Towards an understanding of ramified extensions of structured ring spectra

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Women in Homotopy Theory and Algebraic Geometry

Structured ring spectra

Slogan: Nice cohomology theories behave like commutative rings.
Brown: Cohomology theories can be represented by spectra:
$E^n(X) \cong [X, E_n]$
(E_n): family of spaces with $E_n \simeq \Omega E_{n+1}$.
Since the mid 90's: There are (several) symmetric monoidal model categories whose homotopy categories are Quillen equivalent to the good old stable homotopy category:

- Symmetric spectra (Hovey, Shipley, Smith)
- S-modules (Elmendorf-Kriz-Mandell-May aka EKMM)
- ...

We are interested in commutative monoids (commutative ring spectra) and their algebraic properties.

Examples

You all know examples of such commutative ring spectra:

- Take your favorite commutative ring R and consider singular cohomology with coefficients in R, $H^*(-; R)$. The representing spectrum is the Eilenberg-MacLane spectrum of R, HR. The multiplication in R turns HR into a commutative ring spectrum.
- Topological complex K-theory, $KU^0(X)$, measures how many different complex vector bundles of finite rank live over your space X. You consider isomorphism classes of complex vector bundles of finite rank over X, $Vect_{\mathbb{C}}(X)$. This is an abelian monoid wrt the Whitney sum of vector bundles. Then group completion gives $KU^0(X)$:

$$KU^0(X) = Gr(Vect_{\mathbb{C}}(X)).$$

This can be extended to a cohomology theory $KU^*(-)$ with representing spectrum KU. The tensor product of vector bundles gives KU the structure of a commutative ring spectrum.

- Topological real K-theory, $KO^0(X)$, is defined similarly, using real instead of complex vector bundles.
- Stable cohomotopy is represented by the sphere spectrum S.

Spectra have stable homotopy groups:

- $\pi_*(HR) = H^{-*}(pt; R) = R$ concentrated in degree zero.
- $\pi_*(KU) = \mathbb{Z}[u^{\pm 1}]$, with $|u| = 2$. The class u is the Bott class.
- The homotopy groups of KO are more complicated.

$$\pi_*(KO) = \mathbb{Z}[\eta, y, w^{\pm 1}]/2\eta, \eta^3, \eta y, y^2 - 4w, \quad |\eta| = 1, |w| = 8.$$

The map that assigns to a real vector bundle its complexified vector bundle induces a ring map $c\colon KO \to KU$. Its effect on homotopy groups is $\eta \mapsto 0$, $y \mapsto 2u^2$, $w \mapsto u^4$. In particular, $\pi_*(KU)$ is a graded commutative $\pi_*(KO)$-algebra.

Galois extensions of structured ring spectra

Actually, KU is a commutative KO-algebra spectrum. Complex conjugation gives rise to a C_2-action on KU with homotopy fixed points KO. In a suitable sense KU is unramified over KO: $KU \wedge_{KO} KU \simeq KU \times KU$.
Rognes '08: KU is a C_2-Galois extension of KO.
Definition (Rognes '08) (up to cofibrancy issues..., G finite) A commutative A-algebra spectrum B is a G-Galois extension, if G acts on B via maps of commutative A-algebras such that the maps

- $i\colon A \to B^{hG}$ and
- $h\colon B \wedge_A B \to \prod_G B \qquad (*)$

are weak equivalences.
This definition is a direct generalization of the definition of Galois extensions of commutative rings (due to Auslander-Goldman).

Examples

As a sanity check we have:
Rognes '08: Let $R \to T$ be a map of commutative rings and let G act on T via R-algebra maps. Then $R \to T$ is a G-Galois extension of commutative rings iff $HR \to HT$ is a G-Galois extension of commutative ring spectra.

Let $\mathbb{Q} \subset K$ be a finite G-Galois extension of fields and let $\mathcal{O}_K$ denote the ring of integers in K. Then $\mathbb{Z} \to \mathcal{O}_K$ is never unramified, hence $H\mathbb{Z} \to H\mathcal{O}_K$ is never a G-Galois extension.

$\mathbb{Z} \to \mathbb{Z}[i]$ is wildly ramified at 2, hence $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ is *not* isomorphic to $\mathbb{Z}[i] \times \mathbb{Z}[i]$.
$\mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[i, \frac{1}{2}]$, however, is C_2-Galois.

Examples, continued

We saw $KO \to KU$ already.
Take an odd prime p. Then $KU_{(p)}$ splits as

$$KU_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} L.$$

L is called the Adams summand of KU.
Rognes '08:

$$L_p \to KU_p$$

is a C_{p-1}-Galois extension. Here, the C_{p-1}-action is generated by an Adams operation.

Connective covers

If we want to understand arithmetic properties of a commutative ring spectrum R, then we try to understand its algebraic K-theory, $K(R)$.
$K(R)$ is hard to compute. It can be approximated by easier things like topological Hochschild homology ($THH(R)$) or topological cyclic homology ($TC(R)$).
There are trace maps

$$\begin{tikzcd} K(R) \arrow[r, "trc"] \arrow[dr, "tr"'] & TC(R) \arrow[d] \\ & THH(R) \end{tikzcd}$$

BUT: Trace methods work for connective spectra, these are spectra with trivial negative homotopy groups.

Connective spectra

For any commutative ring spectrum R, there is a commutative ring spectrum r with a map $j\colon r \to R$ such that $\pi_*(j)$ is an isomorphism for all $* \geq 0$.

For instance, we get

$$\begin{tikzcd} ko \arrow[r, "c"] \arrow[d, "j"'] & ku \arrow[d, "j"] \\ KO \arrow[r, "c"] & KU \end{tikzcd}$$

BUT: A theorem of Akhil Mathew tells us, that if $A \to B$ is G-Galois for finite G and A and B are connective, then $\pi_*(A) \to \pi_*(B)$ is étale.

$$\pi_*(ko) = \mathbb{Z}[\eta, y, w]/2\eta, \eta^3, \eta y, y^2 - 4w. \to \pi_*(ku) = \mathbb{Z}[u]$$

is certainly *not* étale.
We have to live with ramification!

Wild ramification

$c\colon ko \to ku$ fails in two aspects:

- ko is not equivalent to ku^{hC_2} (but closely related to...)
- $h\colon ku \wedge_{ko} ku \to \prod_{C_2} ku$ is not a weak equivalence (but $ku \wedge_{ko} ku \simeq ku \vee \Sigma^2 ku$).

Theorem (Dundas, Lindenstrauss, R)
$ko \to ku$ is wildly ramified.

How do we measure ramification?

Relative THH

If we have a G-action on a commutative A-algebra B and if $h\colon B \wedge_A B \to \prod_G B$ is a weak equivalence, then Rognes shows that the canonical map

$$B \to THH^A(B)$$

is a weak equivalence.
What is $THH^A(B)$? Topological Hochschild homology of B as an A-algebra, i.e.,
$THH^A(B)$ is the geometric realization of the simplicial spectrum

$$\cdots \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} B \wedge_A B \wedge_A B \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} B \wedge_A B \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} B$$

$THH^A(B)$ measure the ramification of $A \to B$!
If B is commutative, then we get maps

$$B \to THH^A(B) \to B$$

whose composite is the identity on B.
Thus B splits off $THH^A(B)$. If $THH^A(B)$ is larger than B, then $A \to B$ is ramified.
We abbreviate $\pi_*(THH^A(B))$ with $THH^A_*(B)$.

The $ko \to ku$-case

Theorem (DLR)

- As a graded commutative augmented $\pi_*(ku)$-algebra

$$\pi_*(ku \wedge_{ko} ku) \cong \pi_*(ku)[\tilde{u}]/\tilde{u}^2 - u^2$$

  with $|\tilde{u}| = 2$.
- The Tor spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{\pi_*(ku \wedge_{ko} ku)}_{*,*}(\pi_*(ku), \pi_*(ku)) \Rightarrow THH^{ko}_*(ku)$$

  collapses at the E^2-page.
- $THH^{ko}_*(ku)$ is a square zero extension of $\pi_*(ku)$:

$$THH^{ko}_*(ku) \cong \pi_*(ku) \rtimes \pi_*(ku)/2u\langle y_0, y_1, \ldots \rangle$$

  with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.

Comparison to $\mathbb{Z} \to \mathbb{Z}[i]$

The result is very similar to the calculation of $HH_*(\mathbb{Z}[i]) = THH^{H\mathbb{Z}}(H\mathbb{Z}[i])$ (Larsen-Lindenstrauss):

$$HH_*^{\mathbb{Z}}(\mathbb{Z}[i]) \cong THH_*^{H\mathbb{Z}}(H\mathbb{Z}[i]) = \begin{cases} \mathbb{Z}[i], & \text{for } * = 0, \\ \mathbb{Z}[i]/2i, & \text{for odd } *, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$HH_*^{\mathbb{Z}}(\mathbb{Z}[i]) \cong \mathbb{Z}[i] \rtimes (\mathbb{Z}[i]/2i)\langle y_j, j \geq 0\rangle$$

with $|y_j| = 2j + 1$.

Idea of proof for $ko \to ku$:
Use an explicit resolution to get that the E^2-page is the homology of

$$\ldots \xrightarrow{0} \Sigma^4 \pi_*(ku) \xrightarrow{2u} \Sigma^2 \pi_*(ku) \xrightarrow{0} \pi_*(ku).$$

As $\pi_*(ku)$ splits off $THH_*^{ko}(ku)$ the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.
Use that the spectral sequence is one of $\pi_*(ku)$-modules to rule out additive extensions.
Since the generators over $\pi_*(ku)$ are all in odd degree, and their products cannot hit the direct summand $\pi_*(ku)$ in filtration degree zero, their products are all zero.

Contrast to tame ramification

Consider and odd prime p and

$$\begin{tikzcd} \ell \arrow[r] \arrow[d, "j"'] & ku_{(p)} \arrow[d, "j"] \\ L \arrow[r] & KU_{(p)} \end{tikzcd}$$

$\pi_*(\ell) = \mathbb{Z}_{(p)}[v_1] \to \mathbb{Z}_{(p)}[u] = \pi_*(ku_{(p)})$, $v_1 \mapsto u^{p-1}$ already looks much nicer.

- Rognes: $ku_{(p)} \to THH^{\ell}(ku_{(p)})$ is a $K(1)$-local equivalence.
- Sagave: The map $\ell \to ku_{(p)}$ is log-étale.
- Ausoni proved that the p-completed extension even satisfies Galois descent for THH and algebraic K-theory:

$$THH(ku_p)^{hC_{p-1}} \simeq THH(\ell_p), \quad K(ku_p)^{hC_{p-1}} \simeq K(\ell_p).$$

Tame ramification is visible!

$\ell \to ku_{(p)}$ behaves like a tamely ramified extension:

Theorem (DLR)

$$THH_*^{\ell}(ku_{(p)}) \cong \pi_*(ku_{(p)})_* \rtimes \pi_*(ku_{(p)})\langle y_0, y_1, \ldots \rangle / u^{p-2}$$

where the degree of y_i is $2pi + 3$.

$p - 1$ is a p-local unit, hence no additive integral torsion appears in $THH_*^{\ell}(ku_{(p)})$.

Other important examples

There are ring spectra $E(n)$, called Johnson-Wilson spectra.
$\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$, $|v_i| = 2p^i - 2$.
These are synthetic spectra: For almost all n and p there is no geometric interpretation for $E(n)$.
Exceptions: At an odd prime: $E(1) = L$, $E(2)$ at 2 can be constructed out of $tmf_1(3)_{(2)}$ by inverting a_3. (Similar: $E(2)$ at 3, using a Shimura curve)
All the $E(n)$ for $n \geq 1$ carry a C_2-action that comes from complex conjugation on complex bordism.
Are the $E(n)^{hC_2} \to E(n)$ C_2-Galois extensions?
Yes, for $n = 1, p = 2$. That's the example $KO_{(2)} \to KU_{(2)}$.
$Tmf_0(3) \to Tmf_1(3)$ is C_2-Galois (Mathew, Meier) and closely related to $E(2)^{hC_2} \to E(2)$.
We can control certain quotient maps, e.g. $tmf_1(3)_{(2)} \to ku_{(2)}$.

Open questions

- Problem: We do not know whether the $E(n)$ are commutative ring spectra for all n and p. (Motivic help?)
- Is there more variation than just tame and wild ramification?
- Can there be ramification at chromatic primes rather than integral primes?
- How bad is $tmf_0(3) \to tmf_1(3)$?
- Can we understand the ramification for the extentions $BP\langle n\rangle^{hC_2} \to BP\langle n\rangle$ for higher n? Here, $\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$.
  $BP\langle 2\rangle$ has commutative models at $p = 2, 3$ (Hill, Lawson, Naumann)
- Are ku, ko and ℓ analogues of rings of integers in their periodic versions, i.e., $ku = \mathcal{O}_{KU}$, $ko = \mathcal{O}_{KO}$, $\ell = \mathcal{O}_L$? What is a good notion of $\mathcal{O}_K$ for periodic ring spectra K?
- ???