Exercises in Algebra (master): Homological Algebra

Prof. Dr. Birgit Richter

Summer term 2021

Exercise sheet no 1 for the exercise class on the 14th of April 2021

1 (Ranks of free modules)

a) Let R be a commutative ring. Show that the rank of a free R-module is well-defined: If R^n is isomorphic to R^m, then $n = m$.

For non-commutative rings this is not always the case. If it is, then the ring R is said to have invariant basis number (IBN).

b) Consider a ring R and a free R-module F that does not have a finite basis. Show that the ring of R-endomorphisms $R' = \mathsf{Hom}_R(F, F)$ satisfies

$$R' \cong (R')^2.$$

Prove that one also gets $(R')^n \cong (R')^m$ for all natural numbers n, m.

2 (Group algebras) Let R be a commutative ring.

- Show that the R-algebra $R[\mathbb{Z}]$ is isomorphic to the ring of Laurent polynomials $R[x^{\pm 1}]$. This is the R-algebra whose elements are finite sums $\sum_{i=-N}^{M} r_i x^i$ and whose addition and multiplication is extended from the one in the polynomial algebra $R[x]$.
- Let G and H be two groups. Prove that $R[G] \otimes_R R[H] \cong R[G \times H]$. Is this an isomorphism of R-modules or of R-algebras?

3 (An exact sequence)

Fix a prime p and denote by $\mathbb{Z}_{(p)}$ the p-local integers, *i.e.*, the ring of all rational numbers $\frac{a}{b}$ such that p does not divide b.

Let $\mathbb{Z}/p^\infty$ denote the abelian group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ (with $\mathbb{Z}[\frac{1}{p}] = \{\frac{a}{p^n}, a \in \mathbb{Z}, n \geqslant 1\}$).

Show that there is a short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{i} \mathbb{Q} \xrightarrow{\pi} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \longrightarrow 0$$

where $i\colon \mathbb{Z}_{(p)} \to \mathbb{Q}$ is the inclusion map. What is $\mathbb{Q}/\mathbb{Z}$ in terms of $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$?