$ be the simplicial model of the 1-sphere given by $\Delta_1/\partial\Delta_1$ and let $\mathbb{S}^n$ be defined as $\mathbb{S}^n = \underbrace{\mathbb{S}^1 \wedge \ldots \wedge \mathbb{S}^1}_{n}$. For $A_* \in \mathsf{sAb}$ the spectrum $H(A_*)$ can be defined as $H(A_*)_n := A_* \hat{\otimes} \mathbb{S}^n$ and $H(A_*)_n$ is the n-th term of the diagonal of the bisimplicial set $(A_p \otimes \mathbb{S}^n_q)_{p,q}$. The homotopy groups of the spectrum $H(A_*)$ are naturally isomorphic to the ones of A_*. The symmetry of H is then encoded in the diagram

$$\begin{array}{ccc} H(A_*) \otimes H(B_*) & \longrightarrow & H(A_* \hat{\otimes} B_*) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle H(t)} \\ H(B_*) \otimes H(A_*) & \longrightarrow & H(B_* \hat{\otimes} A_*). \end{array}$$

The horizontal maps are induced by the tensor product of A_* and B_* and the smash product of finite pointed sets (see [10, p.332-333]).

We want to clarify the symmetry of the functor H precomposed with the functor D:

Proposition 6.1 *Given a differential graded commutative algebra R_*, the Eilenberg-MacLane spectrum $H(D(R_*))$ is an E_∞-monoid in the category of $H\mathbb{Z}$-module spectra in symmetric spectra resp. Γ-spaces.*

Proof We already know that $D(R_*)$ is an E_∞-monoid in the category of simplicial abelian groups. The image of our operad $\mathcal{O}_D$ under H is weakly equivalent to $H\mathbb{Z}$ and the functor H is lax symmetric monoidal; hence the action of the operad $\mathcal{O}_D$ on $D(R_*)$ translates to an action of the operad $H(\mathcal{O}_D)$ on $H \circ D(R_*)$ and the operad $H(\mathcal{O}_D)$ is still an E_∞-operad.

□

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References

[1] A.K.Bousfield, D.M.Kan, Homotopy Limits, completions and localizations, Lecture Notes in Mathematics 304, second corrected printing, Springer, (1987)

[2] A.Dold, Über die Steenrodschen Kohomolgieoperationen, Annals of Mathematics 73, (1961)

[3] P.G.Goerss, J.F.Jardine, Simplicial homotopy theory, Birkhäuser, Progress in Mathematics 174, (1999)

[4] V.A. Hinich, V.V. Schechtman, On homotopy limit of homotopy algebras, Lecture Notes in Mathematics 1289, Springer (1987), 240-264

[5] M. Hovey, B. Shipley and J. Smith, Symmetric spectra, J. Amer. Math. Soc. 13, no. 1 (2000), 149–208

[6] I. Kriz, J.P. May, Operads, algebras, modules and motives, Asterisque 233, (1999)

[7] M. Lydakis, Smash products and Γ-spaces, Math. Proc. Cambridge Philos. Soc. 126 (1999), 311-328

[8] D. Quillen, Homotopical Algebra, Lecture Notes in Mathematics 43, Springer (1967)

[9] B. Richter, E_∞-structure for $Q_*(R)$, Math. Ann., 316, 3 (2000), 547-564

[10] S. Schwede, Stable homotopical algebra and Γ-spaces, Math. Proc. Cambridge Philos. Soc. 126 (1999), 329-356

[11] S. Schwede, B. Shipley, Algebras and modules in monoidal model categories, Proc. London Math. Soc. 80 (2000), 491-511

[12] V.A. Smirnov, Homotopy theory of coalgebras, Math. USSR Izv. (1986), 575-592

[13] C. Weibel, An introduction to homological algebra, Cambridge studies in advanced mathematics 38, (1994)