THE COSMIC GALOIS GROUP AS KOSZUL DUAL TO WALDHAUSEN'S $A(*)$

> The world is so full of a number of things
> I'm sure we shall all be as happy as kings.
>
> Robert Louis Stevenson, *A Child's Garden of Verses*

## §I Basic questions

**1.1 Existence:** Why is there something, rather than nothing?

This does not seem to be accessible by current methods. A more realistic goal is

**Classification:** Given that there's something, what could it be?

This suggests a

**Program:** If things fall into **categories** ($\mathcal{A}, \mathcal{B}, \ldots$), hopefully **small** and **stable** enough to be manageable, techniques from K-theory may be useful.

**1.2** More precisely (following [4], see also [10]) there is a Cartesian closed category $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$ of small stable $\infty$-categories, eg $\mathcal{A}, \mathcal{B}, \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B}), \ldots$ and there is then a (similarly Cartesian closed) big **spectral** category of **pre**-motives: with objects as above, and morphism objects

$$\mathrm{Hom}_{\mathsf{Mot}}(\mathcal{A}, \mathcal{B}) := \mathrm{K}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})) \in \mathrm{K}(\$) - \mathrm{Mod}$$

enriched over Waldhausen's $A$-theory spectrum. [The superscript 'ex' signifies functors which preserve finite limits and colimits, and the objects of BGT's category are taken to be idempotent complete (ie, the category is suitably localized with respect to Morita equivalence).]

Such a category has a functorial completion to a pre-**triangulated** category $\mathsf{Mot}$ [6 §4.5], ie whose homotopy category is triangulated; this involves enlarging the set of objects by adjoining suitable cofibers, generalizing the classical Karoubification in the original theory of pure motives.

**1.3** These 'big' categories allow comparisons between objects from quite different areas of mathematics (eg homotopy theory and algebraic geometry), and they raise a host of questions. This talk proposes the motivic (Tannakian? Galois? descent? ) groups of such categories as a tool for sorting out their relations. It is a report on work in progress with **Andrew Blumberg** and **Kathryn Hess**, without whose support it would not be even a fantasy. I also want to thank Michael Ching, Ralph Cohen, Bjorn Dundas, and Bill Dwyer for their help, and in particular for enduring more than their share of foolish questions.

## §II Some examples

**2.1** If $X$ is an algebraic variety over a field $k$, and $\mathcal{A}_X = D^{\rm perf}(\mathbf{o}_X)$ is the derived category of quasicoherent sheaves of $\mathbf{o}_X$ - modules, then we get a version of classical motives, with Hom-objects enriched over $\mathrm{K}(k - \mathrm{Mod})$. A cycle map associates to a subvariety $Z$ of $X$, a resolution of its defining sheaf $\mathcal{I}_Z$ of functions.

**2.2** This example fits in the general framework of $\mathbb{A}^1$ - homotopy theory, but over more general rings the subject is in flux. If $X$ is an **arithmetic** variety, eg over the spectrum of integers of a number field, Deligne and Goncharov [9] have constructed a good category of **mixed Tate** motives over Spec $\mathbb{Z}$, with Hom objects enriched over$\mathrm{K}(\mathbb{Z}) \otimes \mathbb{Q}$.

**2.3** There is a great deal of interest in **noncommutative** motives over a field, perhaps also represented by suitable derived categories of perfect objects [1] . . .

but my concern in this talk is to ask how the most classical example of all,

**2.4 topological spaces**

might fit in this framework: in particular, in this new world of big motives, how does the 'underlying space' or 'Betti' functor

$$X \in \text{Varieties over } \mathbb{Z} \mapsto \mathrm{X}(\mathbb{C}) \in \text{Spaces}$$

behaves? This reality check is the principal motivation for this talk.

## §III Fiber functors and their motivic Galois groups

**3.1** There are dual approaches [3,5,11] to the study of spaces in this context, both involving categories of modules over ring-spectra:

i) $X \mapsto \$[\Omega X_+] = FX \in A_\infty$ - algebras,

and

ii) $X \mapsto [X_+, \$] = DX$ (Spanier-Whitehead dual) $\in E_\infty$ - algebras.

The first leads to Waldhausen's $A(X) = K(\$[\Omega X_+]-\text{Mod})$, while the second leads to Williams' [19] $\forall(X) = K(DX - \text{Mod})$; together these generalize Grothendieck's classical covariant and contravariant versions of K-theory.

Both $DX$ and $FX$ are **supplemented** \$-algebras, and in good cases (ie if $X$ is both finite and simply-connected) then

$$FX \cong \text{Hom}_\$(DX, DX),\ DX \cong \text{Hom}_\$(FX, FY)$$

expresses a kind of 'double centralizer' duality.

**3.2** In these notes I'll work with the second alternative, in the category with finite $CW$-spaces $X, Y$ as objects, and morphisms

$$\text{Hom}_{\mathsf{Mot}}(X, Y) \sim \text{K}(DX \wedge DY^{\text{op}} - \text{Mod})$$

defined by the K-theory spectra of **right-compact** $DX - DY^{\text{op}}$ - bimodules [4 §2.16]. This category can then be made pre-triangulated, as above.

There are many technical variants of this construction: for example, BGT consider both Karoubi-Villamayor or Bass-Thomason K-theory, and in the discussion below we will want to modify categories of this sort by completing their morphism objects in various ways. Eventually we will be interested in constructions based on THH and its relatives (TR, TC, ...); then I'll label the resulting categories by the functors defining their morphism objects. For example, the cyclotomic trace defines a monoidal spectral functor

$$\mathsf{Mot}_{\text{K}} \to \mathsf{Mot}_{\text{TC}}$$

of pre-triangulated categories (and hence a triangulated functor between their homotopy categories).

**3.3** Tannakian analogs of Galois groups are a central topic in the usual theory of motives: complicated categories can sometimes be identified, via some kind of descent, with categories of representations of groups of automorphisms of interesting forgetful (monoidal, 'fiber') functors to simpler categories. Weil cohomologies (Hodge, étale, crystalline) are classical examples, but the following example may be more familiar here:

Ordinary cohomology (with coefficients in $\mathbb{F}_2$ and the grading neglected), viewed as a monoidal functor

$$H : (\mathrm{Spectra}) \ni \mathrm{X} \mapsto \mathrm{H}^*(\mathrm{X}, \mathbb{F}_2) \in (\mathbb{F}_2 - \mathrm{Vect}) \ ,$$

defines a group-valued functor

$$\mathrm{Aut}^H_\otimes : (\mathbb{F}_2 - \mathrm{Alg}) \ni A \mapsto \mathrm{Aut}^A_\otimes(H^*(-, A))$$

which is (co)represented by the dual Steenrod algebra:

$$Aut^A_\otimes(H^*(-, A)) \ \cong \ \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{A}^*, A) \ .$$

The vector-space valued functor $H^*$ thus **lifts** to a functor taking values in representations of a proalgebraic groupscheme, or (in more familiar language), in the category of $\mathcal{A}^*$-comodules.

Here I want to look at (pre-triangulated, spectral, monoidal) categories built by reducing the morphism objects in BGT-style categories modulo the kernel of the Dennis trace $K(\$) \to \$$ (much as we can consider the category obtained from chain complexes over $\mathbb{Z}$ by reducing their internal Hom-objects modulo $p$).

**3.4** Hess's theory of homotopical descent [13] provides us with the needed technology: a cofibrant replacement

$$\begin{array}{ccc} K(\$) & \xrightarrow{\quad \mathrm{tr} \quad} & \$ \\ & {\scriptstyle\tau}\searrow \qquad \nearrow{\scriptstyle\rho} & \\ & Q(\$) & \end{array}$$

(of $\$$ as $K(\$)$-algebra, with $\tau$ a cofibration, and $\rho$ a weak equivalence) associates a 'Hessian' **co-ring** spectrum

$$Q(\$) \wedge_{\mathrm{K}(\$)} Q(\$) \ (= \ \mathrm{THH}_{\mathrm{K}(\$)}(\$) \ )$$

(analogous to a Hopf-Galois object in the sense of Rognes [16]) to Dennis's ring homomorphism. The example

$$\$ \to H\mathbb{F}_2$$

above produces the dual Steenrod algebra

$$Q(H\mathbb{F}_2) \wedge_{\$} Q(H\mathbb{F}_2) \;\sim\; \mathcal{A}^* \;.$$

This leads to a theory of descent relating a $K(\$)$-module spectrum $V$ to a $\mathrm{THH}_{K(\$)}(\$) := \$_{\dagger K(\$)}$ - comodule structure on

$$Q(\$) \wedge_{K(\$)} V \;=\; \mathrm{THH}_{K(\$)}(\$, V) \;:=\; V_{\dagger K(\$)} \;.$$

Then

$$\mathrm{K}(DX \wedge DY^{\mathrm{op}}) \to \mathrm{K}(DX \wedge DY^{\mathrm{op}})_{\dagger K(\$)} := \mathrm{K}_\dagger(DX \wedge DY^{\mathrm{op}})$$

defines a monoidal functor

$$\omega_{\mathrm{K}_\dagger} : \mathsf{Mot}_{\mathrm{K}} \to \mathsf{Mot}_{\mathrm{K}_\dagger} \;,$$

the latter category being enriched over spectra with an $\$_{\dagger K(\$)}$ - comodule action analogous to an action of $\mathrm{Aut}(\omega_{\mathrm{K}_\dagger})$.

We expect a more careful version of this construction to provide **effective** homotopical descent for a category with morphism objects defined by a suitable completion [13 §4, §5.5] of those of $\mathsf{Mot}_{\mathrm{K}}$.

**3.5** The notation above is admittedly unsatisfactory, but I haven't found anything better; it reflects similar difficulties with notation for Koszul duality. In the classical case of a morphism $A \to B$ of algebras over a field $k$, the covariant functor

$$V \mapsto V \otimes^L_A B := V_{\dagger B} : D(A - \mathrm{Mod}) \to D(A_{\dagger B} - \mathrm{Comod})$$

has a contravariant $k$-vector-space dual

$$V \mapsto V^\dagger_B := (V_{\dagger B})^* \cong \mathrm{RHom_A}(V, B)$$

with values in some derived category of $\mathrm{RHom}_A(B,B) \;:=\; A^\dagger_B$-modules [Cartan-Eilenberg VI §5], and in good cases this construction is a (Koszul) duality. In the formulation above, $\$_{\dagger K(\$)}$ is the analog of the algebra of functions on a group object, while

$$\mathrm{RHom}_{K(\$)}(\$,\$) \;=\; \$^{\dagger}_{K\$)}$$

is the analog of its (convolution, $L^1$) group algebra.

## §IV Cyclotomic variants

**4.1** The constructions above have a straightforward analog

$$\mathsf{Mot}_{\mathrm{TC}} \to \mathsf{Mot}_{\mathrm{TC}_\dagger}$$

built from topological cyclic homology; where now

$$\mathrm{TC}_\dagger(-) := \mathrm{THH}_{\mathrm{TC}(\$)}(\$, \mathrm{TC}(-)) \in \mathrm{THH}_{\mathrm{TC}(\$)}(\$) := \$_{\dagger \mathrm{TC}} - \mathrm{Comod}$$

(with profinite completions implicit but suppressed)[1].

The cyclotomic trace

$$\mathrm{K}(\$) \to \mathrm{TC}(\$) \sim \$ \vee \Sigma\mathbb{C}P^\infty_{-1}$$

identifies the K-theory spectrum with $\$ \vee \Sigma\mathbb{H}^\infty_+$ at regular odd primes [14, 17]. The cofibration

$$S^{-1} \to \Sigma\mathbb{C}P^\infty_{-1} \to \Sigma\mathbb{C}P^\infty_+$$

suggests that the Koszul dual of $\mathrm{THH}_{\mathrm{TC}(\$)}\$$ should be close to the tensor $\$$ -algebra [2] $\$[\Omega\Sigma\mathbb{C}P^\infty_+]$ on $\mathbb{C}P^\infty_+$. In any case, $\$_{\dagger K(\$)} \otimes \mathbb{Q}$ can be identified with the algebra of quasisymmetric functions over $\mathbb{Q}$, ie the algebra of functions on a pro-unipotent group with free Lie algebra. The cyclic structure on THH endows this Lie algebra with a $\mathbb{T}$-action and thus with a grading, placing one generator in each odd degree [7].

This is very similar to Deligne's motivic group for the category of mixed Tate motives, itself modeled on Shafarevich's conjectured description of the absolute Galois group of $\mathbb{Q}$ as a profree profinite extension of $\hat{\mathbb{Z}}^\times$.

**4.2** One concern with the constructions discussed here is that neither K nor TC is **linear**, in the sense of the calculus of functors.

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[1] Another interesting variant can be built from THH, regarded as a $\mathbb{T}$-equivariant spectrum.

$\mathrm{THH}_\$(DX)$ is the realization of a cyclic object

$$n \mapsto (DX)^{\wedge(n+1)} \ \sim \ D(X^{n+1})$$

$S$-dual to the totalization of a (cocyclic) cosimplicial space modelling the free loopspace $LX$ (cf [12]; thanks to WD for the reference!). I propose that the homotopy fixed points $\mathrm{THH}_\$(DX)^{h\mathbb{T}}$ can be identified as something like

$$[E\mathbb{T}_+, [LX_+, \$]]^{h\mathbb{T}} = [LX_{h\mathbb{T}+}, \$] = [LX_+, [E\mathbb{T}_+, \$]]^{h\mathbb{T}}$$

and that consequently $TC(DX)$ might be accessible as a homotopy limit of things like $[LX_+, \mathrm{THH}_\$(\$)]^{C_n}$.

This suggests that the inclusion $X \to LX$ of fixed points might define a kind of coassembly [18] map

$$\mathrm{TC}(DX) \to [X_+, \mathrm{TC}(\$)]$$

as a

$$\mathrm{TC}(\mathrm{holim}) \to \mathrm{holim}(\mathrm{TC})$$

interchange. [The classical assembly map defines a composition

$$\mathrm{Hom}_{K(\$)}(K(\$[\Omega X_+], \$) \to \mathrm{Hom}_{K(\S)}(X \wedge K(\$), \$) \sim DX \ \ldots]$$

**4.3** If so, then we might be able to add a third step

$$\mathsf{Mot}_{\mathrm{TC}} \to \mathsf{Mot}_{\mathrm{TC}_\dagger} \to \mathsf{Mot}^{\mathrm{lin}}_{\mathrm{TC}_\dagger}$$

to the sequence of pre-triangulated monoidal functors above, with

$$\mathrm{Hom}^{\mathrm{lin}}_{\mathrm{TC}_\dagger}(X, Y) = \mathrm{THH}_{\mathrm{TC}(\$)}(\$, [DX \wedge DY^{\mathrm{op}}, \mathrm{TC}(\$)]) \in \$_{\dagger\mathrm{TC}} - \mathrm{Comod}\ .$$

Note that

$$\mathrm{Hom}^{\mathrm{lin}}_{\mathrm{TC}_\dagger}(X, Y) \otimes \mathbb{Q} = HH_{\mathrm{TC}_\mathbb{Q}(\$)}(\mathrm{TC}_\mathbb{Q}(\$), H^*(Y \wedge DX))$$
$$= H^*(Y \wedge DX, \mathbb{Q}) = [Y, X]_\mathbb{Q}\ ,$$

so the rationalization of $\mathsf{Mot}^{\mathrm{lin}}_{\mathrm{TC}_\dagger}$ reduces to the (rationalized) category of finite spectra, (conjecturally!) reconciling the motive of an algebraic variety with the stable homotopy type of its underlying space. More generally,

$$[X, K(\$)]_{\dagger K(\$)} \ \sim \ [X, \$] \ldots$$

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