INNER-OUTER FACTORIZATIONS FOR DIFFERENTIAL-ALGEBRAIC SYSTEMS

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Abstract. We consider transfer functions of linear time-invariant differential-algebraic systems. Based on a recently developed differential-algebraic Lur'e equation, we will derive simple formulas for realizations of inner-outer factorizations. Thereby we only assume behavioral stabilizability of the system. We neither assume properness nor (proper) invertibility of the transfer function.

Key words. descriptor systems, differential-algebraic equations, inner-outer factorizations, Lur'e equations, Riccati equations

1. Introduction.

We consider differential-algebraic systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1.1b}$$

where E, $A \in \mathbb{K}^{n\times n}$ (for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) are such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is *regular*, i. e., $\det(sE - A)$ is not the zero polynomial, and $B \in \mathbb{K}^{n\times m}$, $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$. The functions $x : \mathbb{R} \to \mathbb{K}^n$, $u : \mathbb{R} \to \mathbb{K}^m$, and $y : \mathbb{R} \to \mathbb{K}^p$ are called *(generalized) state*, *input*, and *output* of the system, respectively. The *transfer function* of (1.1) is

$$G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}. \tag{1.2}$$

Conversely, we call (1.1) a *realization of* $G(s) \in \mathbb{K}(s)^{p\times m}$ if (1.2) holds true.

In this paper we discuss the construction of *inner-outer factorizations* of $G(s)$, that is

$$\boxed{G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s),}$$

where the rational matrix $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p\times q}$ is inner and $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q\times m}$ is outer (see Def. 2.6).

The crucial role of inner-outer factorizations was recognized in the early days of $\mathcal{H}_\infty$-controller design; see the pioneering textbook by Bruce A. Francis. This has led to various publications on inner-outer factorization for systems governed by ordinary differential equations (i. e., for $E = I_n$) [3–5, 9, 16, 17].

The most simple situation is when the transfer function $G(s)$ of a system with $E = I_n$ has no zeros (see Def. 2.5) on the imaginary axis and D has full column rank; in this case, rather simple realizations of $G_{\mathrm{i}}(s)$ and $G_{\mathrm{o}}(s)$ can be constructed by using the stabilizing solution of a certain algebraic Riccati equation (see also Remark 3.5). If the latter conditions are not fulfilled, the situation becomes more involved. Inner-outer

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factorization for right-invertible transfer functions is considered in [3,17]; the article [5] treats the case of strictly proper transfer functions. The general factorization problem for transfer functions governed by differential-algebraic systems has been examined in [16] as well as [9, 14] which consider the problem from the numerical point of view. All the approaches treating the "non-simple case" have in common that several successive steps are needed to determine the factors; no explicit formulas are given in terms of the realization matrices of the system to be factored.

The novelty of our approach is that we exploit recent results on Lur'e equations for differential-algebraic systems [12] to derive simple formulas for realizations of the inner and outer factors. This yields also new results for systems described by ordinary differential equations; this is possible with the larger framework of DAEs. The only assumption on the realization of the transfer function to be factored will be behavioral stabilizability. Stability, properness, or proper invertibility are not required. Note that parts of this work have already been discussed in the thesis of the second author [15] under the stronger assumption of strong stabilizability.

Notation. We use the notations $\imath$, $\overline{\lambda}$, A^*, A^{-*}, I_n, $0_{m\times n}$ for the imaginary unit, the complex conjugate of $\lambda \in \mathbb{C}$, the conjugate transpose of a complex matrix and its inverse, the identity matrix of size $n\times n$ and the zero matrix of size $m\times n$ (subscripts are omitted if clear from context), respectively. Further, the following notation is used throughout the presented note:

$\mathbb{N}_0$	set of natural numbers including zero
$\mathbb{K}$	either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers
$\mathbb{C}_+$	the open set of complex numbers with positive real part
$\mathbb{K}[s]$, $\mathbb{K}(s)$	the ring of polynomials and the field of rational functions with coefficients in $\mathbb{K}$, resp.
$\Lambda(E,A)$	set of zeros of $\det(sE-A)$ for a matrix pencil $sE-A \in \mathbb{K}[s]^{n\times n}$
$\mathcal{R}^{m\times n}$	the set of $m\times n$ matrices with entries in a ring $\mathcal{R}$
$\mathrm{Gl}_n(\mathcal{R})$	the group of invertible $n\times n$ matrices with entries in a ring $\mathcal{R}$
$\operatorname{rank}_{\mathbb{K}(s)} G(s), \operatorname{im}_{\mathbb{K}(s)} G(s)$, $\ker_{\mathbb{K}(s)} G(s)$	rank, image, and kernel of $G(s) \in \mathbb{K}(s)^{p\times m}$
$\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I},\mathbb{K}^n)$	the set of measurable and locally square integrable functions $f:\mathcal{I}\to\mathbb{K}^n$ on the set $\mathcal{I}\subseteq\mathbb{R}$
$M =_{\mathcal{V}} (\geq_{\mathcal{V}})\, N$	$x^*Mx \;=\; (\geq)\, x^*Nx \quad \forall\, x\in\mathcal{V}$, where $\mathcal{V}\subseteq\mathbb{K}^n$ is a subspace and M, $N\in\mathbb{K}^{n\times n}$ are Hermitian matrices

2. Preliminaries.

2.1. Basic systems theoretic concepts. We denote by $\Sigma_{n,m,p}(\mathbb{K})$ the set of systems (1.1) with E, $A\in\mathbb{K}^{n\times n}$ such that the pencil $sE-A\in\mathbb{K}[s]^{n\times n}$ is regular and $B\in\mathbb{K}^{n\times m}$, $C\in\mathbb{K}^{p\times n}$, $D\in\mathbb{K}^{p\times m}$, and we write $[E,A,B,C,D]\in\Sigma_{n,m,p}(\mathbb{K})$.

The set of control systems (1.1a) with E, A and B as above is denoted by $\Sigma_{n,m}(\mathbb{K})$, and we write $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$.

The *behavior* of $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ is the set of all solutions of (1.1a), that is

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^m) : \tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax + Bu \right\},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the distributional derivative. Note that $(x,u) \in \mathfrak{B}_{[E,A,B]}$ implies that Ex is absolutely continuous, hence the evaluation $Ex(0) := (Ex)(0)$ is well-defined. The behavior of $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ is defined by

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x,u,y) \in \mathfrak{B}_{[E,A,B]} \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^p) : y = Cx + Du \right\}.$$

Next we consider the notion of behavioral stabilizability which has been introduced for a larger class of systems in [10].

Definition 2.1 (Behavioral stabilizability). *We call* $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ behaviorally stabilizable*, if for all* $(x_1,u_1) \in \mathfrak{B}_{[E,A,B]}$*, there exists some* $(x,u) \in \mathfrak{B}_{[E,A,B]}$ *with*

$$(x(t),u(t)) = (x_1(t),u_1(t)) \text{ if } t<0, \text{ and } \lim_{t\to\infty} \operatorname*{ess\,sup}_{\tau>t} \left(\|x(\tau)\| + \|u(\tau)\| \right) = 0.$$

Further, a system $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *is called behaviorally stabilizable if* $[E,A,B]$ *is behaviorally stabilizable.*

Behavioral stabilizability has a simple algebraic characterization [1, Cor. 4.3] (see also [10, Thm. 5.2.30]).

Proposition 2.2 (Algebraic characterization of stabilizability). *The system* $[E,A,B] \in \Sigma_{n,m,p}(\mathbb{K})$ *satisfies:*

$$[E,A,B] \text{ is behaviorally stabilizable} \iff \forall\, \lambda \in \overline{\mathbb{C}_+} : \operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n.$$

Next we introduce two fundamental spaces of a system $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$, namely the system space and the space of consistent initial differential variables.

Definition 2.3 (System space, space of consistent initial differential variables). *Let* $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$ *be given.*

(i) The system space *of* $[E,A,B]$ *is the smallest subspace* $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \subseteq \mathbb{K}^{n+m}$ *such that*

$$\forall\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \quad \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \quad \text{for almost all } t \in \mathbb{R}.$$

(ii) The space of consistent initial differential variables *of* $[E,A,B]$ *is defined by*

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} := \left\{ x_0 \in \mathbb{K}^n : \exists (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \right\}.$$

For a geometric characterization of $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ and $\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}$, we refer to [12, Prop. 2.9 & Prop. 3.3].

Next we introduce some facts and properties of rational matrices. Many properties will be analyzed by means of the Smith-McMillan form; it is a canonical form on $\mathbb{K}(s)^{p\times m}$ under the group action of multiplication from the left and right with unimodular matrices (i.e., units of the ring of square polynomial matrices).

Theorem 2.4 (Smith-McMillan form [8, Sec. 6.5.2]). *For* $G(s) \in \mathbb{K}(s)^{p\times m}$ *with* $\operatorname{rank}_{\mathbb{K}(s)} G(s) = q$*, there exist unimodular matrices* $U(s) \in \mathrm{Gl}_p(\mathbb{K}[s])$ *and* $V(s) \in \mathrm{Gl}_m(\mathbb{K}[s])$ *such that*

$$U^{-1}(s)G(s)V^{-1}(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_q(s)}{\psi_q(s)} \right) \tag{2.1}$$

with unique monic and coprime polynomials $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]\setminus\{0\}$ *such that* $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$ *and* $\psi_{i+1}(s) \mid \psi_i(s)$ *for all* $i \in \{1, \dots, q-1\}$.

Theorem 2.4 gives rise to the following (standard) definitions.

Definition 2.5 (Poles and zeros [18]). *Let* $G(s) \in \mathbb{K}(s)^{p\times m}$ *with* $\operatorname{rank}_{\mathbb{K}(s)} G(s) = q$ *be given. Using the notation of Theorem 2.4,* $\lambda \in \mathbb{C}$ *is called*

(i) *a* zero *of* $G(s)$ *if* $\varepsilon_r(\lambda) = 0$*;*

(ii) *a* pole *of* $G(s)$ *if* $\psi_1(\lambda) = 0$.

Next we introduce the main concepts for this note.

Definition 2.6 (Outer and inner rational functions). *A rational function* $G(s) \in \mathbb{K}(s)^{p\times m}$ *is called*

(i) outer *if* $p = \operatorname{rank}_{\mathbb{K}(s)} G(s)$ *and* $G(s)$ *has no zeros in* $\mathbb{C}_+$*;*

(ii) inner *if* $G(s)$ *has no poles in* $\mathbb{C}_+$ *and* $G^*(-\overline{s})G(s) = I_m$.

Remark 2.7 (Inner and outer functions).

(i) *If* $G(s) \in \mathbb{K}(s)^{p\times m}$ *is inner, then* $G(s)$ *is bounded in* $\mathbb{C}_+$. *Inner functions fulfill* $G(\imath\omega)^*G(\imath\omega) = I_m$ *for all* $\omega \in \mathbb{R}$ *with* $\imath\omega \notin \Lambda(E, A)$. *This means that for a realization* $[E, A, B, C, D]$ *of* $G(s)$*, all frequencies pass equally in gain. For this reason, realizations of inner functions are also called* all-pass filters *[18].*

(ii) *The transfer function* $G(s) \in \mathbb{K}(s)^{p\times m}$ *of* $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ *is outer if*

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C & D \end{bmatrix} = n + p \quad \forall\, \lambda \in \mathbb{C}_+.$$

Further properties of realizations of outer transfer functions have been considered in [6].

The following lemma about realizations of certain fractions of transfer functions will be essential for the construction of the factorizations considered in this article.

Lemma 2.8. *[13, Lem. 3.5] Consider the systems* $[E, A, B, C_1, D_1] \in \Sigma_{n,m,m}(\mathbb{K})$ *and* $[E, A, B, C_2, D_2] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer functions*

$$\begin{aligned} G_1(s) &= C_1(sE - A)^{-1}B + D_1 \in \mathrm{Gl}_m(\mathbb{K}(s)), \\ G_2(s) &= C_2(sE - A)^{-1}B + D_2 \in \mathbb{K}(s)^{p\times m}. \end{aligned}$$

Then the pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C_1 & -D_1 \end{smallmatrix}\right]$ *is regular. Moreover, the transfer function of*

$$\begin{aligned} &[E_\mathrm{e}, A_\mathrm{e}, B_\mathrm{e}, C_\mathrm{e}, D_\mathrm{e}] \\ :=& \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \begin{bmatrix} C_2 & D_2 \end{bmatrix}, 0_{p\times m} \right] \in \Sigma_{n+m,m,p}(\mathbb{K}) \end{aligned}$$

is $G_\mathrm{e}(s) = G_2(s)G_1^{-1}(s)$.

2.2. Lur'e equations. The key ingredient for our inner-outer factorizations are solutions of Lur'e equations for differential-algebraic systems which have been developed in [12].

Definition 2.9 (Lur'e equation). *For* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *and* $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$, *the* Lur'e equation *is given by*

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*. \tag{2.2}$$

A triple $(X,K,L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ *for some* $q \in \mathbb{N}_0$ *is called* solution *of* (2.2), *if* (2.2) *holds with*

$$\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE+A & B \\ K & L \end{bmatrix} = n+q.$$

Further, a solution (X,K,L) *of the Lur'e equation* (2.2) *is called*

(i) stabilizing *if*

$$\operatorname{rank} \begin{bmatrix} -\lambda E+A & B \\ K & L \end{bmatrix} = n+q \quad \forall \lambda \in \mathbb{C}_+;$$

(ii) nonnegative *if* $E^*XE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0$.

Note that (2.2) is a generalization of the standard Lur'e equation for ordinary differential equations [11] which is, on the other hand, a generalization of the famous algebraic Riccati equation. The same is true for the concept of a stabilizing solution.

The following theorem gives a sufficient condition for the existence of a stabilizing solution and summarizes some further implications.

Theorem 2.10. *[12, Thm. 5.3(b), Thm. 5.5(a), Rem. 5.7] Consider a behaviorally stabilizable control system* $[E,A,B] \in \Sigma_{n,m}(\mathbb{K})$. *Further, let the matrices* $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$ *and* $R = R^* \in \mathbb{K}^{m\times m}$ *be given. Assume that there exists some* $P \in \mathbb{K}^{n\times n}$ *that satisfies the* Kalman-Yakubovich-Popov (KYP) inequality

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} 0, \quad P = P^*. \tag{2.3}$$

Then the Lur'e equation (2.2) *has a stabilizing solution* $(X,K,L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$. *This solution has the following properties:*

(i) *It is maximal in the sense that it holds that*

$$E^*XE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} E^*PE$$

for all $P \in \mathbb{K}^{n\times n}$ *fulfilling the KYP inequality* (2.3).

(ii) *It realizes a* spectral factorization *of the* Popov function

$$\Phi(s) := \begin{bmatrix} (-\overline{s}E-A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE-A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}(s)^{m\times m}, \tag{2.4}$$

in the sense that $\Phi(s) = W^*(-\overline{s})W(s)$ *for the outer function*

$$W(s) = K(sE-A)^{-1}B + L \in \mathbb{K}(s)^{q\times m}.$$

(iii) *The number* q *with* $K \in \mathbb{K}^{q\times n}$, $L \in \mathbb{K}^{q\times m}$ *and the Popov function are related by* $q = \operatorname{rank}_{\mathbb{K}(s)} \Phi(s)$.

In the following we show that Lur'e equations can be further used to characterize when a rational function is inner.

Theorem 2.11. *Let* $[E,A,B,C,D] \in \Sigma_{n,m,p}(\mathbb{K})$ *with transfer function* $G(s) \in \mathbb{K}(s)^{p\times m}$ *be given. If there exists some Hermitian* $P \in \mathbb{K}^{n\times n}$ *with*

$$\begin{bmatrix} A^*PE + E^*PA + C^*C & E^*PB + C^*D \\ B^*PE + D^*C & D^*D - I_m \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} 0, \quad E^*PE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0, \tag{2.5}$$

then $G(s)$ is inner.

Proof. We can conclude from the differential-algebraic bounded real lemma [13, Thm. 4.4(a)] that

$$I_m - G^*(\lambda)G(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{C}_+ \setminus \Lambda(E, A).$$

This implies that $G(s)$ has no poles in $\mathbb{C}_+$. By [12, Lem. 3.5] we have

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \subseteq \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \quad \forall \lambda \in \mathbb{C} \setminus \Lambda(E, A),$$

and [12, Eq. (4.12)] yields

$$\begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} = 0.$$

This results in

$$\begin{aligned} 0 &= \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA + C^*C & E^*PB + C^*D \\ B^*PE + D^*C & D^*D - I_m \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - I_m \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \\ &= G^*(-\overline{s})G(s) - I_m, \end{aligned}$$

which shows that $G(s)$ is inner. ∎

3. Construction of inner-outer factorizations. We construct inner-outer factorizations of arbitrary rational matrices. The basis for such a construction will be the Lur'e equation

$$\begin{bmatrix} A^*XE + E^*XA + C^*C & E^*XB + C^*D \\ B^*XE + D^*C & D^*D \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E,A,B]}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*. \tag{3.1}$$

First we present the general idea for our approach: The Popov function corresponding to the Lur'e equation (3.1) is

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} = G(-\overline{s})^*G(s). \tag{3.2}$$

For a stabilizing solution (X, K, L) of (3.1) we obtain from Theorem 2.10 that $\Phi(s) = W(-\overline{s})^*W(s)$ for the outer function $W(s) = K(sE - A)^{-1}B + L$. Assume for convenience that an inner-outer factorization $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s)$ with $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p\times q}$ and $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q\times m}$ exists. Then (3.2) and the property $G_{\mathrm{i}}(-\overline{s})^*G_{\mathrm{i}}(s) = I_q$ implies that

$$G(-\overline{s})^*G(s) = G_{\mathrm{o}}(-\overline{s})^* \underbrace{G_{\mathrm{i}}(-\overline{s})^*G_{\mathrm{i}}(s)}_{=I_q} G_{\mathrm{o}}(s) = G_{\mathrm{o}}(-\overline{s})^*G_{\mathrm{o}}(s).$$

This justifies the ansatz $G_{\mathrm{o}}(s) = W(s) = K(sE-A)^{-1}B+L$. The inner factor will be constructed by $G_{\mathrm{i}}(s) = G(s)G_{\mathrm{o}}(s)^-$, where $G_{\mathrm{o}}(s)^-$ denotes a right inverse of $G_{\mathrm{o}}(s)$. Thereby, we will construct a right inverse of $G_{\mathrm{o}}(s)$ by $Z(G_{\mathrm{o}}(s)Z)^{-1}$, where $Z \in \mathbb{R}^{m\times q}$ is a matrix such that $G_{\mathrm{o}}(s)Z$ is invertible. The realization of $G_{\mathrm{i}}(s) = G(s)G_{\mathrm{o}}(s)^- = G(s)(G_{\mathrm{o}}(s)Z)^{-1}Z$ will be constructed by using Lemma 2.8.

We will show in Theorem 3.3 that the above outlined idea can indeed be used to construct inner-outer factorizations. Note that our construction will be purely based on a realization $[E, A, B, C, D]$ of $G(s)$; no inversions of transfer functions will be involved. As the above idea illustrates, the key ingredients will be the stabilizing solution (X, K, L) of the Lur'e equation (3.1) and a matrix $Z \in \mathbb{R}^{m\times q}$ such that $G_{\mathrm{o}}(s)Z$ is invertible. Before we present our main result on the construction of inner-outer factorizations, we first show that these key ingredients exist.

Proposition 3.1. *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ be behaviorally stabilizable. Then the Lur'e equation* (3.1) *has a stabilizing solution (X, K, L); this solution is nonnegative.*

Proof. Since $P = 0$ solves the KYP inequality associated to (3.1), the result follows immediately from Theorem 2.10. ∎

Proposition 3.2. *Let a system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ with transfer function $G(s) \in \mathbb{K}(s)^{p\times m}$ be given. Denote $q = \operatorname{rank}_{\mathbb{K}(s)} G(s)$. Then there exists some matrix $Z \in \mathbb{K}^{m\times q}$ such that $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$.*
If $q = p$ and Z has the above property, then the following statements are satisfied:
(i) *The pencil $\left[\begin{smallmatrix} -sE+A & BZ \\ C & DZ \end{smallmatrix}\right] \in \mathbb{K}[s]^{n+p\times n+p}$ is regular.*
(ii) *The rational function $\mathcal{P}(s) = Z(G(s)Z)^{-1}G(s) \in \mathbb{K}(s)^{p\times p}$ is a projector with $\ker_{\mathbb{K}(s)} G(s) = \ker_{\mathbb{K}(s)} \mathcal{P}(s)$.*

Proof. Denote the k-th canonical unit vector by $e_k \in \mathbb{R}^m$. Since the column vectors of $G(s)$ can be reduced to a basis of $\operatorname{im}_{\mathbb{K}(s)} G(s)$ and $\dim \operatorname{im}_{\mathbb{K}(s)} G(s) = q$, there exist $i_1, \ldots, i_q \in \{1, \ldots, m\}$ such that $\left\{G(s)e_{i_1}, \ldots, G(s)e_{i_q}\right\}$ is a basis of $\operatorname{im}_{\mathbb{K}(s)} G(s)$. Then the matrix $Z = [e_{i_1}, \ldots, e_{i_q}]$ has the desired property.

Consequently, if $q = p$ then $G(s)Z \in \mathrm{Gl}_p(\mathbb{K}(s))$. Statement (i) can now be concluded from

$$\begin{aligned}\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE+A & BZ \\ C & DZ \end{bmatrix} &= \operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} I_n & 0 \\ C(sE-A)^{-1} & I_q \end{bmatrix} \begin{bmatrix} -sE+A & BZ \\ C & DZ \end{bmatrix} \\ &= \operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE+A & BZ \\ 0 & G(s)Z \end{bmatrix} = n+q.\end{aligned}$$

Statement (ii) follows by simple calculations. ∎

Now we formulate our main result on the construction of inner-outer factorizations.

Theorem 3.3. *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ behaviorally stabilizable with transfer function $G(s) \in \mathbb{K}(s)^{p\times m}$. Let $q = \operatorname{rank}_{\mathbb{K}(s)} G(s)$ and $Z \in \mathbb{K}^{m\times q}$ be a matrix with $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$ (which exists by Proposition 3.2). Let (X, K, L) be a stabilizing solution of the Lur'e equation* (3.1) *(which exists by Proposition 3.1). Then an inner-outer factorization is given by $G(s) = G_{\mathrm{i}}(s)G_{\mathrm{o}}(s)$, where $G_{\mathrm{i}}(s) \in \mathbb{K}(s)^{p\times q}$ is the transfer function of*

$$[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}, C_{\mathrm{i}}, D_{\mathrm{i}}] := \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix}, \begin{bmatrix} 0 \\ -I_q \end{bmatrix}, \begin{bmatrix} C & DZ \end{bmatrix}, 0_{p\times q}\right] \in \Sigma_{n+q,q,p}(\mathbb{K}), \tag{3.3}$$

and $G_o(s) \in \mathbb{K}(s)^{q\times m}$ is the transfer function of

$$[E_{\mathrm{o}}, A_{\mathrm{o}}, B_{\mathrm{o}}, C_{\mathrm{o}}, D_{\mathrm{o}}] := [E, A, B, K, L] \in \Sigma_{n,m,q}(\mathbb{K}). \tag{3.4}$$

Proof. We proceed in several steps.
Step 1: $G_{\mathrm{o}}(s)$ is outer:

This follows from Theorem 2.10 (ii).

Step 2: $\ker_{\mathbb{K}(s)} G(s) = \ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s)$*:*

Theorem 2.10 (ii) together with (3.2) yields

$$G_{\mathrm{o}}(\imath\omega)^* G_{\mathrm{o}}(\imath\omega) = G(\imath\omega)^* G(\imath\omega) \quad \forall \omega \in \mathbb{R} \text{ with } \imath\omega \notin \Lambda(E, A). \tag{3.5}$$

First we show that $\ker_{\mathbb{K}(s)} G(s) \subseteq \ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s)$: Assume that $v(s) \in \ker_{\mathbb{K}(s)} G(s)$. Let $\Gamma \subset \mathbb{C}$ be the (finite) set of poles of $v(s) \in \mathbb{K}(s)^m$. Then, we obtain from (3.5) that for all $\omega \in \mathbb{R}$ with $\imath\omega \notin \Gamma \cup \Lambda(E, A)$ we have

$$\|G_{\mathrm{o}}(\imath\omega)v(\imath\omega)\|_2^2 = \|G(\imath\omega)v(\imath\omega)\|_2^2 = 0.$$

Hence, $\lambda \mapsto G_{\mathrm{o}}(\lambda)v(\lambda)$ is a vector-valued rational function which vanishes on the infinite set $\imath\mathbb{R} \setminus (\Gamma \cup \Lambda(E, A))$. This gives $G_{\mathrm{o}}(s)v(s) = 0$, i.e., $v(s) \in \ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s)$. The proof of the reverse inclusion $\ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s) \subseteq \ker_{\mathbb{K}(s)} G(s)$ is completely analogous and therefore omitted.

Step 3: $G_{\mathrm{o}}(s)Z$ *is invertible:*

We obtain from Step 2 and the outerness of $G_{\mathrm{o}}(s)$ that

$$\operatorname{rank}_{\mathbb{K}(s)} G(s) = \operatorname{rank}_{\mathbb{K}(s)} G_{\mathrm{o}}(s) = q.$$

The outerness of $G_{\mathrm{o}}(s)$ further implies $G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q \times m}$. By (3.5), we obtain

$$(G_{\mathrm{o}}(\imath\omega)Z)^*(G_{\mathrm{o}}(\imath\omega)Z) = (G(\imath\omega)Z)^*(G(\imath\omega)Z) \quad \forall \omega \in \mathbb{R} \text{ with } \imath\omega \notin \Lambda(E, A).$$

The assumption $\operatorname{rank}_{\mathbb{K}(s)} G(s)Z = q$ then leads to $G_{\mathrm{o}}(s)Z \in \mathrm{Gl}_q(\mathbb{K}(s))$.

Step 4: $G_{\mathrm{i}}(s)G_{\mathrm{o}}(s) = G(s)$*:*

Using the statement in Step 1 and the fact that $G_{\mathrm{o}}(s)Z$ is realized by the system $[E, A, BZ, K, LZ]$, Proposition 3.2 (i) leads to regularity of the pencil

$$\begin{bmatrix} -sE + A & BZ \\ K & LZ \end{bmatrix} \in \mathbb{K}[s]^{n+q \times n+q}.$$

Lemma 2.8 then gives rise to

$$G_{\mathrm{i}}(s) = \begin{bmatrix} C & DZ \end{bmatrix} \begin{bmatrix} sE - A & -BZ \\ -K & -LZ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = G(s)Z(G_{\mathrm{o}}(s)Z)^{-1}.$$

Proposition 3.2 (ii) yields that $Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s) \in \mathbb{K}(s)^{q \times q}$ is a projector along $\ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s)$. Since further, by Step 2, $\ker_{\mathbb{K}(s)} G_{\mathrm{o}}(s) = \ker_{\mathbb{K}(s)} G(s)$, we obtain

$$\begin{aligned} G_{\mathrm{i}}(s) \cdot G_{\mathrm{o}}(s) &= G(s)Z(G_{\mathrm{o}}(s)Z)^{-1} \cdot G_{\mathrm{o}}(s) \\ &= G(s) - \underbrace{G(s)(I_q - Z(G_{\mathrm{o}}(s)Z)^{-1}G_{\mathrm{o}}(s))}_{=0} = G(s). \end{aligned}$$

Step 5: $G_{\mathrm{i}}(s)$ *is inner:*

Using Theorem 2.11 and invoking $D_{\mathrm{i}} = 0_{p \times q}$, it suffices to show that there exists some Hermitian $P_{\mathrm{i}} \in \mathbb{K}^{n+q \times n+q}$, such that

$$\begin{bmatrix} A_{\mathrm{i}}^* P_{\mathrm{i}} E_{\mathrm{i}} + E_{\mathrm{i}}^* P_{\mathrm{i}} A_{\mathrm{i}} + C_{\mathrm{i}}^* C_{\mathrm{i}} & E_{\mathrm{i}}^* P_{\mathrm{i}} B_{\mathrm{i}} \\ B_{\mathrm{i}}^* P_{\mathrm{i}} E_{\mathrm{i}} & -I_q \end{bmatrix} =_{\mathcal{V}^{\mathrm{sys}}_{[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}]}} 0, \quad E_{\mathrm{i}}^* P_{\mathrm{i}} E_{\mathrm{i}} \geq_{\mathcal{V}^{\mathrm{diff}}_{[E_{\mathrm{i}}, A_{\mathrm{i}}, B_{\mathrm{i}}]}} 0,$$

with $[E_\mathrm{i}, A_\mathrm{i}, B_\mathrm{i}, C_\mathrm{i}, D_\mathrm{i}] \in \Sigma_{n+q,q,p}(\mathbb{K})$ as in (3.3). By using the block matrix structure in (3.3), the system space and space of consistent initial differential variable of this system may be represented as

$$\begin{aligned}
\mathcal{V}^{\mathrm{sys}}_{[E_\mathrm{i},A_\mathrm{i},B_\mathrm{i}]} &= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+2q} : A_\mathrm{i}x + B_\mathrm{i}u \in \operatorname{im} E_\mathrm{i} \right\} \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} \in \mathbb{K}^{n+2q} : \begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix} \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]} \text{ and } Kx_1 + LZx_2 = u \right\}, \\
\mathcal{V}^{\mathrm{diff}}_{[E_\mathrm{i},A_\mathrm{i},B_\mathrm{i}]} &= \mathcal{V}^{\mathrm{diff}}_{[E,A,B]} \times \mathbb{K}^m.
\end{aligned}$$

Now consider $P_\mathrm{i} = \left[\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Proposition 3.1 gives $E^*XE \geq_{\mathcal{V}^{\mathrm{diff}}_{[E,A,B]}} 0$. The above representation of $\mathcal{V}^{\mathrm{diff}}_{[E_\mathrm{i},A_\mathrm{i},B_\mathrm{i}]}$ leads to

$$E_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} = \begin{bmatrix} E^*XE & 0 \\ 0 & 0 \end{bmatrix} \geq_{\mathcal{V}^{\mathrm{diff}}_{[E_\mathrm{i},A_\mathrm{i},B_\mathrm{i}]}} 0.$$

Let $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E_\mathrm{i},A_\mathrm{i},B_\mathrm{i}]}$. Partitioning $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ according to the block structure of the system $[E_\mathrm{i}, A_\mathrm{i}, B_\mathrm{i}, C_\mathrm{i}, D_\mathrm{i}]$, we obtain from the previous representation of $\mathcal{V}^{\mathrm{sys}}_{[E_\mathrm{i},A_\mathrm{i},B_\mathrm{i}]}$ that $\left(\begin{smallmatrix} x_1 \\ Zx_2 \end{smallmatrix}\right) \in \mathcal{V}^{\mathrm{sys}}_{[E,A,B]}$ and $u = Kx_1 + LZx_2$. By further using that

$$\begin{aligned}
A_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} + E_\mathrm{i}^* P_\mathrm{i} A_\mathrm{i} + C_\mathrm{i}^* C_\mathrm{i} &= \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & BZ \\ K & LZ \end{bmatrix} + \begin{bmatrix} C^* \\ Z^*D^* \end{bmatrix} \begin{bmatrix} C & DZ \end{bmatrix} \\
&= \begin{bmatrix} A^*XE + E^*XA + C^*C & E^*XBZ + C^*DZ \\ Z^*B^*XE + Z^*D^*C & Z^*D^*DZ \end{bmatrix}, \\
E_\mathrm{i}^* P_\mathrm{i} B_\mathrm{i} &= \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -I_q \end{bmatrix} = 0_{n+q \times q},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} + E_\mathrm{i}^* P_\mathrm{i} A_\mathrm{i} + C_\mathrm{i}^* C_\mathrm{i} & E_\mathrm{i}^* P_\mathrm{i} B_\mathrm{i} \\ B_\mathrm{i}^* P_\mathrm{i} E_\mathrm{i} & -I_q \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ Kx_1 + LZx_2 \end{pmatrix}^* \begin{bmatrix} A^*XE + E^*XA + C^*C & E^*XBZ + C^*DZ & 0 \\ Z^*B^*XE + Z^*D^*C & Z^*D^*DZ & 0 \\ 0 & 0 & -I_q \end{bmatrix} \\
&\qquad \cdot \begin{pmatrix} x_1 \\ x_2 \\ Kx_1 + LZx_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix}^* \left(\begin{bmatrix} A^*XE + E^*XA + C^*C & E^*XB + C^*D \\ B^*XE + D^*C & D^*D \end{bmatrix} - \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix} \right) \begin{pmatrix} x_1 \\ Zx_2 \end{pmatrix} \\
&= 0,
\end{aligned}$$

which implies that $G_\mathrm{i}(s)$ is inner. ∎

Next we briefly illustrate the results of this article on two simple examples.

Example 3.4.

(i) *Consider the system* $[E, A, B, C, D] \in \Sigma_{2,1,2}$ *with*

$$sE - A = \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = 0_{2\times 1}. \tag{3.6}$$

The system $[E, A, B, C, D]$ *is behaviorally stabilizable with transfer function*

$$G(s) = \begin{bmatrix} s+1 \\ s \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}.$$

A stabilizing solution of the Lur'e equation (3.1) *is*

$$(X, K, L) = \left(\begin{bmatrix} \sqrt{2}-1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} & -1 \end{bmatrix}, 0 \right).$$

Since $G(s)$ *has full column rank over* $\mathbb{R}(s)$*, we can choose* $Z = 1$*. By using* (3.3) *and* (3.4) *we obtain*

$$G_{\mathrm{o}}(s) = \begin{bmatrix} -\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} -1 & s \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2}s + 1 \in \mathbb{R}(s),$$

$$G_{\mathrm{i}}(s) = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & -1 \\ \sqrt{2} & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{1+\sqrt{2}s} \\ \frac{s}{1+\sqrt{2}s} \end{bmatrix} \in \mathbb{R}(s)^{2\times 1}.$$

It can be verified that $G(s) = G_{\mathrm{i}}(s) G_{\mathrm{o}}(s)$ *and, moreover, that* $G_{\mathrm{i}}(s)$ *is inner and* $G_{\mathrm{o}}(s)$ *is outer.*

(ii) *Consider the system* $[E, A, B, C, D] \in \Sigma_{2,2,1}$ *with*

$$sE - A = \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & 0 \\ 0 & 0 & s \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}, \quad D = 0_{1\times 2}. \tag{3.7}$$

The system $[E, A, B, C, D]$ *is behaviorally stabilizable with transfer function*

$$G(s) = \begin{bmatrix} s-1, & 1 - \frac{1}{s} \end{bmatrix} \in \mathbb{R}(s)^{1\times 2}.$$

A stabilizing solution of the Lur'e equation (3.1) *is*

$$(X, K, L) = \left(\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}, 0_{1\times 2} \right),$$

and we obtain from (3.4) *that the outer factor is given by*

$$G_{\mathrm{o}}(s) = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & s & 0 \\ 0 & -1 & 0 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s+1 & 1 + \frac{1}{s} \end{bmatrix} \in \mathbb{R}(s)^{1\times 2}.$$

The matrix Z *in Theorem 3.3 can be chosen to be* $Z = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$*. Then, in view of* (3.3)*, we obtain that the inner factor reads*

$$G_{\mathrm{i}}(s) = \begin{bmatrix} -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & s & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & s & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \frac{s-1}{s+1} \in \mathbb{R}(s).$$

Remark 3.5 (Inner-outer factorization).

(i) *If* $\det(sE-A)$ *has no zeros in* $\overline{\mathbb{C}_+}$*, then the outer factor* $G_{\mathrm{o}}(s)$ *in* (3.3) *has no poles in in* $\overline{\mathbb{C}_+}$*.*
If $G(s)$ *is proper (that is,* $\lim_{\lambda\to\infty}\|G(\lambda)\|<\infty$*), then it follows from*

$$G_{\mathrm{o}}(s) = G_{\mathrm{i}}(-\overline{s})^* G(s)$$

that $G_{\mathrm{o}}(s)$ *is proper as well.*

(ii) *As we can see from Example 3.4, the realizations* (3.3) *and* (3.4) *of the inner and outer factors are in general not minimal. Of course, minimal realizations can be obtained by a transformation into Kalman decomposition [2, Thm. 8.1] and a subsequent elimination of the uncontrollable and unobservable parts.*

(iii) *In [7], transfer functions of single-input single-output systems (that is,* $m=p=1$*) are considered. It is shown that inner-outer factorizations can be obtained in a rather simple way: The transfer function* $g(s)\in\mathbb{K}(s)$ *is first factorized as*

$$g(s) = \frac{d^+(s)\cdot d^-(s)}{n(s)}$$

for polynomials $d^+(s)$, $d^-(s)$, $n(s)\in\mathbb{K}[s]$ *with the property that all roots of* $d^+(s)$ *are in* $\mathbb{C}^+$ *and all roots of* $d^-(s)$ *are in* $\mathbb{C}\setminus\mathbb{C}^+$*. An inner-outer factorization is then given by* $g(s)=g_{\mathrm{i}}(s)g_{\mathrm{o}}(s)$*, where*

$$g_{\mathrm{i}}(s) = \frac{d^+(s)}{\overline{d^+(-\overline{s})}}, \quad g_{\mathrm{o}}(s) = \frac{\overline{d^+(-\overline{s})}\cdot d^-(s)}{n(s)}.$$

This approach is called Hurwitz reflection.

(iv) *If* $E=I_n$ *and* $\operatorname{rank} D = m$*, then* $\mathcal{V}^{\mathrm{sys}}_{[E,A,B]} = \mathbb{K}^{n+m}$ *and we may choose* $Z=I_m$*. The Lur'e equation* (3.1) *can be reformulated as an algebraic Riccati equation*

$$A^*X + XA + C^*C - \big(XB + C^*D\big)\big(D^*D\big)^{-1}\big(XB+C^*D\big)^* = 0, \quad X = X^*$$

and we may choose $L=\big(D^*D\big)^{1/2}$ *and* $K = L^{-*}\big(B^*X + D^*C\big)$*. In this case we see that the realization of* $G_{\mathrm{i}}(s)$ *reduces to*

$$[I_n, A - BL^{-1}K, BL^{-1}, C - DL^{-1}K, DL^{-1}] \in \Sigma_{n,m,p}(\mathbb{K}).$$

This coincides with the realization obtained in [18, Sect. 13.7].

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