The Hodge decomposition of higher order Hochschild homology

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joint work with Stephanie Ziegenhagen

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Hodge decomposition for Hochschild homology

Pirashvili's Hodge decomposition

E_n-homology

A resolution spectral sequence

Hodge decomposition revisited

Hochschild homology

Assume that A is a commutative, associative and unital k-algebra and let M be a symmetric A-bimodule. Let A and M be k-projective.

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Here, $b = \sum_{i=0}^{n} (-1)^i d_i$ where
$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_n$ for $i < n$ and
$d_n(a_0 \otimes \ldots \otimes a_n) = a_n a_0 \otimes \ldots \otimes a_{n-1}$.

Via the 1-sphere

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$d_i \colon [n] \to [n-1]$,

$$d_i(j) = \begin{cases} j, & j < i \\ i, & j = i < n, \quad (0, \quad j = i = n), \\ j-1, & j > i. \end{cases}$$

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$\mathbb{S}^1$:

$$\begin{tikzcd}
{[0]} \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & {[1]} \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] & {[2]} & \cdots
\end{tikzcd}$$

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Then we can define $\mathcal{L}(A; M)$ which sends $\Gamma \ni [n] \mapsto M \otimes A^{\otimes n}$.
Interpreting $\mathbb{S}^1$ as a functor $\Delta^{op} \to \Gamma$ we get by composition $\mathcal{L}(A; M) \circ \mathbb{S}^1 \colon \Delta^{op} \to k$-mod and

$$HH_*(A; M) = \pi_* \mathcal{L}(A; M)(\mathbb{S}^1).$$

Classical Hodge decomposition

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There are many ways to prove that:

- Collapse of a spectral sequence (Quillen),
- combinatorially (Hain, Gerstenhaber-Schack, Loday),
- using functor homology (Pirashvili),
- ...

From Quillen's spectral sequence one obtains:

$$HH_m^{(\ell)}(A; \mathbb{Q}) \cong H_{m-\ell}(\Lambda^\ell(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q})).$$

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Let $HH_*^{[n]}(A; M)$ denote Hochschild homology of order n:

$$HH_*^{[n]}(A; M) := \pi_*\mathcal{L}(A; M)(\mathbb{S}^n)$$

for $n \geq 1$.

Parity matters – nothing else

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$$HH_{\ell+n}^{[n]}(A; \mathbb{Q}) \cong \bigoplus_{i+nj=\ell+n} HH_{i+j}^{(j)}(A; \mathbb{Q}).$$

Here $HH_*^{(j)}(A; \mathbb{Q})$ is the j-th Hodge summand of ordinary Hochschild homology.

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$$HH_{\ell+n}^{[n]}(A;\mathbb{Q}) \cong \bigoplus_{i+nj=\ell+n} \mathrm{Tor}_i^{\Gamma}(\theta^j, \mathcal{L}(A,\mathbb{Q})).$$

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Here, $\theta^j[n]$ is the dual of the $\mathbb{Q}$-vector space that is generated by the $S \subset \{1, \ldots, n\}$ with $|S| = j$.

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Answers:
1. Stability result for n-Gerstenhaber algebras, that compares n to $n+2$.
2. Yes!

Little n-cubes

Let C_n denote the operad of little n-cubes.
Then $(C_*C_n(r))_r$, $r \geq 1$ is an operad in the category of chain complexes. Let E_n be a cofibrant replacement of C_*C_n.

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Theorem [Fresse 2011] There is an n-fold bar construction for E_n-algebras, B^n, such that

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I.e., E_n-homology is the homology of an n-fold algebraic delooping.

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Setting: In the following k is a field, most of the times $k = \mathbb{Q}$. The underlying chain complex of A_* is non-negatively graded.

n-Lie algebras

Definition An *n-Lie algebra over* $\mathbb{Q}$ is a non-negatively graded $\mathbb{Q}$-vector space, $\mathfrak{g}_*$, together with a Lie bracket of degree n, $[-,-]$:

$$[-,-]\colon \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j+n}, i, j \geq 0.$$

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1. $[-,-]$ satisfies a graded Jacobi relation:

$$(-1)^{pr}[x,[y,z]] + (-1)^{qp}[y,[z,x]] + (-1)^{rq}[z,[x,y]] = 0,$$

2. and graded antisymmetry:

$$[x,y] = -(-1)^{pq}[y,x].$$

Here, $p = |x| + n$, $q = |y| + n$ and $r = |z| + n$.

n-Gerstenhaber algebras

Definition An *n-Gerstenhaber algebra over* $\mathbb{Q}$ is an n-Lie algebra G_* together with a unital commutative $\mathbb{Q}$-algebra structure on G_* and an augmentation $\varepsilon\colon G_* \to \mathbb{Q}$ such that the Poisson relation holds

$$[a, bc] = [a, b]c + (-1)^{q(r-n)} b[a, c], \text{ for all homogeneous } a, b, c \in G_*$$

with $|a| = q - n$, $|b| = r - n$, and such that $\varepsilon[a, b] = 0$.

Free objects and indecomposables

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For $G_* \in nG$ let $Q_{nG}(G_*)$ be the graded vector space of indecomposables.

Homology of free objects

A classical Lemma [Cohen] In characteristic zero:

$$H_*(E_n(\bar{A}_*)) \cong nG(H_*(\bar{A}_*)).$$

Resolution spectral sequence

Theorem There is a spectral sequence

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{nG}(H_*(\bar{A}_*)))_q \Rightarrow H^{E_n}_{p+q}(\bar{A}_*).$$

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d^1 takes homology wrt resolution degree.

Hodge summands as Quillen homology of Gerstenhaber algebras

Theorem Let A be a commutative augmented $\mathbb{Q}$-algebra. For all $\ell, k \geq 1$ and $m \geq 0$:

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- $$\mathrm{Tor}^{\Gamma}_{m-\ell+1}(\theta^\ell, \mathcal{L}(A; \mathbb{Q})) \cong (\mathbb{L}_m Q_{(2k-1)G} \bar{A})_{(\ell-1)(2k-1)}.$$

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$(nG)^{\ell+1}(\bar{A})_{qn} \cong ((n+2)G)^{\ell+1}(\bar{A})_{q(n+2)}$: exchange n-Lie brackets by $(n+2)$-Lie brackets and adjust the internal degrees.
This yields an isomorphism of resolutions and hence an isomorphism on the corresponding homology groups.

Exterior and symmetric powers of derived Kähler differentials

Theorem For every augmented commutative $\mathbb{Q}$-algebra A we can identify the Hodge summands of Hochschild homology of order $2k$ for $k \geq 1$ as

$$\begin{aligned} \mathrm{Tor}^{\Gamma}_{m+1-\ell}(\theta^{\ell}, \mathcal{L}(A; \mathbb{Q})) &\cong (\mathbb{L}_m Q_{(2k-1)}\bar{A})_{(2k-1)(\ell-1)} \\ &\cong H_{m-\ell+1}(\mathrm{Sym}^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q})). \end{aligned}$$

We also recover the identification for Hodge summands of Hochschild homology of odd order:

$$HH^{(\ell)}_{m+1}(A; \mathbb{Q}) \cong \mathbb{L}_m Q_{2kG}(\bar{A})_{2k(\ell-1)} \cong H_{m-\ell+1}(\Lambda^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q})).$$

Proof

Input: For $A = S(V)$ for an n-Lie algebra V and $S(V)$ with the induced n-Gerstenhaber structure one has

$$(\mathbb{L}_p Q_{nG}(S(V)))_q \cong (\mathbb{L}_p Q_{nL}(V))_q.$$

Consider: $C_{*,*} =$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
(SI)^{\circ(3)}(A) \arrow[d] & (nG)((SI)^{\circ(3)}(A)) \arrow[l] \arrow[d] & (nG)^{\circ(2)}((SI)^{\circ(3)}(A)) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
(SI)^{\circ(2)}(A) \arrow[d] & (nG)((SI)^{\circ(2)}(A)) \arrow[l] \arrow[d] & (nG)^{\circ(2)}((SI)^{\circ(2)}(A)) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
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\end{tikzcd}$$

Proof cont.

$$H^h_r(H^v_s(C_{*,*})) \cong \mathbb{L}_r Q_{nG}(\bar{A})$$

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Proof cont.

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concentrated in the $(s = 0)$-line.

$$H_r^v(H_s^h(C_{*,*})) \cong H_r \mathbb{L}_s Q_{nG}(SI)^{\circ(\bullet+1)}(A).$$

This is n-Lie-homology of a trivial n-Lie algebra and this causes the symmetric and exterior powers of $\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q}$ for
$P_t = (SI)^{\circ(t+1)}(A)$.