Commutative Ring Spectra and Spectral Sequences

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Outline

Attitudes toward spectral sequences

Three basic attitudes:

- dismissal
- okay when they collapse or map isomorphically
- hardcore

The hardcore contend with

- nontrivial differentials
- ‘hidden extensions’ or ‘filtration shifts’

My goal is to show how S-algebra or E_∞ stuctures solve some of these problems.

Commutative S-algebras

The product factors through the homotopy orbits

$$\begin{tikzcd} R^{(k)} \arrow[rr, "\mu"] \arrow[dr] & & R \\ & (R^{(k)})_{h\pi} \arrow[ur, "\xi"'] & \end{tikzcd}$$

for any subgroup $\pi \subset \Sigma_k$.

Traditional notation:

$$D^i_\pi R := E\pi^i_+ \wedge_\pi R^{(k)}$$

where $E\pi^i$ is the i-skeleton of the universal π-space $E\pi$.

Let $p = 2$ and $H = H\mathbf{F}_p$ for most examples.
Let $C_2 = \Sigma_2 = \{1, T\}$ and consider the $\mathbf{F}_2[C_2]$ resolution of $\mathbf{F}_2$

$$0 \longleftarrow \mathbf{F}_2 \xleftarrow{\epsilon} \mathcal{W}_0 \xleftarrow{d_0} \mathcal{W}_1 \xleftarrow{d_1} \mathcal{W}_2 \xleftarrow{d_2} \cdots$$

- $\mathcal{W}_i = \langle e_i \rangle \ \cong \ \mathbf{F}_2[C_2]$
- $d_i(e_{i+1}) = (1 + T)e_i$.
- $EC_2 = S(\infty\tau)$ has cellular chains $\mathcal{W}$
- $H_* D_{C_2} R \ \cong \ H(\mathcal{W} \otimes_{C_2} H_*(R) \otimes H_*(R))$.

The S-algebra structure of R then induces *Dyer–Lashof operations* $Q^i : H_n R \longrightarrow H_{n+i} R$ defined by

$$Q^i(x) = \xi_*(e_{i-n} \otimes x \otimes x).$$

Remarkable facts:

- all nonzero operations (any $\pi \subset \Sigma_n$ for any n) can be written in terms of the Q^i.
- all relations between them are generated by the natural map

$$D_{\Sigma_p} D_{\Sigma_p} R \cong D_{\Sigma_p \wr \Sigma_p} R \longrightarrow D_{\Sigma_{p^2}} R$$

This is worked out nicely in J. Peter May's 'A General Algebraic Approach to Steenrod Operations' (Springer LNM V. 168).

Let Ch be the category of $\mathbf{F}_p$ chain complexes and Sym_1 the category of $\mathbf{F}_p$ chain complexes with a homotopy associative product.

May defines a category Sym_∞ and functors

$$\begin{tikzcd}
Top \arrow[dr, "C^*"] & & \\
CocommHopf \arrow[r] & Sym_\infty \arrow[r, "H"] \arrow[dd] & \mathcal{A}-\text{algebras} \arrow[dd] \\
\infty-LoopSp \arrow[ur, "C_*"'] \arrow[d] & & \\
LoopSp \arrow[r, "C_*"'] & Sym_1 \arrow[r, "H"] \arrow[d] & \text{Graded } \mathbf{F}_p-\text{algebras} \arrow[d] \\
 & Ch \arrow[r, "H"] & \text{Graded } \mathbf{F}_p-\text{modules}
\end{tikzcd}$$

where $\mathcal{A}$ is a universal Steenrod algebra.

S-algebras can be treated more directly, as above, but fitting them into this picture through their (cellular) chains elucidates the origin of the properties of the Dyer–Lashof operations.

$\mathcal{A}$ has as quotients

- the Dyer–Lashof algebra acting on ∞-loop spaces,
- the usual Steenrod algebra acting on the cohomology of topological spaces (sSets)
- an extension of the usual Steenrod algebra in which $Sq^0 \neq 1$ acting on the cohomology of cocommutative Hopf algebras.

The last of these acts more generally on $\mathrm{Ext}_C(M, N)$ when

- C is a cocommutative Hopf algebra,
- M is a cocommutative C-coalgebra and
- N is a commutative C-algebra.

Sym_∞ consists of the Cartan and Adem objects in a category $\mathcal{C}(C_p, \mathbf{F}_p)$, with

objects (K, θ):

- K a $\mathbf{Z}$–graded homotopy associative differential $\mathbf{F}_p$-algebra
- $\theta : \mathcal{W} \otimes K^p \longrightarrow K$ a morphism of $\mathbf{F}_p[C_p]$-complexes,

satisfying

- $\theta \,|\, \langle e_0 \rangle \otimes K^p$ is the p-fold iterated product associated in some fixed order, and
- θ is $\mathbf{F}_p[C_p]$-homotopic to a composite

$$\mathcal{W} \otimes K^p \longrightarrow \mathcal{V} \otimes K^p \xrightarrow{\phi} K$$

for some $\mathbf{F}_p[\Sigma_p]$-resolution $\mathcal{V}$ of $\mathbf{F}_p$ and some $\mathbf{F}_p[\Sigma_p]$-morphism ϕ.

morphisms $K \xrightarrow{f} L$,

a morphism of $\mathbf{F}_p$-complexes such that

$$\begin{tikzcd} \mathcal{W} \otimes K^p \arrow[r, "\theta"] \arrow[d, "1 \otimes f^p"'] & K \arrow[d, "f"] \\ \mathcal{W} \otimes L^p \arrow[r, "\theta'"] & L \end{tikzcd}$$

is $\mathbf{F}_p[C_p]$-homotopy commutative.

Then

$$Q^i(x) = \theta_*(e_{i-n} \otimes x \otimes x)$$

defines $Q^i : H_n(K) \longrightarrow H_{n+i}(K)$ if $p = 2$, and similarly for odd p.
Of course, they will not have many desirable properties without additional structure.

Cartan objects

With an evident tensor product in $\mathcal{C}(C_p, \mathbf{F}_p)$ induced by the diagonal $\mathcal{W} \longrightarrow \mathcal{W} \otimes \mathcal{W}$, we say that (K, θ) is a *Cartan object* if the product $(K, \theta) \otimes (K, \otimes) \longrightarrow (K, \theta)$ is a morphism.

If (K, θ) is a Cartan object, the operations in $H(K)$ satisfy the Cartan formula.

Adem objects

Let $\mathcal{Y}$ be an $\mathbf{F}_p[\Sigma_{p^2}]$-resolution of $\mathbf{F}_p$. Say that (K, θ) is an *Adem object* if there exists a Σ_{p^2}-equivariant $\phi : \mathcal{Y} \otimes K^{p^2} \longrightarrow K$ such that

$$\begin{tikzcd}
(\mathcal{W} \otimes \mathcal{W}^p) \otimes K^{p^2} \arrow[r, "w \otimes 1"] \arrow[dd, "1 \otimes \text{shuffle}"] & \mathcal{Y} \otimes K^{p^2} \arrow[dr, "\phi"] & \\
& & K \\
\mathcal{W} \otimes (\mathcal{W} \otimes K^p)^p \arrow[r, "1 \otimes \theta^p"] & \mathcal{W} \otimes K^p \arrow[ur, "\theta"] &
\end{tikzcd}$$

is $C_p \wr C_p$-equivariantly homotopy commutative.

If (K, θ) is an Adem object, the operations in $H(K)$ satisfy the Adem relations.

The Spectral Sequence

(This section is joint work with John Rognes.)
Let $\mathbb{T}$ be the circle group $S(\mathbf{C})$.
Let R be a $\mathbb{T}$-equivariant commutative S-algebra.
E.G., $THH(B)$ for a commutative S-algebra B.
Then

$$R^{h\mathbb{T}} = F(E\mathbb{T}_+, R)^{\mathbb{T}}$$

is again an S-algebra, as are the terms in the limit system

$$\begin{tikzcd}
 & R^{h\mathbb{T}} \arrow[d] \arrow[dr] & & & \\
\cdots \arrow[r] & F(S(\mathbf{C}^n)_+, R)^{\mathbb{T}} \arrow[r] & F(S(\mathbf{C}^{n-1})_+, R)^{\mathbb{T}} \arrow[r] & \cdots \arrow[r] & R
\end{tikzcd}$$

Theorem

There is a natural A_*-comodule algebra spectral sequence

$$E^2_{**}(R) = H^{-*}_{gp}(\mathbb{T}; H_*(R; \mathbf{F}_p)) = P(y) \otimes H_*(R; \mathbf{F}_p)$$

with y in bidegree $(-2, 0)$, converging conditionally to the continuous homology

$$H^c_*(R^{h\mathbb{T}}; \mathbf{F}_p) = \lim_n H_*(F(S(\mathbf{C}^n)_+, R)^{\mathbb{T}}; \mathbf{F}_p)$$

of the homotopy fixed point spectrum $R^{h\mathbb{T}} = F(E\mathbb{T}_+, R)^{\mathbb{T}}$. If $H_*(R; \mathbf{F}_p)$ is finite in each degree, or the spectral sequence collapses at a finite stage, then the spectral sequence is strongly convergent.

Theorem

There are natural Dyer–Lashof operations $\beta^\epsilon Q^i$ acting vertically on this spectral sequence, and they commute with the differentials:

$$d^{2r}(\beta^\epsilon Q^i(x)) = \beta^\epsilon Q^i(d^{2r}(x))$$

Propagation of differentials

Given a d^{2r}, write $d^{2r}x = y^r \delta x$:

$$\begin{tikzcd}
t+2r-1 : & y^r \delta x & \\
t : & & x \arrow[ul]
\end{tikzcd}$$

This is

$$x : S(\mathbf{C})_+ \wedge S^t \longrightarrow H \wedge R$$

which extends to

$$x' : S(\mathbf{C}^r)_+ \wedge S^t \longrightarrow H \wedge R.$$

and induces

$$H \wedge D_p(S(\mathbf{C}^r)_+ \wedge S^t) \longrightarrow H \wedge D_p(H \wedge R) \longrightarrow H \wedge H \wedge R \longrightarrow H \wedge R$$

In the domain, and hence in the codomain, we get d^{2r}s:

$$
\begin{tikzcd}
\vdots & & \\
y^r Q^{t+2r}\delta x & \vdots & \\
y^r Q^{t+2r-1}\delta x & & \\
 & & \vdots \\
 & & Q^{t+2r}x \arrow[uuull] \\
 & & Q^{t+2r-1}x \arrow[uuull] \\
 & & \vdots \\
 & & Q^{t+2}x \\
 & & Q^{t+1}x \\
 & & Q^{t}x
\end{tikzcd}
$$

but also,

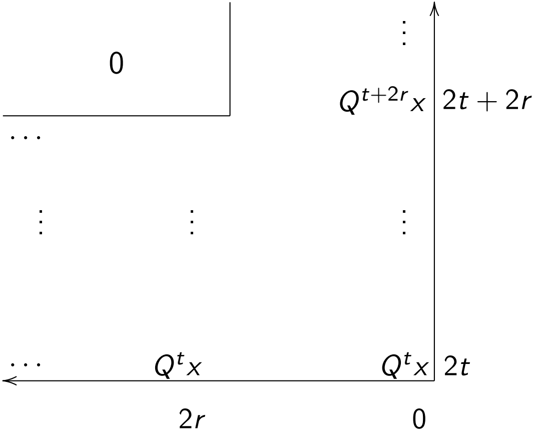
$$
\begin{tikzcd}[row sep=small]
\vdots & & \\
y^r Q^{t+2r}\delta x & \vdots & \\
y^r Q^{t+2r-1}\delta x & & \\
 & & \vdots \\
 & & Q^{t+2r}x \arrow[uuull] \\
 & x\delta x, \arrow[uuul] & Q^{t+2r-1}x \arrow[uuull] \\
 & & \vdots \\
 & & Q^{t+2}x \\
 & & Q^{t+1}x \\
 & & Q^{t}x
\end{tikzcd}
$$

The wonderful thing here is that these classes don't just survive to E^{2r+1}.

Theorem

The elements $Q^t x$, $Q^{t+1} x$, …, $Q^{t+2r-2} x$ and $Q^{t+2r-1} x - x\delta x$ are all infinite cycles.

Proof: At E^{2r+1} in the domain spectral sequence we have nonzero classes only in columns 0 to $2r-2$ and rows $2t$ to $2t+2r-2$:



$\square$

Applications

- RRB and Rognes [A&GT, V. 5] show that the spectral seqence for $H_*^c THH(B)^{hT}$ collapses at E^4 for $B = MU$, BP, ku, ko, and tmf, and calculate the result.
- Caruso, May and Priddy, [Topology V. 26], show that the continuous homology serves as input to an Adams spectral sequence for the homotopy of an inverse limit.
- Similar results hold for R^{hC} for cyclic $C \subset \mathbb{T}$ and for the analogous spectral seqences for the homotopy orbits R_{hC} and $R_{h\mathbb{T}}$ and Tate spectra R^{tC} and $R^{t\mathbb{T}}$.
- Sverre Lunøe-Nielsen's work computes the A_*-comodule structure, with $K(B)$ as the intended goal.

Operations in the Adams Spectral Sequence

- Historically, this came first (late 1960s to 1970s) in the work of D. S. Kahn, R. J. Milgram, J. Mäkinen and RRB.
- In the abutment, we have homotopy operations compatible with the Dyer-Lashof operations in homology under the Hurewicz map.
- Between E_2 and E_∞ we have Steenrod operations in Ext interpolating between these.
- Two extremes of power operations in other cohomology theories, e.g., in the work of N. P. Strickland, C. Rezk, and T. Torii.

Homotopy operations

$$S^n \xrightarrow{x} R$$

$$\begin{tikzcd}
D_G S^n \arrow[r, "D_G x"] & D_G R \arrow[r, "\xi"] & R \\
S^k \arrow[u, "\alpha"] \arrow[urr, "\alpha^*(x)"'] & &
\end{tikzcd}$$

Cup-i operations

We call the operation 'cup-i'

$$S^n \xrightarrow{x} R$$

$$\begin{tikzcd}
D_2S^n \arrow[r, "D_2x"] & D_2R \arrow[r, "\xi"] & R \\
S^{2n+i} \arrow[u, "\cup_i"] \arrow[urr, "\cup_i(x)"'] & &
\end{tikzcd}$$

if

$$\begin{tikzcd}
\cup_i \arrow[d, mapsto] & \in & \pi_{2n+i}D_2S^n \arrow[d] \arrow[r, equal] & \pi_{2n+i}\Sigma^nP_n \\
\text{gen} & \in & H_{2n+i}D_2S^n &
\end{tikzcd}$$

Detection in the Adams spectral sequence

The cohomology of a cocommutative Hopf algebra, such as the Steenrod algebra, has natural operations

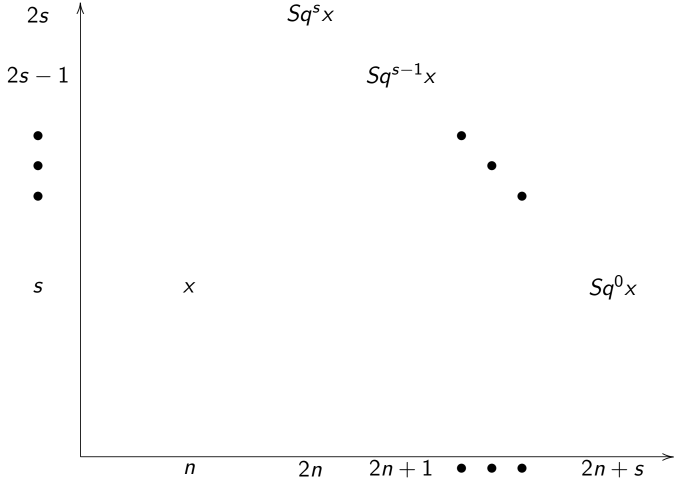
$$Sq^i : \mathrm{Ext}_A^{s,t}(H^*R, \mathbf{F}_2) \longrightarrow \mathrm{Ext}_A^{s+i,2t}(H^*R, \mathbf{F}_2)$$

for $0 \leq i \leq s$ in the cohomological indexing, or

$$Q^i : \mathrm{Ext}_A^{s,t}(H^*R, \mathbf{F}_2) \longrightarrow \mathrm{Ext}_A^{s+t-i,2t}(H^*R, \mathbf{F}_2)$$

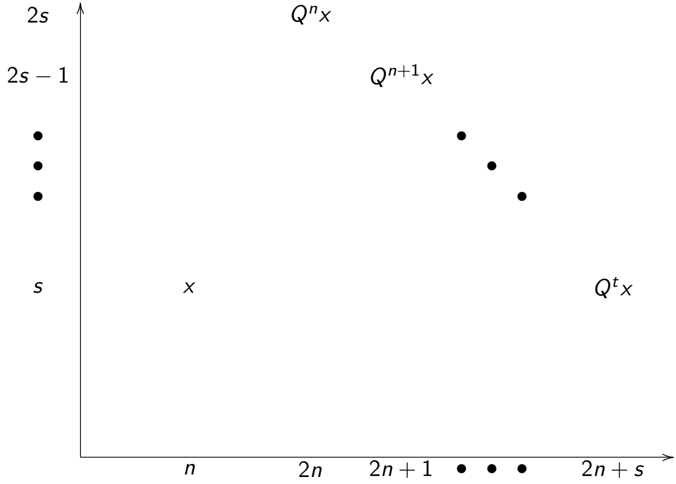
for $t - s \leq i \leq t$ in the homological indexing.

Cohomological indexing:



$$Sq^i : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+i,2t} \quad (n = t - s)$$

Homological indexing:



$$Q^i : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+t-i,2t} \quad (n = t - s)$$

Properties of the cup-i operations

- $\cup_0(x) = x^2$ and always exists.
- $\cup_i : \pi_n \longrightarrow \pi_{2n+i}$ is detected by $Q^{n+i} = Sq^{s-i}$ in Ext
- Each cell of $D_2 S^n$ either defines a $\cup_i$ operation or a relation between lower operations.
- For example, $\cup_1 : \pi_n \longrightarrow \pi_{2n+1}$ exists iff n is even.
- If n is odd then the $2n+1$ cell of $D_2 S^n = \Sigma^n P_n$ instead gives a null-homotopy of $2x^2$.

Manifestation in the Adams spectral sequence

Let

$$R \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$$

be an Adams resolution of R. Taking p-fold smash product, the comparison theorem gives us a map

$$\begin{tikzcd}
 & R & R_1 \arrow[l] & R_2 \arrow[l] & \cdots \arrow[l] \\
D_p R \arrow[ur] & & & & \\
 & R^{(p)} \arrow[ul] \arrow[uu] & B_1 \arrow[l] \arrow[uu] & B_2 \arrow[l] \arrow[uu] & \cdots \arrow[l]
\end{tikzcd}$$

In A_*-comodules this gives maps of resolutions

$$\begin{tikzcd} 0 \arrow[r] & H_*R \arrow[r] & \mathcal{C}_* \\ 0 \arrow[r] & H_*R^{(p)} \arrow[r] \arrow[u] & \mathcal{C}_*^{(p)} \arrow[u] \end{tikzcd}$$

$$\mathcal{C}_i = H_*(\Sigma^i(R_i/R_{i+1}))$$

Standard homological algebra then extends this map of resolutions to a Σ_p-equivariant homomorphism

$$\mathcal{W}_i \otimes \mathcal{C}_s^{(p)} \longrightarrow \mathcal{C}_{s-i}.$$

The spectrum $(E\Sigma_p)_+ \wedge R^{(p)}$ is bifiltered by the $(E\Sigma_p)_+^i \wedge B_s$ with filtration quotients $\mathcal{W}_i \otimes (\mathcal{C}^{(p)})_s$.

When R is an S-algebra, this algebraic map allows us to filter the structure map $D_p R \longrightarrow R$ to give compatible maps

$$(E\Sigma_p)^i_+ \wedge B_s \longrightarrow R_{s-i}$$

geometrically realizing the Steenrod operations in Ext.
This is how we detect the homotopy operations in the Adams spectral sequence. First suppose we have a permanent cycle

$$\begin{tikzcd} & & S^n \arrow[dll, "x"'] \arrow[d] \\ R & \cdots \arrow[l] & R_s \arrow[l] \end{tikzcd}$$

Applying the extended powers, the characteristic map of the $2n+i$-cell of D_2S^n 'carries' $Q^{n+i}(x)$:

$$
\begin{tikzcd}
e^{2n+i} \arrow[d] & S^{2n+i-1} \arrow[l] \arrow[d] \\
\Sigma^n P_n^{n+i} \arrow[d] & \Sigma^n P_n^{n+i-1} \arrow[l] \arrow[d] \\
R_{2s-i} & R_{2s-i+1} \arrow[l]
\end{tikzcd}
$$

For example, if n is odd then $P_n^{n+1} = S^n \cup_2 e^{n+1}$ and we have

$$\begin{tikzcd}
e^{2n+1} \arrow[d] & S^{2n} \arrow[l] \arrow[d, "2"] \arrow[ddr, "h_0 x^2"] & \\
\Sigma^n P_n^{n+1} \arrow[d] & S^{2n} \arrow[l] \arrow[d, "x^2"] & \\
R_{2s-1} & R_{2s} \arrow[l] & R_{2s+1} \arrow[l]
\end{tikzcd}$$

resulting in the differential $d_2(\cup_1(x)) = h_0 x^2$.

Cup-1 of 2 is η

Consider operations on $2 \in \pi_0 S$.

$$\begin{tikzcd} D_2 S \arrow[r, "D_2 2"] & D_2 S \arrow[r, "\xi"] & S \\ S^1 \arrow[u, "\cup_1"] \arrow[urr, "\eta"'] & & \end{tikzcd}$$

This is detected by $Sq^0(h_0) = h_1$ in $\mathrm{Ext}_A(\mathbf{F}_2, \mathbf{F}_2)$.

Cup-1 of η is not defined

$$\begin{tikzcd} D_2 S^1 \arrow[r, "D_2\eta"] & D_2 S \arrow[r, "\xi"] & S \\ S^3 \arrow[u, "\nexists \cup_1"] & & \end{tikzcd}$$

However, we do have $Sq^0(h_1) = h_2$ in Ext.
Restricting to the 3-skeleton,

$$\begin{tikzcd} D_2^1 S^1 \arrow[r, "D_2\eta"] \arrow[d, "\mathrm{top}"'] & D_2^1 S \arrow[r, "\xi"] & S \\ S^3 \arrow[urr, "\nu"'] & & \end{tikzcd}$$

The attaching map of the 3-cell of ΣP_1 has degree 2, giving the Adams spectral sequence differential $d_2(h_2) = h_0 h_1^2 = 0$. There are no possible higher differentials, allowing ν to exist.

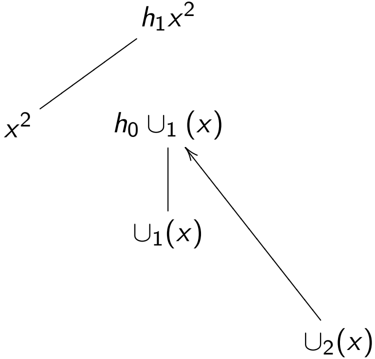
Similarly,

$$\begin{tikzcd} D_2^1 S^3 \arrow[r, "D_2\nu"] \arrow[d, "\mathrm{top}"'] & D_2^1 S \arrow[r, "\xi"] & S \\ S^7 \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

Again, the attaching map has degree 2, and this gives $d_2(h_3) = h_0 h_2^2 = 0$, and there are no possible higher differentials, allowing σ to exist as well.

- After this, the differential $d_2(h_{n+1}) = h_0 h_n^2 \neq 0$, and no higher Hopf maps exist.
- In this sense, η must exist, while ν and σ are 'gifts', or low dimensional accidents.
- The 15 cell carrying h_4 is a null-homotopy of $2\sigma^2$, showing that $2\theta_3 = 0$.
- For higher n, we don't get the implication $2\theta_n = 0$ from the differential $d_2(h_{n+1}) = h_0 h_n^2$, though, because h_n was not a homotopy class to start with and the story is a bit more complicated.
- The boundary of the cell carrying h_n decomposes into a part carrying $h_0 h_n^2$ and a part carrying operations on $h_0 h_{n-1}^2$, effectively setting $2\theta_n$ equal to higher Adams filtration elements which we must analyze.

One more example. Suppose $n \equiv 2 \pmod 4$. Then $P_n^{n+2} = (S^n \vee S^{n+1}) \cup_{(\eta,2)} e^{n+2}$. In the Adams spectral sequence this manifests as

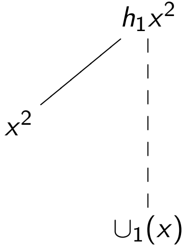


The $d_2(\cup_2(x)) = h_0 \cup_1 (x)$ here reflects the relation

$$2 \cup_1 (x) + \eta x^2 = 0$$

In the Adams spectral sequence this is a 'universally hidden extension':

At E_∞, if $x \in E_\infty$ is in a stem $\equiv 2 \pmod 4$, we have



$2 \cup_1 (x)$ is not detected by $h_0 \cup_1 (x)$, but by $h_1 x^2$. Such relations are ubiquitous.

More generally, if we start with $x \in E_r$, it is realized geometrically by

$$\begin{tikzcd}
e^n \arrow[d, "x"] & & S^{n-1} \arrow[ll] \arrow[d, "d_r x"] \\
R_s & \cdots \arrow[l] & R_{s+r} \arrow[l]
\end{tikzcd}$$

The smash square of a pair $e^n \supset S^{n-1}$ is a Σ_2-equivariant filtration of length 3

$$e^n \wedge e^n \supset e^n \wedge S^{n-1} \cup S^{n-1} \wedge e^n \supset S^{n-1} \wedge S^{n-1}$$

which we abbreviate to $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2$.

The boundary of the top cell of $(E\Sigma_2)^i_+ \wedge \Gamma_0$ decomposes into two pieces:

- one carries a lower operation on x itself (using Γ_0), while
- the other carries an operation on $d_r x$ (using Γ_2).

This results in

$$d_* Sq^i x = Sq^{i+r-1} d_r x \dot{+} \begin{cases} a_j Sq^{i+j} x & j \le s-i \\ a_j x d_r x & j = s-i+1 \\ 0 & j > s-i+1 \end{cases}$$

where j is the vector fields number for P^{n+s-i}, telling how far its top cell compresses.

- May's theorem on Steenrod operations in spectral sequences derived from filtrations.
- Phil Hackney's thesis on operations in the homology of a cosimplicial E_∞-space and Jim Turner's earlier work.
- Kristine Bauer and Laura Scull's results on preservation of operad actions in spectral sequences.
- General idea: S-algebra structures produce operations, differentials, and hidden extensions in spectral sequences. Sean Tilson is working this out for the Kunneth spectral sequence, as we speak, as part of his thesis.

Thank you