Exercises in Algebra (master): Homological Algebra

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Summer term 2021

Exercise sheet no 10 for the exercise class on the 16th of June 2021

1 (Coproducts of groups) For two groups G_1 and G_2 we denote by $G_1 * G_2$ the coproduct of G_1 and G_2 in the category of groups.

(1) Write down an explicit model of $G_1 * G_2$ in terms of finite reduced words in the alphabet $(G_1 \setminus \{1\}) \cup (G_2 \setminus \{1\})$ and prove that this satisfies the universal property.

(2) Consider the augmentation ideals $0 \longrightarrow I(G_i) \xrightarrow{i} \mathbb{Z}[G_i] \xrightarrow{\varepsilon_i} \mathbb{Z} \longrightarrow 0$ and show that

$$I(G_1 * G_2) \cong (I(G_1) \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_1 * G_2]) \oplus (I(G_2) \otimes_{\mathbb{Z}[G_2]} \mathbb{Z}[G_1 * G_2]).$$

(3) Prove a *flat base change for* Tor: Let $f: R_1 \to R_2$ be a ring map such that R_2 is flat as an R_1-module. Then for all R_1^{op}-modules M, all R_2-modules N and all n:

$$\mathsf{Tor}_n^{R_1}(M, f^*N) \cong \mathsf{Tor}_n^{R_2}(M \otimes_{R_1} R_2, N).$$

(4) Use this to show that for all $\mathbb{Z}[G_1 * G_2]$-modules M and all $n \geqslant 2$

$$H_n(G_1 * G_2; M) \cong H_n(G_1; M) \oplus H_n(G_2; M)$$

where on the left hand side we view $M = i_j^* M$ as a G_j-module via the inclusions $i_j: G_j \hookrightarrow G_1 * G_2$. (It is also true that for all $n \geqslant 2$ the inclusion maps induce an isomorphism

$$H^n(G_1 * G_2; M) \cong H^n(G_1; M) \oplus H^n(G_2; M).)$$

(5) What happens for $n = 0$?

2 (Basics on Hochschild homology)

(1) Let A be an associative k-algebra and let M be an A-bimodule over k. Show that $\mathsf{HH}_*(A; M)$ is a module over the center of A, *i.e.*, over $Z(A) = \{a \in A, ab = ba \text{ for all } b \in A\}$.

(2) Let $0 \neq k$ be a commutative ring. Determine $\Omega^1_{k[x_1,\ldots,x_n]|k}$ as a $k[x_1, \ldots, x_n]$-module.

(3) What can you say about $\Omega^1_{\mathbb{Z}[X]/(X^2+1)|\mathbb{Z}} = \Omega^1_{\mathbb{Z}[i]|\mathbb{Z}}$?

3 (Hochschild homology and group homology)

Let G be a group and let M be a $k[G]$-bimodule. Show that

$$\mathsf{HH}_*^k(k[G]; M) \cong H_*(G; M^c)$$

where M^c is the G-module with $g.m := gmg^{-1}$. In particular, if we use the augmentation $\varepsilon: k[G] \to k$ in order to view k as a $k[G]$-bimodule, then $\mathsf{HH}_*^k(k[G]; k) \cong H_*(G; k)$.