Brave New Algebra

Birgit Richter

This talk is based on my book chapter: Commutative ring spectra, to appear in *Stable categories and structured ring spectra*, edited by Andrew J. Blumberg, Teena Gerhardt, and Michael A. Hill, MSRI Book Series, Cambridge University Press.

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However, some authors assumed the existence of such models before that and drew their conclusions and did their calculations earlier.

Some features of working with ring spectra

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In order to stress how large the gap is between S and $H\mathbb{Z}$ or $H\mathbb{F}_p$, we'll see that there is a Galois extension that sits between S and the prime field $H\mathbb{F}_p$.

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The map ι comes from taking the adjoint of the map

$$A \wedge EG_+ \xrightarrow{\mathrm{id}\wedge p} A \wedge S^0 \cong A \longrightarrow B$$

where $p\colon EG_+ \to S^0$ collapses EG to the non-base point of S^0.

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It is also necessary for Galois extensions of discrete commutative rings in order to ensure that the extension is unramified.

For instance $\mathbb{Z} \subset \mathbb{Z}[i]$ satisfies $\mathbb{Z}[i]^{C_2} = \mathbb{Z}$, but $h\colon \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i] \to \mathbb{Z}[i] \times \mathbb{Z}[i]$ is not surjective:

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Important examples of Galois extensions of commutative ring spectra are the following. By C_n we denote the cyclic group of order n.

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- The complexification of real vector bundles gives rise to a map of commutative ring spectra $KO \to KU$ from real to complex topological K-theory. There is a C_2-action on KU corresponding to complex conjugation of complex vector bundles. Rognes shows that this turns $KO \to KU$ into a C_2-Galois extension.

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- At an odd prime p there is a p-adic Adams operation on KU_p that gives rise to a C_{p-1}-action on KU_p such that $L_p \to KU_p \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} L_p$ is a C_{p-1}-Galois extension.

- Let p be an arbitrary prime. The projection map $\pi\colon EC_p \to BC_p = EC_p/C_p$ induces a map on function spectra

$$F(\pi_+, H\mathbb{F}_p)\colon F((BC_p)_+, H\mathbb{F}_p) \to F((EC_p)_+, H\mathbb{F}_p) \sim H\mathbb{F}_p$$

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Beware, this Galois extension is not faithful. This observation is due to Ben Wieland: the Tate construction $H\mathbb{F}_p^{tC_p}$ isn't trivial and it is actually killed by the Galois extension (in the spectral sequence you augment a Laurent generator to zero).

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Issues with commutativity

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In algebra, if someone tells you to check whether a given ring is commutative, you can sit down and check the axiom for commutativity and you should be fine.
In stable homotopy theory the problem is more involved, since strict commutativity may only be satisfied by some preferred point set level model of the underlying associative ring spectrum and the incarnation of commutativity is an extra structure rather than a condition.

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Then it is *not* clear that E is a commutative ring spectrum.

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It can be realized as the image of an idempotent on MU and satisfies

$$\pi_*(BP) \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$$

but now the algebraic generators are spread out in an exponential manner:

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There are even worse examples: If you take the sphere spectrum S and you try to kill the non-regular element $2 \in \pi_0(S)$ then you get the mod-2 Moore spectrum.

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You can also kill all the generators $v_i \in \pi_*(BP)$ including $p = v_0$, leaving only one v_n alive. The resulting spectrum is the connective version of Morava K-theory, $k(n)$. At the prime 2 this isn't even homotopy commutative.

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He showed that for every associative ring R there is an equivalence of categories between the derived category of R, $\mathcal{D}(R)$, and the derived category of the associated Eilenberg-Mac Lane spectrum, $\mathcal{D}(HR)$.

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If R is in addition commutative, then H and L induce a Quillen equivalence between the categories of simplicial R-algebras and HR-algebra spectra.

Here, the functor L is left inverse to H and induces an isomorphism of Γ-spaces

$$\mathrm{Hom}(HA, HB) \cong H(\mathrm{Hom}_{\mathsf{sAb}}(A, B));$$

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Theorem [Shipley 2007] For any commutative ring R, the model categories of unbounded differential graded R-algebras and HR-algebra spectra are Quillen equivalent.

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A concrete example is the differential graded ring A_* which is generated by an element in degree 1, e_1, and has $d(e_1) = 2$ and satisfies $e_1^4 = 0$.

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A concrete example is the differential graded ring A_* which is generated by an element in degree 1, e_1, and has $d(e_1) = 2$ and satisfies $e_1^4 = 0$.
The corresponding $H\mathbb{Z}$-algebra spectrum is equivalent as a ring spectrum to the one on the exterior algebra $B_* = \Lambda_{\mathbb{F}_2}(x_2)$ (with $|x_2| = 2$) but A_* and B_* are *not* quasi-isomorphic.

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Haldun Özgür Bayındır shows that one can find E_∞-differential graded algebras that are not quasi-isomorphic, but whose corresponding commutative HR-algebra spectra are equivalent as commutative ring spectra.

Shipley and Dugger show, that the two maps

$$H\mathbb{Z} \cong H\mathbb{Z} \wedge S \to H\mathbb{Z} \wedge H\mathbb{F}_2$$

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