Real Hochschild homology as an equivariant Loday construction

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joint work with Ayelet Lindenstrauss and Foling Zou

Algebraic Topology Seminar Princeton, 5th of February 2026

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$\mathrm{THR}(A)$ can be identified with $N_{D_2}^{O(2)}(A)$ and $N_{D_2}^{O(2)}(A)$ can be modelled by a dihedral bar construction (Angelini-Knoll, Gerhardt, Hill).

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Goal for today: Find geometric descriptions of Real Hochschild homology (and of $i_{D_{2m}}^{O(2)}(\mathrm{THR}(A))$) for all m.

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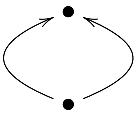

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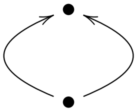

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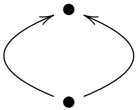

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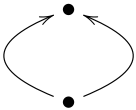

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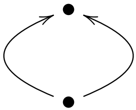

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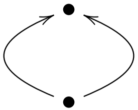

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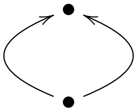

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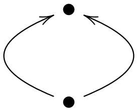

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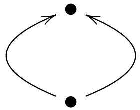

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Kristen Mazur 2013 ($G = C_{p^n}$), Rolf Hoyer 2014 (general G) showed: G-commutative monoids are G-Tambara functors.

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$\underline{A}$ is initial in Tamb_G and a unit for the so-called box product of G-Mackey functors, $\Box$.

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1. For all X and Y in $\mathrm{Sets^f}_G$ and $\underline{R}$, $\underline{T}$ in Tamb_G, there are natural isomorphisms $(X \amalg Y) \otimes \underline{R} \cong (X \otimes \underline{R}) \Box (Y \otimes \underline{R})$ and $X \otimes (\underline{R} \Box \underline{T}) \cong (X \otimes \underline{R}) \Box (X \otimes \underline{T})$.
2. There is a natural isomorphism $X \otimes (Y \otimes \underline{R}) \cong (X \times Y) \otimes \underline{R}$.

The main technical input in the equivariant context is the following result:

Theorem [Kristen Mazur 2013, Rolf Hoyer 2014] There is a functor

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3. On the category with objects finite sets with trivial G-action and morphisms consisting only of isomorphisms, the functor restricts to exponentiation $X \otimes \underline{R} = \square_{x \in X} \underline{R}$.

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The pair (N_H^G, i_H^G) is an adjoint functor pair.

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Angelini-Knoll, Gerhardt, Hill 2025:

The Real D_{2m}-Hochschild homology of $\underline{R}$ is the graded D_{2m}-Mackey functor

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For our Loday construction we would a priori need a D_{2m}-Tambara functor.

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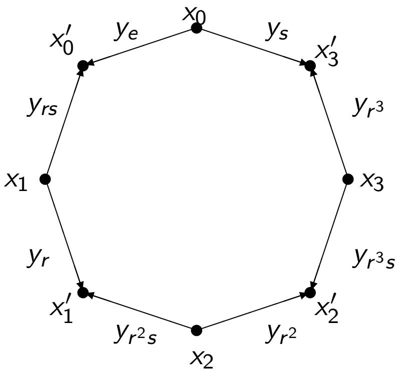

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Lemma Assume that G is a finite group and that H is a subgroup of G. Let φ be an automorphism of G and let $\underline{R}$ be a $\varphi(H)$-Tambara functor. Then the assignment

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Idea of proof: Set $(\varphi^* \underline{R})(H/K) := \underline{R}(\varphi H/\varphi K)$ and diligently show that this is alright. □

So it remains to show that $\mathcal{L}_{P_{2m}}^{D_{2m}}(\underline{R})$ is well-defined even if $\underline{R}$ is just a D_2-Tambara functor (even just a discrete E_σ-ring).

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Proposition [LRZ] Let $\underline{R}$ be a discrete E_σ-ring. Then $\mathcal{L}^{D_{2m}}_{P_{2m}}(\underline{R})$ is defined for all $m \geq 1$ and coincides with the Real D_{2m}-Hochschild homology of $\underline{R}$.

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We consider simplicial models of the 1-skeleta of the nth permutohedra and call them P_{Σ_n}.

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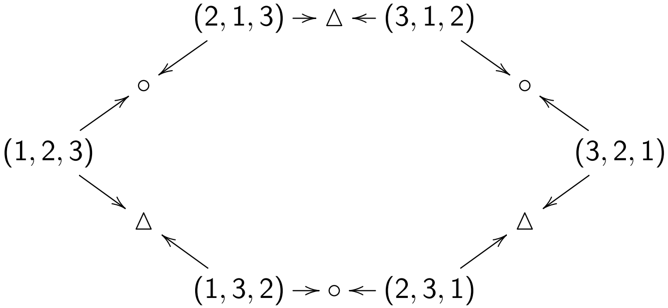

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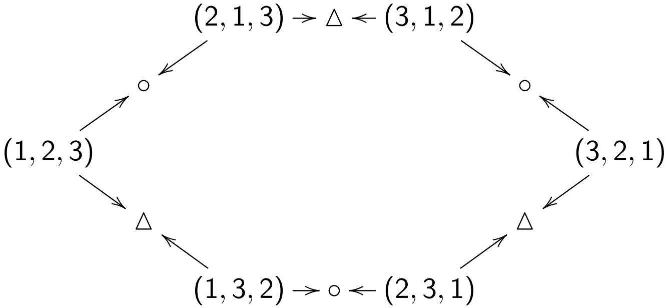
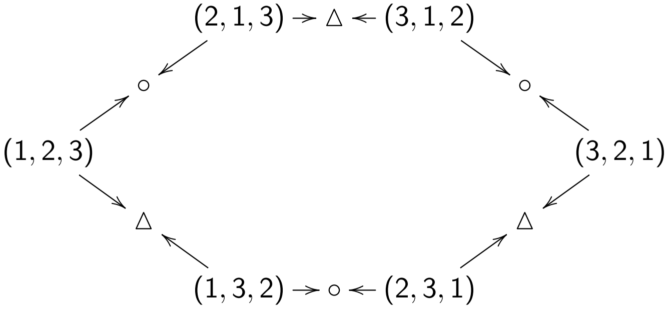


Here the vertices labelled with (i, j, k) for $i, j, k \in \{1, 2, 3\}$ constitute a free orbit Σ_3/e.

P_{Σ_3}:



Here the vertices labelled with (i, j, k) for $i, j, k \in \{1, 2, 3\}$ constitute a free orbit Σ_3/e.
If we choose as representatives for the orbits the points in the segment $(1, 2.5, 2.5) = \triangle \longleftarrow (1, 2, 3) \longrightarrow \circ = (1.5, 1.5, 3)$, then the vertices labelled with $\circ$ give rise to a $\Sigma_3/\langle(1, 2)\rangle$-orbit and the $\triangle$-vertices assemble into a $\Sigma_3/\langle(2, 3)\rangle$-orbit.

We can define $\mathcal{L}_{P_{\Sigma_3}}^{\Sigma_3}(\underline{R})$ for every $\langle(1,2)\rangle$-Tambara functor $\underline{R}$.

We can define $\mathcal{L}^{\Sigma_3}_{P_{\Sigma_3}}(\underline{R})$ for every $\langle(1,2)\rangle$-Tambara functor $\underline{R}$.

With $C_2 := \langle(1,2)\rangle$ this gives $\mathcal{L}^{\Sigma_3}_{P_{\Sigma_3}}(\underline{R})$ as a pushout of

$$\begin{array}{ccc} N_e^{\Sigma_3} i_e^{C_2} \underline{R} & \longrightarrow & B(N_e^{\Sigma_3} i_e^{C_2} \underline{R}, N_e^{\Sigma_3} i_e^{C_2} \underline{R}, N_{C_2}^{\Sigma_3} \underline{R}) \\ \big\downarrow & & \\ B(N^{\Sigma_3}_{\langle(2,3)\rangle}(c_{(1,3)})^* \underline{R}, N_e^{\Sigma_3} i_e^{C_2} \underline{R}, N_e^{\Sigma_3} i_e^{C_2} \underline{R}) & & \end{array}$$