# A spectral sequence for the homology of a finite algebraic delooping

Birgit Richter
joint work in progress with Stephanie Ziegenhagen

Fourth Arolla Conference
on Algebraic Topology
2012

$E_n$-homology

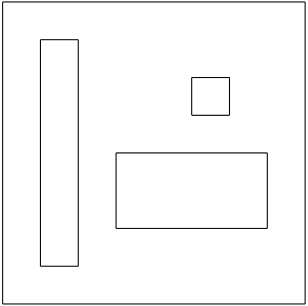
A resolution spectral sequence

A Blanc-Stover spectral sequence

Examples

# Little $n$-cubes

Let $C_n$ denote the operad of little $n$-cubes. $C_n(r)$, $r \geq 0$.
$n = 2, r = 3$:



$C_n$ acts on and detects $n$-fold based loop spaces.

# $E_n$-homology

$(C_* C_n(r))_r$, $r \geq 1$ is an operad in the category of chain complexes.
Let $E_n$ be a cofibrant replacement of $C_* C_n$.
For an augmented $E_n$-algebra $A_*$ let $\bar{A}_*$ denote the augmentation ideal.

The $s$th $E_n$-homology group of $\bar{A}_*$, $H_s^{E_n}(\bar{A}_*)$ is then the $s$th derived functor of indecomposables of $\bar{A}_*$.

Theorem [Fresse 2011]
There is an $n$-fold bar construction for $E_n$-algebras, $B^n$, such that

$$H_s^{E_n}(\bar{A}_*) \cong H_s(\Sigma^{-n} B^n(\bar{A}_*)).$$

I.e., $E_n$-homology is the homology of an $n$-fold algebraic delooping.

# Some results

Cartan (50s): $H_*^{E_n}$ of polynomial algebras, exterior algebras and some more.
Fresse (2011): $X$ a nice space: $B^n(C^*(X))$ determines the cohomology of $\Omega^n X$.
Livernet-Richter (2011): Functor homology interpretation for $H_*^{E_n}$ for augmented commutative algebras.
$H_*^{E_n}(\bar{A}_*) \cong HH_{*+n}^{[n]}(A_*)$, Hochschild homology of order $n$ in the sense of Pirashvili.
What is $H_*^{E_n}(\bar{A}_*)$ in other interesting cases such as Hochschild cochains, $A_* = C^*(B, B)$, or $A_* = C_*(\Omega^n X)$?

# Setting

In the following $k$ is a field, most of the times $k = \mathbb{F}_2$ or $k = \mathbb{Q}$.
The underlying chain complex of $A_*$ is non-negatively or non-positively graded.
Over $\mathbb{F}_2$: $n = 2$; for $\mathbb{Q}$: arbitrary $n$.

# 1-restricted Lie algebras

## Definition

A 1-*restricted Lie algebra over* $\mathbb{F}_2$ is a non-negatively graded $\mathbb{F}_2$-vector space, $\mathfrak{g}_*$, together with two operations, a Lie bracket of degree one, $[-,-]$ and a restriction, $\xi$:

$$\begin{aligned} [-,-]\colon \quad & \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j+1}, \quad & i,j \geq 0, \\ \xi\colon \quad & \mathfrak{g}_i \to \mathfrak{g}_{2i+1} & i \geq 0. \end{aligned}$$

These satisfy the relations

1. The bracket is bilinear, symmetric and satisfies the Jacobi relation

   $[a,[b,c]]+[b,[c,a]]+[c,[a,b]] = 0$ for all homogeneous $a,b,c \in \mathfrak{g}_*$.

2. The restriction interacts with the bracket as follows: $[\xi(a), b] = [a,[a,b]]$ and $\xi(a+b) = \xi(a) + \xi(b) + [a,b]$ for all homogeneous $a, b \in \mathfrak{g}_*$.

1-rL: The category of 1-restricted Lie algebras.

# 1-restricted Gerstenhaber algebras

## Definition

A 1-*restricted Gerstenhaber algebra over* $\mathbb{F}_2$ is a 1-restricted Lie algebra $G_*$ together with an augmented commutative $\mathbb{F}_2$-algebra structure on $G_*$ such that the multiplication in $G_*$ interacts with the restricted Lie-structure as follows:

- (Poisson relation)

$$[a, bc] = b[a, c] + [a, b]c, \text{ for all homogeneous } a, b, c \in G_*.$$

- (multiplicativity of the restriction)

$$\xi(ab) = a^2\xi(b)+\xi(a)b^2+ab[a, b] \text{ for all homogeneous } a, b \in G_*.$$

1-rG: the category of 1-restricted Gerstenhaber algebras.
In particular, the bracket and the restriction annihilate squares: $[a, b^2] = 2b[a, b] = 0$ and $\xi(a^2) = 2a^2\xi(a) + a^2[a, a] = 0$. Thus if 1 denotes the unit of the algebra structure in $G_*$, then $[a, 1] = 0$ for all $a$ and $\xi(1) = 0$.

# Free objects and indecomposables

For a graded vector space $V_*$ let $1rL(V_*)$ be the free 1-restricted Lie algebra on $V_*$.
The free commutative algebra $S(1rL(V_*))$ has a well-defined 1-rG structure and is the free 1-restricted Gerstenhaber algebra generated by $V_*$:

$$1rG(V_*) = S(1rL(V_*)).$$

For $G_* \in 1rG$ let $Q_{1rG}(G_*)$ be the graded vector space of indecomposables.
Note: $Q_{1rG}(G_*) = Q_{1rL}(Q_a(G_*))$.

# Homology of free objects

Lemma

$$H_*(E_2(\bar{A}_*)) \cong 1rG(H_*(\bar{A}_*)).$$

Proof: Let $X$ be a space. F. Cohen desribes $H_*(C_2(X); \mathbb{F}_2)$.
Observation by Haynes Miller: $H_*(C_2(X); \mathbb{F}_2) \cong 1rG(\bar{H}_*(X; \mathbb{F}_2))$.
(Dyer-Lashof operations only give algebraic operations. )
Take $X$ with $\bar{H}_*(X; \mathbb{F}_2) \cong H_*(\bar{A}_*)$, then
$H_*(E_2(\bar{A}_*)) \cong H_*(C_2(X); \mathbb{F}_2)$.

# Resolution spectral sequence

**Theorem**

There is a spectral sequence

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{1rG}(H_*(\bar{A}_*)))_q \Rightarrow H^{E_2}_{p+q}(\bar{A}_*)$$

.

Proof: Standard resolution $E_2^{\bullet+1}(\bar{A}_*)$.

$E^1_{p,q} : H^{E_2}_q(E_2^{p+1}(\bar{A}_*)) \cong H_q(E_2^p(\bar{A}_*))$

$$H_q(E_2^p(\bar{A}_*)) \cong 1rG^p(H_*\bar{A}_*)_q \cong Q_{1rG}(1rG^{p+1}(H_*\bar{A}_*))_q.$$

$d^1$ takes homology wrt resolution degree.

# Example

For $X$ connected:
$(\mathbb{L}_p Q_{1rG}(H_*(C_*(\Omega^2\Sigma^2 X;\mathbb{F}_2)))_* = (\mathbb{L}_p Q_{1rG}(1rG(\bar{H}_*(X;\mathbb{F}_2)))_*$.
This reduces to $\bar{H}_q(X;\mathbb{F}_2)$ in the $(p=0)$-line and

$$H_q^{E_2}(C_*(\Omega^2\Sigma^2 X;\mathbb{F}_2)) \cong \bar{H}_q(X;\mathbb{F}_2).$$

# Rational case

The rational case is much easier:

$$H_*(E_{n+1}\bar{A}_*) \cong nG(H_*(\bar{A}_*)),$$

the free $n$-Gerstenhaber algebra generated by the homology of $\bar{A}_*$.
We get:

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{nG}(H_*(\bar{A}_*)))_q \Rightarrow H^{E_{n+1}}_{p+q}(\bar{A}_*)$$

for every $E_{n+1}$-algebra $\bar{A}_*$ over the rationals.

# General Blanc-Stover setting

Let $\mathcal{C}$ and $\mathcal{B}$ be some categories of graded algebras (*e.g.*, Lie, Com, $n$-Gerstenhaber etc.) and let $\mathcal{A}$ be a concrete category (such as graded vector spaces) and $T\colon \mathcal{C} \to \mathcal{B}$, $S\colon \mathcal{B} \to \mathcal{A}$.
If $TF$ is is $S$-acyclic for every free $F$ in $\mathcal{C}$, then there is a Grothendieck composite functor spectral sequence for all $C$ in $\mathcal{C}$

$$E^2_{s,t} = (\mathbb{L}_s \bar{S}_t)(\mathbb{L}_* T)C \Rightarrow (\mathbb{L}_{s+t}(S \circ T))C.$$

- Note: $T, S$ non-additive.
- $\bar{S}_t(\pi_* B) = \pi_t(SB)$ if $B$ is free simplicial; otherwise it is defined as a coequaliser.
- $\bar{S}$ takes the homotopy operations on $\pi_* B$ into account ($B$ a simplicial object in $\mathcal{B}$): $\pi_* B$ is a $\Pi$-$\mathcal{B}$-algebra.
- $\mathcal{B} = Com$: $\pi_*(B)$ has divided power operations. $\mathcal{B} = rLie$: $\pi_* B$ inherits a Lie bracket and has some extra operations.

# In our case

Theorem

- $k = \mathbb{F}_2$: For any $C \in 1rG$:

$$E^2_{s,t} = \mathbb{L}_s((\bar{Q}_{1rL})_t)(AQ_*(C|\mathbb{F}_2, \mathbb{F}_2)) \Rightarrow \mathbb{L}_{s+t}(Q_{1rG}).$$

- For $k = \mathbb{Q}$ we get for all $n$-Gerstenhaber algebras $C$:

$$\mathbb{L}_s((\bar{Q}_{nL})_t)(AQ_*(C|\mathbb{Q}, \mathbb{Q})) \Rightarrow \mathbb{L}_{s+t}(Q_{nG}).$$

# Hochschild cochains, rational case

Let $V$ be a vector space. Then $C^*(TV, TV)$ is an $E_2$-algebra. How close is $H_*^{E_2}(\bar{C}^*(TV, TV)$ to $V$, *i.e.*, how free is $C^*(TV, TV)$ as an $E_2$-algebra?

Proposition

For $V = \mathbb{Q}$, *i.e.* $TV = \mathbb{Q}[x]$ the $E_2$-homology of the reduced Hochschild cochain complex is non-trivial in all degrees $r \geq -1$, more precisely

$$H_r^{E_2}(\bar{C}^*(\mathbb{Q}[x], \mathbb{Q}[x])) \cong \mathbb{Q}$$

for all $r \geq -1$.

Thus in this case $E_2$-homology of the Hochschild cochains on $T\mathbb{Q}$ is much larger than the vector space $\mathbb{Q}$ we started with.

The calculations uses the equivalence of categories of $n$-Lie algebras and graded Lie-algebras given by $n$-fold (de)suspension. Thus (up to suspension) we have to calculate ordinary Lie-homology of $AQ_*(HH^*(\mathbb{Q}[x], \mathbb{Q}[x]))$ and this is concentrated in homological degree zero and there it is $\mathbb{Q}\langle x_0, y_{-1}\rangle$ with trivial 1-Lie structure.

# $\infty > dim(V) \geq 2$

In these cases we can determine the input for the Blanc-Stover spectral sequence:

$$AQ_*(HH^*(TV, TV)|\mathbb{Q}; \mathbb{Q}) \cong HH^{(1)}_{*+1}(\mathbb{Q} \rtimes M(-1); \mathbb{Q})$$

and the first Hodge summand $HH^{(1)}_*(\mathbb{Q} \rtimes M(-1); \mathbb{Q})$ is additively isomorphic to the free graded Lie-algebra generated by the graded vector space $M(-1)$.
However, $M(-1) = HH^1(TV, TV)$ is *not* free as a Lie-algebra.

# Chains on iterated loop spaces

Let $k$ be $\mathbb{Q}$ and let $X$ be an $(n+1)$-connected nice topological space.
Then $H_*(\Omega^{n+1}X;\mathbb{Q}) \cong S(\Sigma^{-n}\pi_*(\Omega X)\otimes\mathbb{Q})$ as $n$-Gerstenhaber algebras.

Proposition

$$\mathbb{L}_s(Q_{nG})(H_*(\Omega^{n+1}X;\mathbb{Q}))_q \cong \mathrm{Tor}^{H_*(\Omega X;\mathbb{Q})}_{s+1,q+n}(\mathbb{Q},\mathbb{Q}).$$

Conjecture

This does not just look like a shifted version of the Rothenberg-Steenrod $E^2$-term, but there is an underlying isomorphism of spectral sequences. The RS-spsq converges to $H_*(X;\mathbb{Q})$.