

# Some notes on port-Hamiltonian systems on Banach spaces

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**Abstract:** We consider port-Hamiltonian systems from a functional analytic perspective. Dirac structures and Hamiltonians on Banach spaces are introduced, and an energy balance is proven. Further, we consider port-Hamiltonian systems on Banach manifolds, and we present some physical examples that fit into the presented theory.

*Keywords:* port-Hamiltonian systems, partial differential-algebraic systems, Dirac structures, Banach manifold, infinite dimensional systems

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## 1. INTRODUCTION

The port-Hamiltonian approach to energy-based modeling has quite recently boomed in the mathematical community, see Jeltsema and van der Schaft (2014) for an introductory overview. In the author's opinion, this is due to its mathematical elegance and a nearly unlimited applicability in various (multi-)physical domains. The class of port-Hamiltonian systems moreover closed under power-conserving network interconnection, that is, the coupling of port-Hamiltonian systems again results into a port-Hamiltonian system, see Cervera et al. (2007). Hence this approach is in particular suitable for coupled systems. Recently, tremendous progress has been made in differential-algebraic port-Hamiltonian systems and therefore enables to also treat constrained multibody systems as well as electrical circuits, see Maschke and van der Schaft (2018); Mehl et al. (2018); Gernandt et al. (2020b). Beyond that, the physical principles which lead to port-Hamiltonian modelling hold all the more for systems which are spatially distributed. This leads to the urgent need of a theory for port-Hamiltonian systems governed by partial differential equations and partial differential-algebraic equations. Such a theory is presented in Maschke and van der Schaft (2002) from a differential geometric perspective by utilizing differential forms. On the other hand, functional analysis and operator theory are fruitful tools for treating systems governed by partial differential equations, in particular for the study of existence, regularity and asymptotics of the solutions. Note that, though a functional analytic approach to port-Hamiltonian systems to partial differential equations has been discussed in Jacob and Zwart (2012), the theory presented therein, however, restricts to the very limited class of coupled systems of linear transport equations.

The aim of this article is to give impulses for research on a functional analytic treatment. Besides presenting a definition and proving elementary results for port-Hamiltonian systems on Banach spaces, we discuss some examples which fit into this framework.

This article is organized as follows: In the subsequent Section 2, we present the definition of port-Hamiltonian systems after introducing Dirac structures, resistive relations and Hamiltonians. As an example, we show that Maxwell's equations with nonlinear material laws fit into this framework. In Section 3, we generalize the setup to Banach manifolds, which allows to treat practical examples from constrained continuum mechanics. In particular, we introduce the concept of *modulated Dirac structure* and, as an example, we present a non-elastic rope. In Section 4 we discuss some further possible, in particular in conjunction with gradient systems and Lagrangian manifolds, and we discuss some possible research topics like solvability and regularity.

Throughout this article, all considered spaces are real. The Euclidean norm on  $\mathbb{R}^n$  is denoted by  $\|\cdot\|_2$ , and  $X'$  stands for dual of a normed space  $X$ . We write  $\langle x, x' \rangle_{X, X'}$  or  $\langle x', x \rangle_{X', X}$  for the evaluation of  $x' \in X'$  at  $x \in X$ . If the spaces are clear from context, then we neglect the subindices indicating the spaces. The Cartesian product  $X_1 \times X_2$  of two normed spaces becomes a normed space by taking the norm  $\|(x_1, x_2)\|_{X_1 \times X_2} = \|(\|x_1\|_{X_1}, \|x_2\|_{X_2})\|_{\mathbb{R}^2}$ , where  $\|\cdot\|_{\mathbb{R}^2}$  is a norm on  $\mathbb{R}^2$ . In particular,  $X_1 \times X_2$  becomes a Hilbert space, if both  $X_1$  and  $X_2$  are Hilbert spaces, and  $\|\cdot\|_{\mathbb{R}^2} = \|\cdot\|_2$ . It follows that the dual of  $X_1 \times X_2$  is given by  $X_1' \times X_2'$  with  $\langle (x_1, x_2), (x_1', x_2') \rangle = \langle x_1, x_1' \rangle + \langle x_2, x_2' \rangle$ . Moreover, the symbol  $\mathcal{L}(X, Y)$  stands for the space of linear operators mapping from  $X$  to  $Y$ .

Lebesgue and first order Sobolev spaces of functions defined on  $\Omega \subseteq \mathbb{R}^m$  and with values in a Banach space  $X$  are respectively denoted by  $L^p(\Omega; X)$  and  $W^{1,p}(\Omega; X)$ , and we shortly write  $L^p(\Omega)$  and  $W^{1,p}(\Omega)$  when  $X = \mathbb{R}$ . Especially, when  $\Omega = \mathbb{I} \subset \mathbb{R}$  is an interval, we also consider the space  $L^p_{\text{loc}}(\mathbb{I}; X)$  which consists of all (equivalence classes of) functions  $f : \mathbb{I} \rightarrow X$  such that  $f \in L^p(\mathbb{K}; X)$  for all compact intervals  $\mathbb{K} \subseteq \mathbb{I}$ . Similarly, one defines  $W^{1,p}_{\text{loc}}(\mathbb{I}; X)$ .

## 2. DEFINITION OF PORT-HAMILTONIAN SYSTEMS ON BANACH SPACES

Port-Hamiltonian systems are composed of so-called *Dirac structures*, *resistive relations* and *Hamiltonians*, which are successively introduced in the sequel.

### 2.1 Dirac structures

An important concept is the Dirac structure which describes the power preserving energy-routing of the system.

*Definition 1.* (Dirac structure). *Let  $X$  be a Banach space. A subspace  $\mathcal{D} \subset X \times X'$  is called a Dirac structure, if for all  $f \in X$ ,  $e \in X'$ , it holds*

$$(f, e) \in \mathcal{D} \iff (\langle f, \hat{e} \rangle + \langle \hat{f}, e \rangle = 0 \text{ for all } (\hat{f}, \hat{e}) \in \mathcal{D}).$$

Hereby,  $X$  is called the space of flows, whereas  $X'$  is called the space of efforts.

Dirac structures on Hilbert spaces have been considered in Behrndt et al. (2010), and their structure has been analysed by using the theory of Krein spaces.

*Remark 1.* a) If  $J \in \mathcal{L}(X', X)$  be a skew-dual operator in the sense that it fulfills  $\langle Jv, w \rangle = -\langle Jw, v \rangle$  for all  $v, w \in X'$ , then, by using the Hahn-Banach theorem (Alt, 2016, Sec. 6.14), it can be shown that  $\mathcal{D} = \{(\mathcal{J}e, e) : e \in X'\}$  is a Dirac structure, see Reis and Stykel (2021). It is even more straightforward to prove that for  $J \in \mathcal{L}(X, X')$  fulfills  $\langle Jv, w \rangle = -\langle Jw, v \rangle$  for all  $v, w \in X$ , then the graph of  $J$ , i.e.,  $\mathcal{D} = \{(f, Jf) : f \in X\}$ . Note that this generalizes the well-known statement that graphs of skew-symmetric matrices define Dirac structures, see Maschke and van der Schaft (2018); Gernandt et al. (2020a).

- b) Dirac structures are closed subspaces of  $X \times X'$ , and they are therefore complete.
- c) Let  $\mathcal{D} \subset X \times X'$  be a Dirac structure. If  $X$  is reflexive, i.e., the canonical embedding  $X \rightarrow X''$ ,  $x \mapsto (x' \mapsto \langle x, x' \rangle)$  (which allows to regard  $X$  as a subspace of  $X''$ ) is surjective, then

$$\mathcal{D}_{\text{swap}} \subset X' \times X'' = \{(e, f) : (f, e) \in \mathcal{D}\}$$

is a Dirac structure as well. This is no longer true in general, if  $X$  is not reflexive. For instance, consider a non-reflexive space  $X$  and the Dirac structure  $\mathcal{D} = X \times \{0\}$ . Then  $\mathcal{D}_{\text{swap}} \subset X' \times X''$  is not a Dirac structure, since any  $x'' \in X'' \setminus X$  fulfills  $\langle 0, x \rangle + \langle 0, x'' \rangle = 0$  for all  $x \in X$ , but  $(x'', 0) \notin \mathcal{D} = X \times \{0\}$ . On the other hand, the Hahn-Banach theorem implies that - independent on reflexivity - for any Banach space  $X$  the swap of the Dirac structure  $\mathcal{D} = \{0\} \times X'$  is a Dirac structure as well.

### 2.2 Hamiltonians

Now we introduce functionals which express the energy storage in a system. We first review some needed differentiability and continuity concepts.

*Definition 2.* Let  $X, Y$  be Banach spaces,  $U \subset X$  be open. Then a function  $f : U \rightarrow Y$  is called

- a) *locally Lipschitz continuous*, if for all  $x \in U$ , there exists some neighborhood  $V$  and some  $L > 0$ , such that

$$\forall y, z \in U \cap V : \|f(y) - f(z)\| \leq L \|y - z\|.$$

- b) *Gâteaux differentiable* at  $x \in U$ , if the *Gâteaux derivative*

$$Df(x) = \left( y \mapsto \lim_{t \rightarrow 0} \frac{1}{t} (f(x + ty) - f(x)) \right) \in \mathcal{L}(X, Y)$$

exists.

- c) *Gâteaux differentiable*, if it is Gâteaux at any  $x_0 \in U$ .

The *Hamiltonian* is a real-valued mapping with the above properties.

*Definition 3.* (Hamiltonian). Let  $X$  be a Banach space and  $U \subset X$  be open. We call  $\mathcal{H} : U \rightarrow \mathbb{R}$  a *Hamiltonian*, if it is locally Lipschitz continuous and Gâteaux differentiable.

Note that the Gâteaux derivative of the Hamiltonian is mapping to the dual of  $X$ , i.e.,  $D\mathcal{H} : X \rightarrow X'$ . In our context, the most important property of the Hamiltonian is that it fulfills the weak form of the chain rule. Before presenting the result, we declare a manner of speaking: It follows from weak form of the fundamental theorem of calculus (Alt, 2016, E3.6) that for any  $x \in W_{\text{loc}}^{1,p}(\mathbb{I}; X)$  and almost all  $t_0, t_1 \in \mathbb{I}$  with  $t_0 \leq t_1$ , the integral of  $\frac{d}{dt}x(t)$  equals to the difference between  $x(t_1)$  and  $x(t_0)$ . Consequently,  $x$  possesses a continuous representative, which is moreover unique since the complements of null sets are dense. Hence, by writing  $x(t)$  for some  $x \in W_{\text{loc}}^{1,p}(\mathbb{I}; X)$  and  $t \in \mathbb{I}$ , we mean the evaluation of the continuous representative at  $t$ .

*Proposition 1.* Let  $X$  be a Banach space, let  $U \subset X$  be open, let  $\mathcal{H} : U \rightarrow \mathbb{R}$  be a Hamiltonian and, for some interval  $I \subset \mathbb{R}$  and  $p \in [1, \infty]$ , let  $x \in W_{\text{loc}}^{1,p}(\mathbb{I}; X)$  with  $x(t) \in U$  for all  $t \in \mathbb{I}$ . Then the mapping  $\mathcal{H} \circ x : t \mapsto \mathcal{H}(x(t))$  is in  $W_{\text{loc}}^{1,p}(\mathbb{I})$ . In particular, the weak derivative of  $\mathcal{H} \circ x$  fulfills the *weak chain rule*

$$\frac{d}{dt} \mathcal{H}(x(t)) = \left\langle \frac{d}{dt} x, D\mathcal{H}(x) \right\rangle \text{ for almost all } t \in I.$$

Before presenting the proof, we note that “ $x(t) \in U$  for all  $t \in \mathbb{I}$ ” means that the continuous representative of  $x$  has this property. In case of compact  $\mathbb{I}$ , this is equivalent to the existence of a neighborhood  $V \subset U$  of the trace of  $x$ . Be aware that “ $x(t) \in U$  for almost all  $t \in \mathbb{I}$ ” does not guarantee the result since the trace of  $x$  may kiss the boundary of  $U$ .

**Proof.** The result for  $U = X$  and  $\mathcal{H}$  Lipschitz continuous (that is, the constant  $L$  in Definition 2 does not depend on  $x$ ) has been shown in (Arendt and Kreuter, 2018, Theorem 4.2). A careful inspection of the proof yields that this statement also holds, if  $\mathcal{H}$  is Gâteaux differentiable and Lipschitz continuous on some open subset  $U \subset X$ , and  $x \in W_{\text{loc}}^{1,p}(\mathbb{I}; X)$  fulfills  $x(t) \in U$  for all  $t \in \mathbb{I}$ .

Now we show that the result also holds in the case where  $\mathcal{H}$  is Gâteaux differentiable and locally Lipschitz continuous: First note that - by restricting to a suitable subinterval - it is no loss to assume that  $\mathbb{I}$  is compact. Then, by continuity of  $x$ , the trace of  $x$ , namely  $\text{tr } x := \{x(t) : t \in \mathbb{I}\}$ , is compact. Then, by a covering argument, there exists some open set  $V \subset U$  with  $\text{tr } x \subset V$ , such that the restriction of  $\mathcal{H}$  to  $V$  is Lipschitz continuous. Then the result follows from the argumentation at the beginning of this proof.

*Remark 2.* a) If  $H \in \mathcal{L}(X, X')$  is self-dual in the sense that it fulfills  $\langle Hv, w \rangle = \langle Hw, v \rangle$  for all  $v, w \in X$ , then  $\mathcal{H} : X \rightarrow \mathbb{R}$ ,  $x \mapsto \frac{1}{2} \langle x, Hx \rangle$  is a Hamiltonian. In particular, the Gâteaux derivative reads  $D\mathcal{H}(x) = Hx$  for all  $x \in X$ .

b) Assume that  $\mathcal{H} : U \rightarrow \mathbb{R}$  is Gâteaux differentiable, where  $U$  is an open subset of a Banach space  $X$ . Consider  $x, y \in U$  such that all convex combinations of  $x$  and  $y$  are still in  $U$ . The chain rule applied to the function  $[0, 1] \rightarrow \mathbb{R}$ ,  $t \mapsto \mathcal{H}(tx + (1-t)y)$  implies that

$$\mathcal{H}(y) - \mathcal{H}(x) = \int_0^1 \langle (x-y), D\mathcal{H}(tx + (1-t)y) \rangle dt.$$

Consequently, Lipschitz continuity of  $\mathcal{H}$  is guaranteed, if  $D\mathcal{H} : U \rightarrow X'$  is *locally bounded*, that is, for all  $x \in U$  there exists some neighborhood  $V \subset U$  of  $x$  such that the restriction of  $\mathcal{H}$  to  $V$  is bounded. Local boundedness of  $D\mathcal{H}$  is for instance guaranteed, if  $D\mathcal{H} : U \rightarrow X'$  is a continuous function. Note that the latter implies the even stronger concept of *Fréchet differentiability* (Zeidler, 1986a, §4.2).

### 2.3 Resistive relations

Another ingredient for port-Hamiltonian systems are *resistive relations*, which are defined below.

*Definition 4.* (Resistive relation). *Let  $X$  be a Banach space. A relation  $\mathcal{R} \subset X \times X'$  is called resistive, if*

$$\langle f, e \rangle \leq 0 \text{ for all } (f, e) \in \mathcal{R}.$$

*Remark 3.* a) Assume that  $\mathcal{R} \subset X \times X'$  is a resistive relation. By using the canonical embedding of  $X$  into its bidual, we see that

$$\mathcal{R}_{\text{swap}} \subset X' \times X'' = \{(e, f) : (f, e) \in \mathcal{R}\}$$

is again a resistive relation.

- b) The closure of a resistive relation is again a resistive relation. Any subset of a resistive relation is a resistive relation.
- c) If the mapping  $R : X \rightarrow X'$  is *dissipative*, i.e.,  $\langle x, R(x) \rangle \leq 0$ , then the graph of  $R$ , i.e.,  $\mathcal{R} = \{(f, R(f)) : f \in X\}$  is a resistive relation. Likewise, if  $G : X' \rightarrow X$  is dissipative in the sense that  $\langle x', G(x') \rangle \leq 0$  for all  $x' \in X'$ , then the relation  $\mathcal{R} = \{(G(e), e) : e \in X'\}$  is resistive.

### 2.4 Port-Hamiltonian systems

Having defined Dirac structures, Hamiltonians and resistive relations, we are now ready to introduce port-Hamiltonian systems.

*Definition 5.* (Port-Hamiltonian system). Let  $X_S, X_R$  and  $X_P$  be Banach spaces. A *port-Hamiltonian system* is a triple  $(\mathcal{D}, \mathcal{H}, \mathcal{R})$ , where  $\mathcal{D} \subset (X_S \times X_R \times X_P) \times (X'_S \times X'_R \times X'_P)$  is a Dirac structure,  $\mathcal{H} : U \rightarrow \mathbb{R}$  (with  $U \subset X_S$  open) is a Hamiltonian, and  $\mathcal{R} \subset X_R \times X'_R$  is a resistive relation. The *behavior* of the port-Hamiltonian system on an interval  $\mathbb{I} \subset \mathbb{R}$  is consisting of all  $(x, f_R, f_P, e_R, e_P)$  with  $x \in W_{\text{loc}}^{1,2}(\mathbb{I}; X_S)$ , and  $x(t) \in U$  for all  $t \in \mathbb{I}$ ,  $(f_R, e_R) \in L_{\text{loc}}^2(\mathbb{I}; X_R \times X'_R)$ ,  $(f_P, e_P) \in L_{\text{loc}}^2(\mathbb{I}; X_P \times X'_P)$  that fulfill the differential inclusion

for almost all  $t \in \mathbb{I}$ :

$$\begin{aligned} \left(-\frac{d}{dt}x(t), f_R(t), f_P(t), D\mathcal{H}(x(t)), e_R(t), e_P(t)\right) &\in \mathcal{D}, \\ (f_R(t), e_R(t)) &\in \mathcal{R}. \end{aligned} \tag{1}$$

The object  $x(t)$  is called *state at  $t \in \mathbb{I}$* . The spaces  $X_S \times X'_S$ ,  $X_R \times X'_R$  and  $X_P \times X'_P$  are referred to as *energy storage port*, *resistive port* and *external port*, respectively.

The definition of the Dirac structure together with Proposition 1 yields that any element  $(x, f_R, f_P, e_R, e_P)$  of the behavior of a port-Hamiltonian system on the interval  $\mathbb{I} \subset \mathbb{R}$  fulfills

$$\begin{aligned} 0 &= -\left\langle \frac{d}{dt}x(t), D\mathcal{H}(x(t)) \right\rangle + \langle f_R(t), e_R(t) \rangle + \langle f_P(t), e_P(t) \rangle \\ &= -\frac{d}{dt}\mathcal{H}(x(t)) + \langle f_R(t), e_R(t) \rangle + \langle f_P(t), e_P(t) \rangle \\ &\leq -\frac{d}{dt}\mathcal{H}(x(t)) + \langle f_P(t), e_P(t) \rangle. \end{aligned}$$

Hence, an integration on  $[t_0, t_1] \subset \mathbb{I}$  yields

$$\begin{aligned} &\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \\ &= \int_{t_0}^{t_1} \langle f_R(t), e_R(t) \rangle dt + \int_{t_0}^{t_1} \langle f_P(t), e_P(t) \rangle dt \\ &\leq \int_{t_0}^{t_1} \langle f_P(t), e_P(t) \rangle dt. \end{aligned} \tag{2}$$

In practical situations, this is an energy balance: The expression  $\mathcal{H}(x(t))$  stands for the energy of the system at time  $t \in \mathbb{I}$ , whereas  $\int_{t_0}^{t_1} \langle f_P(t), e_P(t) \rangle dt$  is the energy which is put into the system, and  $\int_{t_0}^{t_1} \langle f_R(t), e_R(t) \rangle dt$  stands for the energy that is dissipated from the system in the interval  $[t_0, t_1]$ .

### 2.5 Example: An eddy current model

Eddy current models occur as a simplification of Maxwell's equations in which the contribution of the dynamics of the electric field is small compared to the dynamics of the magnetic field. For an interval  $\mathbb{I}$ , we consider the functions  $\mathbf{D}, \mathbf{B}, \mathbf{E}, \mathbf{H}, \mathbf{J} : \mathbb{R}^3 \times \mathbb{I} \rightarrow \mathbb{R}^3$  which are referred to as *electric displacement*, *magnetic flux intensity*, *electric field intensity*, *magnetic field intensity* and *electric current density*. Assuming that there are no electric charges, Maxwell's equations are given by

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial}{\partial t} \mathbf{B} & \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D}, \end{aligned}$$

where  $\nabla \cdot$  stands for the divergence and  $\nabla \times$  denotes the curl of a vector field with respect to the spatial variable  $\xi \in \mathbb{R}^3$ . The simplification caused by staticity of the electric field means that the time derivative of the electric displacement vanishes, i.e.,  $\frac{\partial}{\partial t} \mathbf{D} = 0$ , see e.g. Cortes Garcia et al. (2018); Chill et al. (2021). In addition, the above variables fulfill *constitutive relations*, which are determined by the physical properties of the medium. The constitutive relations are, in the quasilinear and isotropic case, of the form

$$\begin{aligned} \mathbf{H}(\xi, t) &= \nu(\xi, \|\mathbf{B}(\xi, t)\|_2) \mathbf{B}(\xi, t), \\ \mathbf{J}(\xi, t) &= \sigma(\xi, \|\mathbf{E}(\xi, t)\|_2) \mathbf{E}(\xi, t) + \mathbf{J}_{\text{ext}}(\xi, t) \end{aligned}$$

for some measurable and bounded nonnegative functions  $\nu, \sigma : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  which respectively express the electric permittivity, magnetic reluctivity and electric conductivity of the material, and  $\mathbf{J}_{\text{ext}}$  stands for the externally

injected currents. We assume that the latter is induced by a winding

$$\mathbf{J}_{\text{ext}}(\xi, t) = \chi(\xi) \mathbf{i}(t), \quad (3)$$

where  $\mathbf{i} : \mathbb{I} \rightarrow \mathbb{R}$  is the injected current, and the divergence-free function  $\chi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  expresses the geometry of the winding. The voltage at the winding is defined by

$$\mathbf{u}(t) = \int_{\mathbb{R}^3} \chi^\top(\xi) \mathbf{E}(\xi, t) d\xi. \quad (4)$$

For the magnetic reluctivity  $\nu$ , we consider the function  $\vartheta : \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  given by

$$\vartheta(\xi, \varrho) = \frac{1}{2} \int_0^\varrho \nu(\xi, \sqrt{\zeta}) d\zeta = \int_0^{\sqrt{\varrho}} \nu(\xi, \zeta) \zeta d\zeta, \quad (5)$$

which stands for the magnetic energy density. Consequently, the magnetic energy is the spatial integral of the magnetic energy density. That is, we consider the functional

$$\mathcal{H}(\mathbf{B}) = \int_{\mathbb{R}^3} \vartheta(\xi, \|\mathbf{B}(\xi)\|_2^2) d\xi \quad (6)$$

on a space which has to be further specified in the following. First note that the boundedness and measurability of the magnetic reluctivity implies that  $\mathcal{H}(\mathbf{B})$  is well-defined for any  $\mathbf{B} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ .

Now consider the space  $L^2(\text{div}=0)$  consisting of all square integrable functions whose distributional divergence vanishes, where we identify functions which coincide on sets of measure zero. Note that  $L^2(\text{div}=0)$  is a closed subspace of the space  $L^2$  of all square-integrable  $\mathbb{R}^3$ -valued functions on  $\mathbb{R}^3$ , and it is thus a Hilbert space provided with respect to the natural inner product on  $L^2$ . Further, we set

$$H(\text{curl}, \text{div}=0) = \{\mathbf{A} \in L^2(\text{div}=0), \text{curl } \mathbf{A} \in L^2(\text{div}=0)\},$$

where  $\text{curl } \mathbf{A}$  stands for the distributional curl of  $\mathbf{A} \in L^2(\text{div}=0)$ . This space is again a Hilbert space, now provided with the inner product which is given by the sum of the  $L^2$ -inner product of the functions and  $L^2$ -inner product of the curl of the given functions. To this end, note that  $\text{curl } \mathbf{A} \in L^2(\text{div}=0)$  is fulfilled if, and only if,  $\text{curl } \mathbf{A}$  is a square integrable function.

Next we formulate our eddy current model as a port-Hamiltonian system. Hereby, the external port is supposed to be consisting of the injected current  $\mathbf{i}$  together with the voltage  $\mathbf{u}$  at the winding. More precisely, we set  $X_{\mathcal{P}} = \mathbb{R}$  and  $e_{\mathcal{P}}(t) = \mathbf{u}(t)$ ,  $f_{\mathcal{P}}(t) = \mathbf{i}(t)$ . Furthermore, we consider the space  $X_{\mathcal{S}} = L^2(\text{div}=0)$  and  $X_{\mathcal{R}} = H(\text{curl}, \text{div}=0)'$ , where the latter is given by the dual of  $H(\text{curl}, \text{div}=0)$  with respect to the pivot space  $L^2(\text{div}=0)$ . Note that, by using the integration by parts formula for the curl operator, curl extends to an operator from  $L^2(\text{div}=0)$  to  $H(\text{curl}, \text{div}=0)'$  via

$$\text{curl } \mathbf{B} = \left( \mathbf{F} \mapsto \int_{\mathbb{R}^3} (\text{curl } \mathbf{F})^\top \mathbf{B} d\xi \right). \quad (7)$$

Since Hilbert spaces are reflexive, and the dual space of  $\mathbb{R}$  can be canonically identified with itself, we have  $X_{\mathcal{S}}' = L^2(\text{div}=0)$ ,  $X_{\mathcal{R}}' = H(\text{curl}, \text{div}=0)$  and  $X_{\mathcal{P}}' = \mathbb{R}$ . By setting  $X = X_{\mathcal{S}} \times X_{\mathcal{R}} \times X_{\mathcal{P}}$ , we consider the set

$$\mathcal{D} = \left\{ \left( \begin{pmatrix} f_{\mathcal{S}} \\ f_{\mathcal{R}} \\ f_{\mathcal{P}} \end{pmatrix}, \begin{pmatrix} e_{\mathcal{S}} \\ e_{\mathcal{R}} \\ e_{\mathcal{P}} \end{pmatrix} \right) \in X \times X' : f_{\mathcal{S}} = \text{curl } e_{\mathcal{R}}, \right. \\ \left. f_{\mathcal{R}} = -\text{curl } e_{\mathcal{S}} - \chi e_{\mathcal{P}}, f_{\mathcal{P}} = \int_{\mathbb{R}^3} \chi^\top e_{\mathcal{R}} d\xi \right\}.$$

It can be seen that this set is of the form  $\mathcal{D} = \{(\mathcal{J}e, e) : e \in X'\}$  for some skew-dual operator  $\mathcal{J} \in \mathcal{L}(X', X)$ , whence we obtain from Remark 1a) that  $\mathcal{D}$  is a Dirac structure. It can be further seen that the nonnegativity of the electric conductivity implies that

$$\mathcal{R} = \{(f_{\mathcal{R}}, e_{\mathcal{R}}) \in X_{\mathcal{R}} \times X_{\mathcal{R}}' : \\ f_{\mathcal{R}}(\xi) = -\sigma(\xi, \|e_{\mathcal{R}}(\xi)\|_2) e_{\mathcal{R}}(\xi) \forall \xi \in \mathbb{R}^3\}.$$

is a resistive structure. As Hamiltonian, we take  $\mathcal{H} : X_{\mathcal{S}} = L^2(\text{div}=0) \rightarrow \mathbb{R}$  defined by the relation (6). A straightforward calculation implies that the Gâteaux derivative fulfills

$$D\mathcal{H}(\mathbf{B}) = \nu(\cdot, \|\mathbf{B}(\cdot)\|_2) \mathbf{B}, \quad (8)$$

where, for some function  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $\|\mathbf{B}(\cdot)\|_2$  stands for the pointwise evaluation of the Euclidean norm of  $\mathbf{B}$ . Next, we show that the port-Hamiltonian system  $(\mathcal{D}, \mathcal{H}, \mathcal{R})$  indeed represents the previously introduced eddy current model. Denote the state  $x(t)$  by  $\mathbf{B}(t) \in L^2(\text{div}=0)$ , and  $f_{\mathcal{R}} = \mathbf{J}$ ,  $e_{\mathcal{R}} = \mathbf{E}$ ,  $f_{\mathcal{P}} = \mathbf{u}$  and  $e_{\mathcal{P}} = \mathbf{i}$ . Then

$$\left( -\frac{d}{dt} \mathbf{B}(t), \mathbf{J}(t), \mathbf{v}(t), D\mathcal{H}(\mathbf{B}(t)), \mathbf{E}(t), \mathbf{i}(t) \right) \in \mathcal{D}, \\ (\mathbf{J}, \mathbf{E}(t)) \in \mathcal{R}. \quad (9)$$

By using the definition of  $\mathcal{D}$  and  $\mathcal{R}$  and the representation (8) of the Gâteaux derivative of the Hamiltonian, we obtain

$$-\frac{d}{dt} \mathbf{B}(t) = \text{curl } \mathbf{E}(t), \\ -\sigma(\cdot, \|\mathbf{E}(\cdot, t)\|_2) \mathbf{E}(t) = -\text{curl}(\nu(\cdot, \|\mathbf{B}(\cdot, t)\|_2) \mathbf{B}(t)) \\ - \chi \mathbf{i}_{\text{wind}}(t), \\ \mathbf{u}(t) = \int_{\mathbb{R}^3} \chi^\top(\xi) \mathbf{E}(\xi, t) d\xi,$$

which is exactly the previously introduced eddy current model. Note that the second relation in an equality in  $H(\text{curl}, \text{div}=0)'$ , and the identity (7) for the extension of the curl operator to  $L^2(\mathbb{R}^3; \mathbb{R}^3)$  means that it is equivalent to

$$\forall \mathbf{F} \in H(\text{curl}, \text{div}=0) :$$

$$\int_{\mathbb{R}^3} \mathbf{F}(\xi)^\top (\sigma(\xi, \|\mathbf{E}(\xi, t)\|_2) \mathbf{E}(\xi, t) + \chi(\xi) \mathbf{i}_{\text{wind}}(t)) \\ + (\text{curl } \mathbf{F}(\xi))^\top (\nu(\xi, \|\mathbf{B}(\xi, t)\|_2) \mathbf{B}) + \mathbf{F}(\xi)^\top d\xi = 0,$$

which indeed corresponds to a weak formulation.

### 3. PORT-HAMILTONIAN SYSTEMS ON BANACH MANIFOLDS

The Banach space setup is oftentimes not capable for physical systems, in particular for those involving ideal constraints. Instead, the state evolves in a *manifold*, i.e., a topological spaces which has at least locally the structure of a Banach space.

*Definition 6.* (Banach manifold). Let  $\mathcal{M}$  be a topological space. An *atlas of class  $C^1$*  on  $\mathcal{M}$  is a family of pairs (called *charts*)  $(U_i, \varphi_i)_{i \in I}$ , such that

- $U_i \subset \mathcal{M}$  for all  $i \in I$ , and  $\bigcup_{i \in I} U_i = \mathcal{M}$ ;

- $\varphi_i$  is a homeomorphism from  $U_i$  onto an open subset  $\varphi_i(U_i)$  of some Banach space  $X_i$ , and for any  $i, j \in I$ , the *crossover map*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$$

is continuously differentiable, i.e., its Fréchet derivative exists and is a continuous function with respect to the operator norm topology on  $\mathcal{L}(X_i, X_j)$ .

Two atlases  $(U_i, \varphi_i)_{i \in I}$  and  $(V_j, \psi_j)_{j \in J}$  are called *compatible*, if for all  $i \in I, j \in J$ , the map

$$\psi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \psi_j(U_i \cap U_j)$$

is continuously differentiable (note that compatibility defines an equivalence relation on the set of atlases of class  $C^1$  on  $\mathcal{M}$ ).

A *differentiable manifold structure on  $\mathcal{M}$* , is an equivalence class of compatible atlases of class  $C^1$  on  $\mathcal{M}$ .

We refer to (Zeidler, 1986b, Chapter 73) for further details on Banach manifolds. For sake of brevity, we will call  $\mathcal{M}$  itself a differentiable manifold instead of the equivalence class of compatible atlases. Next we introduce the tangent space. Hereby we use the concept of *differentiable curve*  $x : (-1, 1) \rightarrow \mathcal{M}$ , which is defined to be a continuous mapping with the property that for all charts  $(U, \varphi)$ , it holds that  $\varphi \circ \gamma : \gamma^{-1}(U) \rightarrow \varphi(U)$  is continuously differentiable.

*Definition 7.* (Tangent space). Let  $\mathcal{M}$  be a differentiable manifold and let  $x_0 \in \mathcal{M}$ . Two differentiable curves  $x_1, x_2 : (-1, 1) \rightarrow \mathcal{M}$  with  $x_1(0) = x_2(0) = x_0$  are called *equivalent at  $x_0$* , if for some (and hence any) chart  $(U, \varphi)$  with  $x_0 \in U$ , the derivative fulfills  $\frac{d}{dt}(\varphi \circ x_1)(0) = \frac{d}{dt}(\varphi \circ x_2)(0)$ . The set of all equivalent curves at  $x_0$  is called a *tangent vector*. The set of tangent vectors is called *tangent space at  $x \in \mathcal{M}$* , and will be denoted by  $TM_{x_0}$ .

The definition of the tangent space allows to write  $\frac{d}{dt}x(t) \in TM_{x(t)}$  for all  $t \in \mathbb{I}$  and all differentiable curves  $x : \mathcal{I} \rightarrow \mathcal{M}$ . Addition and scalar multiplication on the set of tangent vectors is well-defined by addition and scalar multiplication in the space defined by the vectors  $\frac{d}{dt}(\varphi \circ x)(0)$ . Further, note that, for two charts  $(U_1, \varphi_1), (U_2, \varphi_2)$ , the norms  $\|\cdot\|_i : [x] \mapsto \|\frac{d}{dt}(\varphi_i \circ x)(0)\|$ ,  $i = 1, 2$  are equivalent, where  $[x]$  stands for the class of all curves being equivalent to  $x$  at  $x_0 \in \mathcal{M}$ . Hence, the tangent space is a topological vector space with topology induced by a norm. As a consequence, the *cotangent space  $TM'_x$  at  $x \in \mathcal{M}$*  is well-defined by the dual of the tangent space at  $x \in \mathcal{M}$ .

The previously introduced concepts allow to introduce Dirac structures on Banach manifolds.

*Definition 8.* (Modulated Dirac structure). Let  $\mathcal{M}$  be a Banach manifold. A *modulated Dirac structure on  $\mathcal{M}$*  is a family  $(\mathcal{D}_x)_{x \in \mathcal{M}}$ , where for each  $x \in \mathcal{M}$ ,  $\mathcal{D}_x$  is a Dirac structure with  $\mathcal{D}_x \subset TM_x \times TM'_x$ .

*Definition 9.* (Hamiltonian on Banach manifold). Let  $\mathcal{M}$  be a Banach manifold. We call  $\mathcal{H} : \mathcal{M} \rightarrow \mathbb{R}$  a *Hamiltonian*, if is locally Lipschitz continuous and Gâteaux differentiable. That is, for any chart  $(\varphi, U)$  of  $\mathcal{M}$ , the mapping  $\mathcal{H} \circ \varphi^{-1}$  is locally Lipschitz continuous and Gâteaux differentiable.

Let  $\mathcal{H} : \mathcal{M} \rightarrow \mathbb{R}$  be a Hamiltonian. Then for any chart  $(\varphi, U)$  of  $\mathcal{M}$ , we have that the mapping  $\mathcal{H} \circ \varphi^{-1}$  is locally Lipschitz continuous and Gâteaux differentiable. Furthermore, for any differentiable curve  $x : I \rightarrow \mathcal{M}$  and any chart  $(\varphi, U)$  of  $\mathcal{M}$ , Proposition 1 implies that the derivative of the function  $t \mapsto \mathcal{H}(x(t))$  fulfills

$$\begin{aligned} \forall x^{-1}(U) \subset I : \quad \frac{d}{dt}\mathcal{H}(x(t)) &= \frac{d}{dt}(\mathcal{H} \circ \varphi^{-1}) \circ (\varphi \circ x(t)) \\ &= \langle D(\mathcal{H} \circ \varphi^{-1})(x(t)), \frac{d}{dt}\varphi \circ x(t) \rangle. \end{aligned}$$

The definition of the tangential space now leads to the fact that the Gâteaux derivative  $D\mathcal{H} : \mathcal{M} \rightarrow TM'_x$  of  $\mathcal{H}$  at  $x \in \mathcal{M}$  is well- and completely defined by

$$\begin{aligned} \langle D\mathcal{H}(x(t)), \frac{d}{dt}x(t) \rangle &:= \frac{d}{dt}(\mathcal{H} \circ \varphi^{-1}) \circ (\varphi \circ x(t)) \\ &= \langle D(\mathcal{H} \circ \varphi^{-1})(x(t)), \frac{d}{dt}\varphi \circ x(t) \rangle \end{aligned}$$

for some (and hence any) chart  $(\varphi, U)$  with  $x(t) \in U$ . Combining the previous identity with Proposition 1, we see that this also holds, if the curve  $x$  is only weakly differentiable.

Next we introduce port-Hamiltonian systems involving Banach manifolds. To this end, we note that the Cartesian product of Banach manifolds is canonically again a Banach manifold. Further, for a Banach manifold  $\mathcal{M}$ , we say that  $x : \mathbb{I} \rightarrow \mathcal{M}$  is in  $W_{loc}^{1,2}(\mathbb{I}; \mathcal{M})$ , if for any  $t_0 \in \mathbb{I}$ , there exists some chart  $(U, \varphi)$  for  $\varphi^{-1}(U) \subset X$  with some Banach space  $X$ , and some relative neighborhood  $\mathbb{J} \subset \mathbb{I}$ , such that  $x(\mathbb{J}) \subset U$ , and  $\varphi^{-1} \circ x|_{\mathbb{J}} \in W_{loc}^{1,2}(\mathbb{J}; X)$ .

*Definition 10.* Let  $\mathcal{M}$  be a Banach manifold, and let  $X_{\mathcal{R}}$  and  $X_{\mathcal{P}}$  be Banach spaces. A *port-Hamiltonian system* is a triple  $(\mathcal{D}, \mathcal{H}, \mathcal{R})$ , where  $\mathcal{D} = (\mathcal{D}_z)_{z \in \mathcal{M} \times X_{\mathcal{R}} \times X_{\mathcal{P}}}$  is a modulated Dirac structure on  $\mathcal{M} \times X_{\mathcal{R}} \times X_{\mathcal{P}}$ ,  $\mathcal{H} : \mathcal{M} \rightarrow \mathbb{R}$  is a Hamiltonian, and  $\mathcal{R} \subset X_{\mathcal{R}} \times X'_{\mathcal{R}}$  is a resistive relation. The *behavior* of the port-Hamiltonian system on an interval  $\mathbb{I} \subset \mathbb{R}$  is consisting of all  $(x, f_{\mathcal{R}}, f_{\mathcal{P}}, e_{\mathcal{R}}, e_{\mathcal{P}})$  with  $x \in W_{loc}^{1,2}(\mathbb{I}; \mathcal{M})$ ,  $(f_{\mathcal{R}}, e_{\mathcal{R}}) \in L_{loc}^2(\mathbb{I}; X_{\mathcal{R}} \times X'_{\mathcal{R}})$ ,  $(f_{\mathcal{P}}, e'_{\mathcal{P}}) \in L_{loc}^2(\mathbb{I}; X_{\mathcal{P}} \times X'_{\mathcal{P}})$  that fulfill the differential inclusion

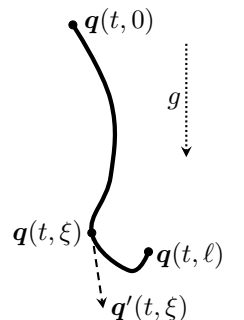
for almost all  $t \in \mathbb{I}$  :

$$\begin{aligned} \left(-\frac{d}{dt}x(t), f_{\mathcal{R}}(t), f_{\mathcal{P}}(t), D\mathcal{H}(x(t)), e_{\mathcal{R}}(t), e_{\mathcal{P}}(t)\right) &\in \mathcal{D}_{x(t)}, \\ (f_{\mathcal{R}}(t), e_{\mathcal{R}}(t)) &\in \mathcal{R}. \end{aligned}$$

The notions of state, energy storage port, resistive port and external port are used as for conventional port-Hamiltonian systems. The energy balance (2) can be inferred by using the definition of the Gâteaux derivative of Hamiltonians on Banach manifolds and the argumentation as in Section 2.4.

### 3.1 Example: A (not necessarily heavy) rope

Consider a non-elastic and undamped rope of fixed length  $\ell$ . The rope at time  $t$  is described by a curve  $\mathbf{q}(\cdot, t) : [0, \ell] \rightarrow \mathbb{R}^2$ ,  $\xi \mapsto \mathbf{q}(\xi, t)$ , which is parameterized by arc length, that is  $\|\mathbf{q}'(\xi, t)\|_2 = 1$  for all  $\xi \in [0, \ell]$ , where the prime stands for the derivative with respect to  $\xi$ . The rope has a mass density  $\rho : [0, \ell] \rightarrow \mathbb{R}_{>0}$  per unit length, which is further assumed to be measurable, bounded, and bounded from below by some positive constant.



The acceleration due to gravity is assumed to be  $\begin{pmatrix} 0 \\ g \end{pmatrix}$ , whence the potential energy of the rope is given by  $E_{\text{pot}}(\mathbf{q}(t)) = \int_0^\ell \rho(\xi) \mathbf{q}(\xi, t)^\top \begin{pmatrix} 0 \\ g \end{pmatrix} d\xi$ . The kinetic energy reads  $E_{\text{kin}}(\frac{d}{dt} \mathbf{q}(t)) = \frac{1}{2} \int_0^\ell \rho(\xi) \|\frac{d}{dt} \mathbf{q}(\xi, t)\|_2^2 d\xi$ . By using the Lagrangian function (see Jeltsema and van der Schaft (2014)),

$$\begin{aligned} L(\mathbf{q}(t), \frac{d}{dt} \mathbf{q}(t)) \\ = E_{\text{pot}}(\mathbf{q}(t)) - E_{\text{kin}}(\frac{d}{dt} \mathbf{q}(t)) + \int_0^\ell \lambda(\xi, t) (\|\mathbf{q}'(\xi, t)\|_2^2 - 1), \end{aligned}$$

the Lagrange formalism leads to the partial differential-algebraic equation

$$\begin{aligned} \frac{d}{dt} \mathbf{q} &= \frac{1}{\rho} \mathbf{p}, \\ \frac{d}{dt} \mathbf{p} &= (\lambda \mathbf{q}')' - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rho g, \\ 0 &= \|\mathbf{q}'\|_2^2 - 1, \end{aligned} \quad (10)$$

where the  $\mathbb{R}^2$ -valued  $\mathbf{p}$  stands for the infinitesimal momentum at the rope.

Our aim is now to embed this model into the port-Hamiltonian framework: Due to the constraint  $\|\mathbf{q}'\|_2^2 = 1$ , we choose the manifold

$$\mathcal{M}^{\text{pos}} = \{x \in W^{1,\infty}([0, \ell]; \mathbb{R}^2) : \|\mathbf{q}'(\xi)\|_2^2 = \ell \forall \xi \in [0, 1]\}$$

for the positions  $\mathbf{q}$  of the rope. It can be seen that the mapping

$$\begin{aligned} \mathcal{F} : \mathbb{R}^2 \times L^\infty([0, \ell]) &\rightarrow \mathcal{M}^{\text{pos}}, \\ (\mathbf{q}_0, \vartheta) &\mapsto \mathbf{q}_0 + \int_0^\xi \begin{pmatrix} \cos(\vartheta(\zeta)) \\ \sin(\vartheta(\zeta)) \end{pmatrix} d\zeta \end{aligned}$$

is onto and continuous. Denoting the  $r$ -ball centered in  $\mathbf{q}$  by  $U_r(\mathbf{q})$ , an atlas on  $\mathcal{M}^{\text{pos}}$  is given by the family

$$(\varphi_{(\mathbf{q}_0, \vartheta)}, U_{(\mathbf{q}_0, \vartheta)}) \in \mathbb{R}^2 \times L^\infty([0, \ell])$$

with

$$\begin{aligned} \varphi_{(\mathbf{q}_0, \vartheta)} &= \left( \mathcal{F}|_{U_1(\mathbf{q}_0) \times U_\pi(\vartheta)} \right)^{-1}, \\ U_{(\mathbf{q}_0, \vartheta)} &= \mathcal{F}(U_1(\mathbf{q}_0) \times U_\pi(\vartheta)). \end{aligned}$$

Since the crossover map between two charts is simply the identity, we further see that  $\mathcal{M}^{\text{pos}}$  (and thus also  $\mathcal{M}$ ) is a differentiable Banach manifold. Note that  $\mathcal{M}$  equipped with the relative topology in  $W^{1,p}([0, \ell]; \mathbb{R}^2)$  for  $p < \infty$  is not a Banach manifold, since there do not exist any charts due to possible unboundedness of the weak derivative of functions in  $W^{1,p}([0, \ell]; \mathbb{R}^2)$ .

By using that  $\mathcal{M}^{\text{pos}}$  is embedded in the Banach space  $W^{1,\infty}([0, \ell]; \mathbb{R}^2)$ , we can directly identify the tangential vectors in  $\mathbf{q} \in \mathcal{M}^{\text{pos}}$  by the derivatives  $\frac{d}{dt} \mathbf{q}(0)$  of the differentiable curves in  $W^{1,\infty}([0, \ell]; \mathbb{R}^2)$  with  $\mathbf{q}(t) \in \mathcal{M}^{\text{pos}}$  for all  $t$  in some neighborhood of zero. Note that, by formally differentiating this constraint we obtain that any curve  $\mathbf{q} : \mathbb{I} \rightarrow \mathcal{M}^{\text{pos}}$  fulfills  $(\mathbf{q}'(\xi, t))^\top \frac{d}{dt} \mathbf{q}(\xi, t) = 0$  for all  $\xi \in [0, \ell]$ . Indeed, it can be shown that

$$\begin{aligned} T\mathcal{M}_{\mathbf{q}}^{\text{pos}} &= \{z \in W^{1,\infty}([0, \ell]; \mathbb{R}^2) : \\ &(\mathbf{q}'(\xi))^\top z'(\xi) = 0 \forall \xi \in [0, \ell]\}. \end{aligned}$$

By  $\frac{d}{dt} \mathbf{q} = \frac{1}{\rho} \mathbf{p}$ , we see that the momentum fulfills  $\mathbf{p}(t) \in T\mathcal{M}_{\mathbf{q}(t)}^{\text{pos}}$ . Hence, we have  $x(t) := (\mathbf{q}(t), \mathbf{p}(t)) \in \mathcal{M}$  with

$$\mathcal{M} = \{(\mathbf{q}, \mathbf{p}) \in \mathcal{M} \times W^{1,\infty}([0, \ell]; \mathbb{R}^2) : \mathbf{p} \in T\mathcal{M}_{\mathbf{q}}^{\text{pos}}\},$$

which is also known as the *tangent bundle* of  $\mathcal{M}^{\text{pos}}$ . Since the tangent bundle is itself a Banach manifold by (Zeidler,

1986b, Prop. 73.17), we obtain that  $\mathcal{M}$  is a Banach manifold.

Our resistive structure is chosen to be trivial, whereas the external port is a pair of four-dimensional spaces, i.e.,  $X_{\mathcal{P}} = \mathbb{R}^4$ . As Dirac structure, we choose

$$\mathcal{D}_{(\mathbf{q}, \mathbf{p})} \subset (T\mathcal{M}_{(\mathbf{q}, \mathbf{p})} \times \mathbb{R}^4) \times (T\mathcal{M}'_{(\mathbf{q}, \mathbf{p})} \times \mathbb{R}^4)$$

with the property that

$$\left( \begin{pmatrix} f_S \\ f_{\mathcal{P}} \end{pmatrix}, \begin{pmatrix} e_S \\ e_{\mathcal{P}} \end{pmatrix} \right) \in \mathcal{D}_{(\mathbf{q}, \mathbf{p})}$$

if, and only if, there exists some measurable  $\lambda \in L^\infty([0, \ell])$  with  $\lambda \mathbf{q}' \in W^{1,\infty}([0, \ell]; \mathbb{R}^2)$  and

$$f_S = \begin{pmatrix} f_{S1} \\ f_{S2} \end{pmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \underbrace{\begin{pmatrix} e_{S1} \\ e_{S2} \end{pmatrix}}_{=e_S} + \begin{pmatrix} 0 \\ \lambda \mathbf{q}' \end{pmatrix}$$

as well as

$$f_{\mathcal{P}} = \begin{pmatrix} f_{\mathcal{P}1} \\ f_{\mathcal{P}2} \end{pmatrix} = \begin{pmatrix} (\lambda \mathbf{q}')^{(0)} \\ -(\lambda \mathbf{q}')^{(\ell)} \end{pmatrix}, \quad e_{\mathcal{P}} = \begin{pmatrix} e_{\mathcal{P}1} \\ e_{\mathcal{P}2} \end{pmatrix} = \begin{pmatrix} e_{\mathcal{P}1}^{(0)} \\ e_{\mathcal{P}2}^{(0)} \end{pmatrix}.$$

Note that the physical interpretation of  $\lambda \mathbf{q}'$  is the force acting along the rope. Further note that, in the above Dirac structure, we have implicitly used the canonical embedding  $W^{1,\infty}([0, \ell]; \mathbb{R}^2) \subset W^{1,\infty}([0, \ell]; \mathbb{R}^2)'$  via

$$\begin{aligned} W^{1,\infty}([0, \ell]; \mathbb{R}^2) &\ni z \\ &\cong \left( \bar{z} \mapsto \int_0^\ell z(\xi)^\top \bar{z}(\xi) d\xi \right) \in W^{1,\infty}([0, \ell]; \mathbb{R}^2)' \end{aligned} \quad (11)$$

As Hamiltonian chose the sum of kinetic and potential energy, i.e.,

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \int_0^\ell \frac{1}{2\rho(\xi)} \|\mathbf{p}(\xi)\|_2^2 + \rho(\xi) \mathbf{q}(\xi)^\top \begin{pmatrix} 0 \\ g \end{pmatrix} d\xi,$$

and it can be directly seen that it indeed meets the requirements on Hamiltonians. In particular, by using the canonical embedding (11), we see that the Gâteaux derivative reads

$$D\mathcal{H}(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix} \\ \frac{1}{\rho} \mathbf{p} \end{pmatrix},$$

and a straightforward calculation yields that the behavior of the resulting port-Hamiltonian system exactly reflects the model (10).

#### 4. CONCLUDING REMARKS

We have presented some aspects of port-Hamiltonian systems theory from a functional analytic viewpoint. By presenting a Banach space (manifold) framework to (modulated) Dirac structures, Hamiltonians and resistive structures, we have seen that several examples from electromagnetism and continuum mechanics can be formulated in this manner. However, a functional analytic approach to port-Hamiltonians is still in its infancy, and a quite variety of tasks can be interesting subjects of future research. Such topics are listed below.

- a) **Implicit energy storage:** Port-Hamiltonian systems theory on Banach spaces can be extended by using so-called *Lagrangian manifolds* instead of Hamiltonians, see Maschke and van der Schaft (2018) for the linear and finite-dimensional case. These are submanifolds  $\mathcal{L}$  of the cotangent bundle  $\{(x, x') : x \in \mathcal{M}, x' \in T\mathcal{M}'_x\}$  with the property that  $(x, x') \in \mathcal{L}$  if, and only if,  $\langle x, y' \rangle - \langle y, x' \rangle = 0$  for all  $(y, y') \in \mathcal{L}$ . It

can be shown that the graph of a Hamiltonian defines a Lagrangian manifold. The extension to Lagrangian manifolds which are no graphs allows to incorporate energy storage which further exerts constraints to the systems. For instance, this extension allows to relax the assumption on the mass density per unit length in the rope model in Section 3.1 to be bounded from below by some positive constant.

- b) **Input-output structures:** This refers to a split of the external ports via  $X_{\mathcal{P}} = X_{\mathcal{P}_i} \times X_{\mathcal{P}_o}$  and  $f_{\mathcal{P}} = (f_{\mathcal{P}_i}, f_{\mathcal{P}_o})$ ,  $e_{\mathcal{P}} = (e_{\mathcal{P}_o}, e_{\mathcal{P}_i})$ . The *input* of the system is defined by  $u(\cdot) = (f_{\mathcal{P}_i}(\cdot), e_{\mathcal{P}_i}(\cdot)) : \mathbb{I} \rightarrow X_{\mathcal{P}_i} \times X'_{\mathcal{P}_o}$ , whereas the *output* is  $y(\cdot) = (e_{\mathcal{P}_o}(\cdot), f_{\mathcal{P}_o}(\cdot)) : \mathbb{I} \rightarrow X'_{\mathcal{P}_i} \times X_{\mathcal{P}_o}$ . The most important problem in conjunction with input-output structures is existence of and qualitative behavior of solutions for prescribed inputs.
- c) **Non-smooth systems:** Recall that we have obtained the energy balance by the weak chain rule from Proposition 1 together with Gâteaux differentiability of the Hamiltonian. On the other hand, a generalization of the Gâteaux derivative is given by the *subdifferential* as considered in Barbu (2010) from the perspective of nonlinear evolution equations. This approach is applicable to a class of Hamiltonians which are further allowed to map to  $\mathbb{R}_{\geq 0} \cup \{\infty\}$ , and typically results into a subdifferential which is set-valued and only defined on some subset of  $X_{\mathcal{S}}$ . In other words, subdifferentials give rise to subsets of  $X_{\mathcal{S}} \times X'_{\mathcal{S}}$ , which can be also analyzed from the viewpoint of Lagrangian manifolds. Under the additional assumption that  $X_{\mathcal{S}}$  is a Hilbert space, it is shown in (Barbu, 2010, Lemma 4.4) that the weak chain rule also holds, if the Gâteaux derivative is replaced by a subdifferential. Since this is the essential ingredient used in (2), the incorporation of subdifferentials is a further possible generalization of our approach to infinite-dimensional port-Hamiltonian systems.

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