Systems theoretic properties of linear RLC circuits

Timo Reis * **Fedor Glazov** **

* *Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 22083 Hamburg, Germany (e-mail: timo.reis@uni-hamburg.de).*
** *Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany (e-mail: glazov@campus.tu-berlin.de)*

Abstract: We consider the differential-algebraic systems obtained by modified nodal analysis of linear RLC circuits from a systems theoretic viewpoint. We derive expressions for the set of consistent initial values and show that the properties of controllability at infinity and impulse controllability do not depend on parameter values but rather on the interconnection structure of the circuit. We further present circuit topological criteria for behavioral stabilizability.

Keywords: electrical circuits, stabilizability, system space, consistent initial values, controllability at infinity, impulse controllability

1. INTRODUCTION

Modified nodal analysis (MNA) is a widely used technique for modelling RLC circuits. It has been first introduced in Ho et al. (1975). It is based on regarding a circuit as a graph, and results in a differential-algebraic model. This model provides a structure which allows a mathematically elegant analysis of essential properties and their physical interpretation. Among these properties is the *index*, i.e., the order of smoothness of perturbations entering the solution of the differential-algebraic equation, see Lamour et al. (2013); Kunkel and Mehrmann (2006); it is shown in Estévez Schwarz and Tischendorf (2000); Günther and Feldmann (1999a,b), see also Bächle (2007); März et al. (2003); Estévez Schwarz (2002); Estévez Schwarz and Lamour (2001); Freund (2005); Reis (2014); Riaza (2013); Takamatsu and Iwata (2010), that the index is not dependent on system parameters (such as values of resistances, capacitances and inductances), but rather on the interconnection structure, i.e., the topology, of the circuit. Further important possible properties of the circuit system are *stability* and *asymptotic stability*. Whereas MNA models of RLC circuits are always stable as long as the parameter values of resistances, capacitances and inductances are positive, asymptotic stability requires some further conditions. It is shown in Riaza and Tischendorf (2010, 2007); Riaza (2006) that asymptotical stability is guaranteed, if certain parameter-independent criteria on the circuit interconnection structure are fulfilled. The general idea of these articles is used in Berger and Reis (2014), where topological criteria for asymptotic stability and autonomy of the zero dynamics are presented for the purpose of adaptive tracking control of circuits.

In this article, we analyse further systems theoretic properties of the MNA equations. Besides presenting sufficient topological criteria for behavioral stabilizability, we derive expressions for the system space and the space of consistent initial values, and conclude topological conditions for controllability at infinity and impulse controllability.

In Sec. 2 we present the required tools from graph theory and Sec. 3 collects the basics on RLC circuit models. Sec. 4 and Sec. 5 contain the results on stability and stabilizability of the circuit model and their topological interpretation. Sec. 6 is devoted to the system space of the MNA equations, whereas we specify the space of consistent initial values and give topological conditions for controllability at infinity and impulse controllability in Sec. 7 and Sec. 8.

1.1 Nomenclature

$\mathbb{N}_0$ is the set of nonnegative integers, $\mathbb{R}(s)$ is the field of real rational functions, and $\mathbb{C}_+$, $\overline{\mathbb{C}_+}$ are, resp., the open and closed complex right half planes. For a field K, $K^{n\times m}$ is the set of $n\times m$ matrices with entries in K. We use $\operatorname{rk}_K M$, $\ker_K M$, $\operatorname{im}_K M$ for the rank, kernel and image of a matrix M over K. If $K=\mathbb{R}$, we omit the subindex indicating the underlying field. Further, $M^\top$ and M^* resp. stand for the transpose and conjugate transpose of a matrix M, and by writing $M>0$ ($M\geq 0$), we mean that the square matrix M is symmetric positive (semi-)definite. The identity matrix of size $n\times n$ is denoted by I_n and the zero matrix of size $m\times n$ by $0_{m,n}$. We omit the subindices, if they are clear from context. $\mathcal{V}^\perp$ denotes the orthogonal space of a subspace $\mathcal{V}\subset\mathbb{R}^n$, and we call the matrix Z a *basis matrix* of $\mathcal{V}$, if $\ker Z=\{0\}$ and $\operatorname{im} Z=\mathcal{V}$.

2. GRAPH THEORETIC PRELIMINARIES

For the purpose of this article, we consider finite and loop-free directed graphs, see Diestel (2017). We present some basics of graphs and incidence matrices along with some results about the correspondence between the topological structure of a graph and properties of its incidence matrix.

Definition 1. (Graph theoretic concepts). A *directed graph* is a quadruple $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ consisting of a *vertex set* V, a *edge set* E and two maps $\mathrm{init}, \mathrm{ter} : E \to V$ assigning to each edge e an *initial vertex* $\mathrm{init}(e)$ and a *terminal vertex* $\mathrm{ter}(e)$. The edge e is said to be *directed from* $\mathrm{init}(e)$ *to* $\mathrm{ter}(e)$. $\mathcal{G}$ is said to be *loop-free*, if $\mathrm{init}(e) \neq \mathrm{ter}(e)$ for all $e \in E$. Let $V' \subset V$ and $E' \subset E$ with

$$E' \subset E|_{V'} := \{e \in E : \mathrm{init}(e) \in V' \wedge \mathrm{ter}(e) \in V'\}.$$

Then the triple $(V', E', \mathrm{init}|_{E'}, \mathrm{ter}|_{E'})$ is called a *subgraph of* $\mathcal{G}$. If $E' = E|_V'$, then the subgraph is called the *induced subgraph* on V'. If $V' = V$, then the subgraph is called *spanning*. Additionally a *proper subgraph* is one where $E' \neq E$. $\mathcal{G}$ is called *finite*, if V and E are finite.

For each $e \in E$ define $-e \notin E$ as an edge with $\mathrm{init}(-e) = \mathrm{ter}(e)$ and $\mathrm{ter}(-e) = \mathrm{init}(e)$. Define $\overline{E}$ to be the set which contains all $e \in E$ and all corresponding $-e$. An r-tuple $e = (e_1, \dots, e_r) \in \overline{E}^r$ is called a *path from* v *to* w, if

$$\begin{aligned}&\mathrm{init}(e_1), \dots, \mathrm{init}(e_r) \text{ are distinct},\\ &\mathrm{ter}(e_i) = \mathrm{init}(e_{i+1}) \ \ \forall i \in \{1, \dots, r-1\},\\ &\mathrm{init}(e_1) = v \wedge \mathrm{ter}(e_r) = w.\end{aligned}$$

A *cycle* is a path from v to v. Two vertices v, w are *connected*, if there is a path from v to w. This gives is an equivalence relation on the vertex set. A graph is called *connected*, if there is only one equivalence class. The induced subgraph on an equivalence class of connected vertices gives a *connected component* of the graph.

A spanning subgraph $\mathcal{K} = (V, E', \mathrm{init}|_{E'}, \mathrm{ter}|_{E'})$ of a directed graph $\mathcal{G} = (V, E, \mathrm{init}, \mathrm{ter})$ is called a *cut* of $\mathcal{G}$, if $\mathcal{G} - \mathcal{K} := (V, E \backslash E', \mathrm{init}|_{E\backslash E'}, \mathrm{ter}|_{E\backslash E'})$ has two connected components.

Consider a directed graph $\mathcal{G}$ with spanning subgraph $\mathcal{K}$. We call a subgraph $\mathcal{L}$ of $\mathcal{G}$ a *$\mathcal{K}$-cut*, if $\mathcal{L}$ is a cut of $\mathcal{K}$. Further, we call a path in $\mathcal{G}$ a *$\mathcal{K}$-cycle*, if it is a cycle in $\mathcal{K}$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two spanning subgraphs $\mathcal{G}$, then $\mathcal{K}_1\mathcal{K}_2$ denotes the spanning subgraph obtained by taking the union of the edges $\mathcal{K}_1$ and $\mathcal{K}_2$.

Essential ingredients of the circuit model are incidence matrices.

Definition 2. (Incidence matrix). Let $G = (V, E, \mathrm{init}, \mathrm{ter})$ be a finite and loop-free directed graph. Let $E = \{e_1, \dots, e_{n_e}\}$ and $V = \{v_1, \dots, v_{n_v}\}$. Then the *all-vertex incidence matrix* of $\mathcal{G}$ is $A_0 = (a_{ij}) \in \mathbb{R}^{n_e \times n_v}$ with

$$a_{ij} = \begin{cases} 1 & \text{if } \mathrm{init}(e_j) = v_i,\\ -1 & \text{if } \mathrm{ter}(e_j) = v_i,\\ 0 & \text{otherwise.}\end{cases}$$

The rows of A_0 sum up to zero, so we can delete an arbitrary row to obtain an *incidence matrix* $A_0 = (a_{ij}) \in \mathbb{R}^{(n_e-1)\times n_v}$ of $\mathcal{G}$.

Starting with an incidence matrix A of a finite and loop-free directed graph $\mathcal{G}$, along with a spanning subgraph $\mathcal{K}$ of $\mathcal{G}$, it is possible to obtain an incidence matrix of $\mathcal{K}$ by deleting all of columns corresponding to edges of $\mathcal{G} - \mathcal{K}$. By rearranging the columns, it follows that the matrix A is of the form

$$A = [A_{\mathcal{K}} \ A_{\mathcal{G}-\mathcal{K}}]. \tag{1}$$

Next we collect some auxiliary results on incidence matrices corresponding to subgraphs from Estévez Schwarz and Tischendorf (2000). Note that this reference has wording which slightly differs from ours, as, for instance, cycles are called *loops* therein. Our notation is oriented by the standard reference Diestel (2017) for graph theory.

Proposition 1. (Estévez Schwarz and Tischendorf, 2000, Lem. 2.1 & 2.3) Let $\mathcal{G}$ be a finite and loop-free connected graph with incidence matrix A. Furthermore let $\mathcal{K}$ be a spanning subgraph, and assume that the incidence matrix is partitioned as in (1). Then the following holds:

(i) $\mathcal{G}$ does not contain any $\mathcal{K}$-cuts if, and only if, $\ker A_{\mathcal{G}-\mathcal{K}}^\top = \{0\}$.
(ii) $\mathcal{G}$ does not contain any $\mathcal{K}$-cycles if, and only if, $\ker A_{\mathcal{K}} = \{0\}$.

Let $\mathcal{G}$ be a connected graph with incidence matrix A. Let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and $\mathcal{L}$ a spanning subgraph of $\mathcal{K}$. Then, as in (1), we can, after possibly rearranging the columns, assume that the incidence matrix of $\mathcal{G}$ reads

$$A = [A_{\mathcal{L}} \ A_{\mathcal{K}-\mathcal{L}} \ A_{\mathcal{G}-\mathcal{K}}], \ A_{\mathcal{K}} = [A_{\mathcal{L}} \ A_{\mathcal{K}-\mathcal{L}}]. \tag{2}$$

Proposition 2. [(Riaza and Tischendorf, 2007, Prop. 4.4 & 4.5)] Let $\mathcal{G}$ be a finite and loop-free connected graph with incidence matrix A. Let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and $\mathcal{L}$ a spanning subgraph of $\mathcal{K}$. Further assume that the incidence matrix A of $\mathcal{G}$ is partitioned as in (2). Then the following holds:

(i) $\mathcal{G}$ does not contain $\mathcal{K}$-cycles except for $\mathcal{L}$-cycles if, and only if, $\ker A_{\mathcal{K}} = \ker A_{\mathcal{L}} \times \{0\}$.
(ii) $\mathcal{G}$ does not contain $\mathcal{K}$-cuts except for $\mathcal{L}$-cuts if, and only if, $\ker A_{\mathcal{G}-\mathcal{K}}^\top = \ker A_{\mathcal{G}-\mathcal{L}}^\top$.

3. CIRCUIT EQUATIONS

The MNA of a linear RLC circuit is given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t) \tag{3}$$

with state being composed of vertex potentials, inductive currents, and currents through voltage sources, i.e., $x = (\eta^\top i_{\mathcal{L}}^\top i_{\mathcal{V}}^\top)^\top$ and input consisting of voltages at voltage sources and currents at current sources, i.e., $u = (v_{\mathcal{V}}^\top i_{\mathcal{I}}^\top)^\top$. The matrices E, A, B in (3) are given by

$$sE - A = \begin{bmatrix} sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} & A_{\mathcal{V}}\\ -A_{\mathcal{L}}^\top & s\mathcal{L} & 0\\ -A_{\mathcal{V}}^\top & 0 & 0\end{bmatrix}, B = \begin{bmatrix} -A_{\mathcal{I}} & 0\\ 0 & 0\\ 0 & -I_{n_{\mathcal{V}}}\end{bmatrix}, \tag{4}$$

where s has to be regarded as a formal variable. The expression $sE - A$ is called a *matrix pencil*. Here, $\mathcal{G} \in \mathbb{R}^{n_{\mathcal{G}} \times n_{\mathcal{G}}}$, $\mathcal{L} \in \mathbb{R}^{n_{\mathcal{L}} \times n_{\mathcal{L}}}$, $\mathcal{C} \in \mathbb{R}^{n_{\mathcal{C}} \times n_{\mathcal{C}}}$ are the conductance, inductance and capacitance matrix, and

$$\begin{aligned}&A_{\mathcal{R}} \in \mathbb{R}^{n_e \times n_{\mathcal{R}}}, \quad A_{\mathcal{L}} \in \mathbb{R}^{n_e \times n_{\mathcal{L}}}, \quad A_{\mathcal{C}} \in \mathbb{R}^{n_e \times n_{\mathcal{C}}},\\ &A_{\mathcal{V}} \in \mathbb{R}^{n_e \times n_{\mathcal{V}}}, \quad A_{\mathcal{I}} \in \mathbb{R}^{n_e \times n_{\mathcal{I}}}\end{aligned}$$

are the element-specific incidence matrices with sizes $n = n_e + n_{\mathcal{L}} + n_{\mathcal{V}}$, $m = n_{\mathcal{I}} + n_{\mathcal{V}}$. The matrices $\mathcal{G}$, $\mathcal{L}$, $\mathcal{C}$ contain the parameters of capacitances, resistances, and inductances. Further, $A_{\mathcal{R}}$ is an incidence matrix of the spanning subgraph consisting of all vertices that contain resistances. Similarly, the incidence matrices $A_{\mathcal{L}}, A_{\mathcal{C}}, A_{\mathcal{V}}, A_{\mathcal{I}}$ then resp. correspond to the spanning subgraphs with the edges to inductances, capacitances, voltage and current source. An incidence matrix of the finite and loop-free directed graph modeling the circuit is consequently given by $A = [A_{\mathcal{R}} \ A_{\mathcal{L}} \ A_{\mathcal{C}} \ A_{\mathcal{V}} \ A_{\mathcal{I}}]$. It is also reasonable to assume that the circuit graph is connected, as any connected component corresponds to a subcircuit which does not physically interact with the remaining components, so one may simply consider the connected components separately. We consider circuits with *passive* devices. This leads to the assumption that the conductance matrix is dissipative, whereas the inductance and capacitance matrices are positive definite. Altogether, this means

$$\operatorname{rk}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}\ A_{\mathcal{I}}] = n_e, \tag{5a}$$

$$\mathcal{G} + \mathcal{G}^\top > 0, \quad \mathcal{L} = \mathcal{L}^\top > 0, \quad \mathcal{C} = \mathcal{C}^\top > 0. \tag{5b}$$

4. REGULARITY AND STABILITY

This section will take a closer look at the properties of the properties of the pencil $sE - A$ with matrices as in (4). First we recall some results from Berger and Reis (2014).

Proposition 3. Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then there exist invertible $W, T \in \mathbb{R}^{n \times n}$ with

$$W(sE - A)T = \operatorname{diag}(sI - \tilde{A}, sN - I, 0_{n_0,n_0}), \tag{6}$$

where $n_0 \in \mathbb{N}_0$, N is nilpotent with $N^2 = 0$, and $\tilde{A}$ is a square matrix with the property that all its eigenvalues have nonpositive real part. Further, all eigenvalues of $\tilde{A}$ on the imaginary axis are semi-simple (i.e., their respective geometric and algebraic multiplicities coincide). The pencil $sE - A$ further fulfills

$$\begin{aligned} &\ker_{\mathbb{R}(s)}(sE - A) \\ =&\ker_{\mathbb{R}(s)}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_{\mathcal{V}}, \\ &\operatorname{im}_{\mathbb{R}(s)}(sE - A) \\ =&\operatorname{im}_{\mathbb{R}(s)}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}] \times \mathbb{R}(s)^{n_{\mathcal{L}}} \times \operatorname{im}_{\mathbb{R}(s)} A_{\mathcal{V}}^\top. \end{aligned} \tag{7}$$

Proof. Since (5) implies $E = E^\top \geq 0$ and $A + A^\top \leq 0$, the existence of invertible $W, T \in \mathbb{R}^{n \times n}$ with (6) follows from (Berger and Reis, 2014, Lem. 2.6), whereas (7) is a consequence of (Berger and Reis, 2014, Thm. 4.3). □

A direct consequence of Prop. 3 is that

$$\begin{aligned} \forall \lambda \in \mathbb{C}_+ :\ &\ker_{\mathbb{C}}(\lambda E - A) \\ &= \ker_{\mathbb{C}}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top \times \{0\} \times \ker_{\mathbb{C}} A_{\mathcal{V}}, \\ \forall \lambda \in \mathbb{C}_+ :\ &\operatorname{im}_{\mathbb{C}}(\lambda E - A) \\ &= \operatorname{im}_{\mathbb{C}}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}] \times \mathbb{R}^{n_{\mathcal{L}}} \times \ker_{\mathbb{C}} A_{\mathcal{V}}^\top. \end{aligned} \tag{8}$$

We further characterize *regularity*, i.e., the invertibility of $sE - A$ in $\mathbb{R}(s)^{n \times n}$. Note that regularity translates to the property of a differential-algebraic equation having a solution for all smooth right hand sides, which is moreover unique by specification of the initial condition, see Kunkel and Mehrmann (2006). Prop. 1 and Prop. 3 allow to characterize regularity in terms of the circuit topology.

Corollary 4. Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then the pencil $sE - A$ is regular, if and only if, the underlying circuit neither contains $\mathcal{V}$-cycles nor $\mathcal{I}$-cuts; equivalently (by Prop. 1)

$$\ker [A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top = \{0\}\ \wedge\ \ker A_{\mathcal{V}} = \{0\}.$$

Next we consider *generalized eigenvalues* of $sE - A$. This is a complex number λ with $\operatorname{rk}_{\mathbb{C}} \lambda E - A < \operatorname{rk}_{\mathbb{R}(s)} sE - A$. We see from Prop. 3 that all generalized eigenvalues of $sE - A$ have nonpositive real part. In the following we discuss the possible absence of purely imaginary generalized eigenvalues. The absence of generalized eigenvalues on $\overline{\mathbb{C}_+}$ corresponds to stabilizability of the circuit equation $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$. The latter refers to the properties that for all $x_0 \in \mathbb{R}^n$ such that there exists a solution x of $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ with $Ex(0) = Ex_0$, there also exists a solution x of $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ with $Ex(0) = Ex_0$ which vanishes at infinity, see (Berger and Reis, 2013, Sec. 5).

Proposition 5. [(Berger and Reis, 2014, Thm. 4.6)] Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then all generalized eigenvalues of $sE - A$ have negative real part, if at least one of the following two assertions holds:

(i) The circuit neither contains $\mathcal{LV}$-cycles except for $\mathcal{V}$-cycles, nor $\mathcal{LCI}$-cuts except for $\mathcal{LI}$-cuts; equivalently (by Prop. 2)

$$\begin{aligned} &\ker [A_{\mathcal{L}}\ A_{\mathcal{V}}] = \{0\} \times \ker A_{\mathcal{V}}, \\ \wedge\ &\ker [A_{\mathcal{R}}\ A_{\mathcal{V}}]^\top = \ker [A_{\mathcal{R}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top. \end{aligned}$$

(ii) The circuit neither contains $\mathcal{CI}$-cuts except for $\mathcal{I}$-cuts, nor $\mathcal{LCV}$-cycles except for $\mathcal{CV}$-cycles; equivalently (by Prop. 2)

$$\begin{aligned} &\ker [A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top = \ker [A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{V}}]^\top, \\ \wedge\ &\ker [A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}] = \{0\} \times \ker [A_{\mathcal{C}}\ A_{\mathcal{V}}]. \end{aligned}$$

Prop. 5 slightly generalizes (Riaza and Tischendorf, 2007, Thm. 5.2), where regularity (i.e., the absence of $\mathcal{V}$-cycles and $\mathcal{I}$-cuts) is presumed. Now we combine Prop. 3 with Prop. 5 to show a condition for $\ker_{\mathbb{C}} \lambda E - A = \{0\}$ for all $\lambda \in \overline{\mathbb{C}_+}$. The latter refers to *asymptotic stability*, i.e., all solutions of $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ vanish at infinity.

Proposition 6. Let $E, A \in \mathbb{R}^{n \times n}$ as in (4) and assume that (5) holds. Then $\ker_{\mathbb{C}} \lambda E - A = \{0\}$ for all $\lambda \in \overline{\mathbb{C}_+}$, if at least one of the following two assertions holds:

(i) The circuit neither contains $\mathcal{LV}$-cycles, nor $\mathcal{LCI}$-cuts except for $\mathcal{LI}$-cuts which are no $\mathcal{I}$-cuts; equivalently (by Prop. 1 & Prop. 2)

$$\begin{aligned} &\ker [A_{\mathcal{L}}\ A_{\mathcal{V}}] = \{0\}, \\ \wedge\ &\ker [A_{\mathcal{R}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top = \ker [A_{\mathcal{R}}\ A_{\mathcal{V}}]^\top, \\ \wedge\ &\ker [A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]^\top = \{0\}. \end{aligned}$$

(ii) The circuit neither contains $\mathcal{CI}$-cuts, nor $\mathcal{LCV}$-cycles except for $\mathcal{CV}$-cycles which are no $\mathcal{V}$-cycles; equivalently (by Prop. 1 & Prop. 2)

$$\begin{aligned} &\ker [A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{V}}]^\top = \{0\}, \\ \wedge\ &\ker [A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}] = \{0\} \times \ker [A_{\mathcal{C}}\ A_{\mathcal{V}}], \\ \wedge\ &\ker A_{\mathcal{V}} = \{0\}. \end{aligned}$$

5. BEHAVIORAL STABILIZABILITY

Loosely speaking, behavioral stabilizability of a differential-algebraic control system (3) means that x can always be asymptotically steered to zero by a suitable choice of the input u. More precisely, for any $x_0 \in \mathbb{R}^n$ for which there exists a control u such that a solution x of (3) with initial conditions $Ex(0) = Ex_0$ exists, there especially exists some control u such that a solution x of (3) with initial condition $Ex(0) = Ex_0$ exists which vanishes at infinity. It is proven in (Berger and Reis, 2013, Sec. 5) that this is equivalent to

$$\forall\ \lambda \in \overline{\mathbb{C}_+} :\quad \operatorname{rk}_{\mathbb{C}} [\lambda E - A\ \ B] = \operatorname{rk}_{\mathbb{C}} [\lambda E - A\ \ B]\,. \tag{9}$$

Now consider the circuit model $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Then

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}(s)}[sE-A\ B] = \operatorname{im}_{\mathbb{R}(s)}(sE-A) \ + \ \operatorname{im}_{\mathbb{R}(s)} B \\ \overset{\text{Prop.3}}{=} \ &\operatorname{im}_{\mathbb{R}(s)}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}] \times \mathbb{R}^{n_{\mathcal{L}}} \times \operatorname{im}_{\mathbb{R}(s)} \\ &\quad + \operatorname{im}_{\mathbb{R}(s)} A_I \times \{0\} \times \mathbb{R}^{n_{\mathcal{L}}} \\ = \quad &\operatorname{im}_{\mathbb{R}(s)}[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}\ A_I] \times \mathbb{R}^{n_{\mathcal{L}}} \times \mathbb{R}^{n_{\mathcal{V}}} \overset{(5a)}{=} \mathbb{R}(s)^n. \end{aligned}$$

Likewise, by using (8), the circuit model (4) with assumption (5) fulfills

$$\forall \lambda \in \overline{\mathbb{C}_+}: \quad \operatorname{im}_{\mathbb{C}}[\lambda E - A\ B] = \mathbb{C}^n. \tag{10}$$

As a consequence, the circuit model is behaviorally stabilizable if, and only if, $\operatorname{rk}_{\mathbb{C}}[\imath\omega E - A\ B] = n$ for all $\omega \in \mathbb{R}$. This is used in the following result, where we present sufficient conditions for behavioral stabilizability in terms of the circuit topology.

Proposition 7. Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Then (3) is behaviorally stabilizable, if at least one of the below two statements holds:

(i) The circuit neither contains $\mathcal{L}$-cycles, nor $\mathcal{LC}$-cuts except for $\mathcal{L}$-cuts; equivalently (by Prop. 1 & Prop. 2)

$$\begin{aligned} \ker A_{\mathcal{L}} \quad &= \{0\}, \\ \wedge\ \ker\left[A_{\mathcal{R}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}\ A_I\right]^\top &= \ker\left[A_{\mathcal{R}}\ A_{\mathcal{V}}\ A_I\right]^\top. \end{aligned}$$

(ii) The circuit neither contains $\mathcal{C}$-cuts, nor $\mathcal{LC}$-cycles except for $\mathcal{C}$-cycles; equivalently (by Prop. 1 & Prop. 2)

$$\begin{aligned} \ker\left[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{V}}\ A_I\right]^\top &= \{0\}, \\ \wedge\ \ker\left[A_{\mathcal{L}}\ A_{\mathcal{C}}\right] \quad &= \{0\} \times \ker A_{\mathcal{C}}. \end{aligned}$$

Proof. By the findings prior to this proposition, it suffices to show that the aforementioned topological conditions imply that for all $\omega \in \mathbb{R}$

$$\ker_{\mathbb{C}} \begin{bmatrix} \imath\omega E^\top - A^\top \\ B^\top \end{bmatrix} = \{0\}. \tag{11}$$

Let $\omega \in \mathbb{R}$ and $x = (x_1^\top x_2^\top x_3^\top)^\top \in \ker_{\mathbb{C}} \begin{bmatrix} \imath\omega E^\top - A^\top \\ B^\top \end{bmatrix}$ be partitioned according to the blocks in E and A, i.e.,

$$\begin{bmatrix} \imath\omega A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} & A_{\mathcal{V}} \\ -A_{\mathcal{L}}^\top & \imath\omega L & 0 \\ -A_{\mathcal{V}}^\top & 0 & 0 \\ -A_I^\top & 0 & 0 \\ 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

This gives $x_3 = 0$, $x_1 \in \ker\left[A_{\mathcal{V}}\ A_I\right]^\top$ and

$$\begin{bmatrix} \imath\omega A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top & \imath\omega\mathcal{L} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

A multiplication of the latter equation with $(x_1^* \ \ x_2^*)$ and taking the real part, one arrives at $0 = x_1^* A_{\mathcal{R}}(\mathcal{G} + \mathcal{G}^\top)A_{\mathcal{R}}^\top x_1$. Then $\mathcal{G}+\mathcal{G}^\top > 0$ gives $x_1 \in \ker A_{\mathcal{R}}^\top$. Altogether, $x \in \ker_{\mathbb{C}} \begin{bmatrix} \imath\omega E^\top - A^\top \\ B^\top \end{bmatrix}$ leads to

$$x_1 \in \begin{bmatrix} A_{\mathcal{R}}^\top \\ A_{\mathcal{V}}^\top \\ A_I^\top \end{bmatrix} \ \wedge\ x_3 = 0 \ \wedge\ \begin{bmatrix} \imath\omega A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top & A_{\mathcal{L}} \\ -A_{\mathcal{L}}^\top & \imath\omega\mathcal{L} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{12}$$

First assume that (i) holds. Since we obtain from (12) that $x_1 \in \ker[A_{\mathcal{R}}\ A_{\mathcal{V}}\ A_I]^\top$, (i) leads to $x_1 \in \ker A_{\mathcal{C}}^\top$, whence (12) gives rise to $A_{\mathcal{L}}x_2 = 0$. Again making use of (i), we obtain $x_2 = 0$, and thus $A_{\mathcal{L}}^\top x_1 = 0$. We altogether have $x_2 = 0$, $x_3 = 0$ and $x_1 \in \ker[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}\ A_I]^\top$, and we again obtain $x_1 = 0$ by (5a), whence $x = 0$.

Now assume that (ii) holds: We use (12) to see that $\imath\omega A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top x_1 + A_{\mathcal{L}}x_2 = 0$, i.e.,

$$\begin{bmatrix} \imath\omega\mathcal{C}A_{\mathcal{C}}^\top x_1 \\ x_2 \end{bmatrix} \in \ker[A_{\mathcal{C}}\ A_{\mathcal{L}}] = \ker A_{\mathcal{C}} \times \{0\},$$

and thus $x_2 = 0$. Then (12) gives $A_{\mathcal{L}}^\top x_1 = 0$, and we obtain $x_1 \in \ker[A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{V}}\ A_I]^\top$. The latter space is trivial by (ii). Consequently, $x_1 = 0$, and thus $x = 0$.

6. SYSTEM SPACE

A useful space to understand differential-algebraic systems is the *system space*, which is the minimal subspace $V \subset \mathbb{R}^{n+m}$ in which all solutions $(x(t)^\top\ u(t)^\top)^\top$ of (3) evolve pointwisely. This space plays a crucial role, for instance in optimal control and dissipativity analysis of differential-algebraic systems, see Reis and Voigt (2015, 2019).

The main result in this section is an expression for the system space of the MNA equations (4).

Theorem 8. Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Let $Z_{\mathcal{C}}$ and $Z_{\mathcal{RCV}I}$ be basis matrices of $\ker A_{\mathcal{C}}^\top$ and, resp., $\ker[A_{\mathcal{C}}\ A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{V}}\ A_I]^\top$. Then the system space of (3) is given by

$$\ker \begin{bmatrix} Z_{\mathcal{C}}^\top A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & Z_{\mathcal{C}}A_{\mathcal{L}} & Z_{\mathcal{C}}^\top A_{\mathcal{V}} & Z_{\mathcal{C}}^\top A_I & 0 \\ A_{\mathcal{V}}^\top & 0 & 0 & 0 & -I_{n_{\mathcal{V}}} \\ Z_{\mathcal{RCV}I}^\top A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^\top & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thm. 8 means that a vector $(x_1^\top\ x_2^\top\ x_3^\top\ u_1^\top\ u_2^\top)^\top$ partitioned according to the blocks in $[A\ \ B]$ as in (4) is in the system space of (3) if, and only if, it satisfies

$$\begin{aligned} Z_{\mathcal{C}}^\top(A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top x_1 + A_{\mathcal{L}}x_2 + A_{\mathcal{V}}x_3 + A_I u_1) &= 0, \\ A_V^\top x_1 - u_2 &= 0, \\ Z_{\mathcal{RCV}I}^\top A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{C}}^\top x_1 &= 0. \end{aligned}$$

The remaining part is devoted to the proof of Thm. 8 along with some preparatory results. We first recall a geometric characterization of the system space.

Lemma 9. (Reis et al., 2015, Prop. 3.3) Let $E, A \in \mathbb{R}^{k\times n}$ and $B \in \mathbb{R}^{k\times m}$. Consider the sequence $(V_i)_{i\in\mathbb{N}_0}$ of subspaces of $\mathbb{R}^{n+m}$ with $\mathcal{V}_0 = \mathbb{R}^{n+m}$ and

$$\mathcal{V}_{i+1} = \left\{ \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathbb{R}^{n+m} : Ax + Bu \in [E\ 0]\cdot\mathcal{V}_i \right\}\ \forall i \in \mathbb{N}_0.$$

Then $\mathcal{V}_i \supset \mathcal{V}_{i+1}$ for all $i \in \mathbb{N}_0$. Further, there exists some $i_0 \in \mathbb{N}_0$ $\mathcal{V}_{i_0} = \mathcal{V}_{i_0+1}$ for some $i_0 \in N_0$. Then the system space of (3) is $\mathcal{V}_{i_0}$.

Remark 1. Consider the matrices $\mathcal{A} = [A\ B] \in \mathbb{R}^{n\times(n+m)}$, $\mathcal{E} = [E\ 0] \in \mathbb{R}^{n\times(n+m)}$. Then $\mathcal{V}_{i+1}$ is the preimage of $\mathcal{E}V_i$ under $\mathcal{A}$, i.e., $\mathcal{V}_{i+1} = \mathcal{A}^{-1}(\mathcal{E}V_i)$.

To determine the system space, we advance some helpful results.

Lemma 10. ((Basile and Marro, 1992, Property 3.1.3)). Let $M \in \mathbb{R}^{k\times l}$ and $\mathcal{V} \subset \mathbb{R}^k$ a subspace. Then

$$(M^\top\mathcal{V})^\perp = M^{-1}(\mathcal{V}^\perp).$$

By taking $\mathcal{V} = \mathbb{R}^k$, Lem. 10 implies

$$\operatorname{im} M^\top = (\ker M)^\perp \quad \forall M \in \mathbb{R}^{k\times l}. \tag{13}$$

Lemma 11. Let $E, A \in \mathbb{R}^{k\times n}$, $B \in \mathbb{R}^{k\times m}$ and consider the sequence $(\mathcal{V}_i)$ as in Lem. 9. Then $(\mathcal{W}_i) := (\mathcal{V}_i^\top)$ fulfills $\mathcal{W}_0 = \{0\}$ and

$$\mathcal{W}_{i+1} = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \cdot \left(\begin{bmatrix} E^\top \\ 0 \end{bmatrix}^{-1} \mathcal{W}_i \right) \ \forall i \in \mathbb{N}_0. \tag{14}$$

Proof. We prove the statement via induction on i. The induction start $i=0$ is fulfilled by $\mathcal{W}_0=\{0\}$. For the induction step, assume that $i\in\mathbb{N}_0$ with $\mathcal{V}_i=\mathcal{W}_i^\perp$. Then

$$\begin{aligned}\mathcal{V}_{i+1}&=[A\ B]^{-1}\left([E\ 0]\cdot\mathcal{V}_i\right)=[A\ B]^{-1}\left([E\ 0]\cdot\mathcal{W}_i^\perp\right)\\&\overset{\text{Lem.10}}{=}[A\ B]^{-1}\left(\begin{bmatrix}E^\top\\0\end{bmatrix}^{-1}\mathcal{W}_i\right)^\perp\\&\overset{\text{Lem.10}}{=}\left(\begin{bmatrix}A^\top\\B^\top\end{bmatrix}\left(\left([E\ 0]^\top\right)^{-1}\mathcal{W}_i\right)\right)^\perp=\mathcal{W}_{i+1}^\perp.\qquad\square\end{aligned}$$

Lemma 12. Consider an electrical circuit with incidence matrices as in (4). Let $Z_\mathcal{C}$ and $Z_{\mathcal{RCVI}}$ be basis matrices of $\ker A_\mathcal{C}^\top$ and, resp., $\ker[A_\mathcal{R}\ A_\mathcal{L}\ A_\mathcal{C}\ A_\mathcal{V}\ A_\mathcal{I}]^\top$. Then there exists a basis matrix $Z_{\mathcal{RVI-C}}$ of $\ker[A_\mathcal{R}\ A_\mathcal{V}\ A_\mathcal{I}]^\top Z_\mathcal{C}$ such that $Z_{\mathcal{RCVI}}=Z_\mathcal{C}Z_{\mathcal{RVI-C}}$.

Proof. We have $\operatorname{im}Z_{\mathcal{RCVI}}\subset\operatorname{im}Z_\mathcal{C}$ by definition. Hence there exists a matrix $Z_{\mathcal{RVI-C}}$ with $Z_{\mathcal{RCVI}}=Z_\mathcal{C}Z_{\mathcal{RVI-C}}$. Then $\ker Z_{\mathcal{RCVI}}=\{0\}$ implies $\ker Z_{\mathcal{RVI-C}}=\{0\}$. Then, with $k:=\dim\ker A_\mathcal{C}^\top$, the result follows from

$$\begin{aligned}\operatorname{im}Z_{\mathcal{RVI-C}}&=\{z\in\mathbb{R}^k\ :\ Z_\mathcal{C}z\in\operatorname{im}Z_{\mathcal{RCVI}}\}\\&=\{z\in\mathbb{R}^k\ :\ Z_\mathcal{C}z\in\ker[A_\mathcal{R}\ A_\mathcal{C}\ A_\mathcal{V}\ A_\mathcal{I}]^\top\}\\&=\ker[A_\mathcal{R}\ A_\mathcal{C}\ A_\mathcal{V}\ A_\mathcal{I}]^\top Z_\mathcal{C}.\qquad\square\end{aligned}$$

Now we present a proof of Thm. 8. In doing so, we use the subspace iteration in Lem. 9. Instead of a direct calculation, we determine the orthogonal space via Lem. 11.

Proof of Thm. 8. Let $(\mathcal{W}_i)$ be a sequence of subspaces as in Lem. 11. For $i\in\mathbb{N}_0$, define

$$\mathcal{Z}_{i+1}:=\begin{bmatrix}E^\top\\0\end{bmatrix}^{-1}\mathcal{W}_i.$$

Then $\mathcal{W}_i=\begin{bmatrix}A^\top\\B^\top\end{bmatrix}\cdot\mathcal{Z}_i$ for all $i\in\mathbb{N}_0$ with $i\geq1$. Further, let $Z_{\mathcal{RVI-C}}$ be a basis matrix of $\ker[A_\mathcal{C}\ A_\mathcal{R}\ A_\mathcal{V}\ A_\mathcal{I}]^\top Z_\mathcal{C}$, such that $Z_{\mathcal{RCVI}}=Z_\mathcal{C}Z_{\mathcal{RVI-C}}$ (which exists by Lem. 12).

Step 1: We determine $\mathcal{W}_1$: By $\mathcal{W}_0=\{0\}$ and $E=E^\top$, we have $\mathcal{Z}_1=\left[\begin{smallmatrix}E^\top\\0\end{smallmatrix}\right]^{-1}\mathcal{W}_0=\ker E$. By incorporating $\mathcal{C}>0$, we obtain that the latter space equals to $\operatorname{im}Z_\mathcal{C}\times\{0\}\times\mathbb{R}^{n_\mathcal{V}}$, and thus

$$\mathcal{W}_1=\begin{bmatrix}A^\top\\B^\top\end{bmatrix}\cdot\ker E=\operatorname{im}\begin{bmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}&A_\mathcal{V}\\A_\mathcal{L}^\top Z_\mathcal{C}&0\\A_\mathcal{V}^\top Z_\mathcal{C}&0\\A_\mathcal{I}^\top Z_\mathcal{C}&0\\0&I_{n_\mathcal{V}}\end{bmatrix}.$$

Step 2: We show that $\mathcal{Z}_2$ fulfills

$$\mathcal{Z}_2=\operatorname{im}\begin{bmatrix}Z_\mathcal{C}&0&0\\0&0&\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}\\0&I_{n_\mathcal{V}}&0\end{bmatrix}:\tag{15}$$

"$\supset$": Let z be in the space on the right hand side of (15), and partition $z=(z_1^\top\ z_2^\top\ z_3^\top)^\top$ with $z_1\in\mathbb{R}^{n_e},z_2\in\mathbb{R}^{n_\mathcal{L}},z_3\in\mathbb{R}^{n_\mathcal{V}}$ Then there exist vectors v_1,v_2 with $z_1=Z_\mathcal{C}v_1$ and $z_2=\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}v_2$. By using $A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top z_1=A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top Z_\mathcal{C}v_1=0$, we obtain

$$\begin{aligned}\begin{bmatrix}E^\top\\0\end{bmatrix}z&=\begin{bmatrix}A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top&0&0\\0&\mathcal{L}&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\A_\mathcal{L}^\top Z_{\mathcal{RCVI}}v_2\\0\\0\\0\end{pmatrix}\\&=\begin{bmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}&A_\mathcal{L}\\A_\mathcal{L}^\top Z_\mathcal{C}&0\\A_\mathcal{V}^\top Z_\mathcal{C}&0\\A_\mathcal{I}^\top Z_\mathcal{C}&0\\0&I_{n_\mathcal{V}}\end{bmatrix}\begin{pmatrix}Z_{\mathcal{RVI-C}}v_2\\0\end{pmatrix}\in\mathcal{W}_1.\end{aligned}$$

"$\subset$": Let $z_1\in\mathbb{R}^{n_e},z_2\in\mathbb{R}^{n_\mathcal{L}},z_3\in\mathbb{R}^{n_\mathcal{V}}$ with

$$\begin{bmatrix}A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top&0&0\\0&\mathcal{L}&0\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}\in\mathcal{W}_1=\operatorname{im}\begin{bmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}&A_\mathcal{V}\\A_\mathcal{L}^\top Z_\mathcal{C}&0\\A_\mathcal{V}^\top Z_\mathcal{C}&0\\A_\mathcal{I}^\top Z_\mathcal{C}&0\\0&I_{n_\mathcal{V}}\end{bmatrix}.\tag{16}$$

We have to show that $z_1\in\operatorname{im}Z_\mathcal{C}$ and $z_2\in\operatorname{im}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}$: We obtain from (16) that there exist vectors w_1,w_2 with

$$\begin{pmatrix}A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top z_1\\\mathcal{L}z_2\\0\\0\\0\end{pmatrix}=\begin{pmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}w_1+A_\mathcal{V}w_2\\A_\mathcal{L}^\top Z_\mathcal{C}w_1\\A_\mathcal{V}^\top Z_\mathcal{C}w_1\\A_\mathcal{I}^\top Z_\mathcal{C}w_1\\-w_2\end{pmatrix}.\tag{17}$$

Hence $w_2=0$ and $A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top z_1=A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}w_1$, and a multiplication with $Z_\mathcal{C}^\top$ results in $0=Z_\mathcal{C}^\top A_\mathcal{R}(\mathcal{G}+\mathcal{G}^\top)A_\mathcal{R}^\top Z_\mathcal{C}w_1$. Then $\mathcal{G}+\mathcal{G}^\top>0$ gives $A_\mathcal{R}^\top Z_\mathcal{C}w_1=0$. Thus $w_1\in\ker[A_\mathcal{R}\ A_\mathcal{V}\ A_\mathcal{I}]^\top Z_\mathcal{C}=\operatorname{im}Z_{\mathcal{RVI-C}}$. That is, $w_1=Z_{\mathcal{RVI-C}}y$ for a vector y, and (17) leads to

$$\begin{pmatrix}A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top z_1\\\mathcal{L}z_2\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\A_\mathcal{L}^\top Z_\mathcal{C}Z_{\mathcal{RVI-C}}y\\0\\0\\0\end{pmatrix}.$$

Thus $z_1\in\ker A_\mathcal{C}^\top=\operatorname{im}Z_\mathcal{C}$ and $z_2\in\operatorname{im}\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}$.

Step 3: We conclude that

$$\begin{aligned}\mathcal{W}_2&=\begin{bmatrix}A^\top\\B^\top\end{bmatrix}\mathcal{Z}_2\\&=\begin{bmatrix}-A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top&A_\mathcal{L}&A_\mathcal{V}\\-A_\mathcal{L}^\top&0&0\\-A_\mathcal{V}^\top&0&0\\A_\mathcal{I}^\top&0&0\\0&0&I_{n_\mathcal{V}}\end{bmatrix}\cdot\operatorname{im}\begin{bmatrix}Z_\mathcal{C}&0&0\\0&0&\mathcal{L}^{-1}A_\mathcal{C}^\top Z_{\mathcal{RCVI}}\\0&I_{n_\mathcal{V}}&0\end{bmatrix}\\&=\operatorname{im}\begin{bmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}&A_\mathcal{V}&A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}\\A_\mathcal{L}^\top Z_\mathcal{C}&0&0\\A_\mathcal{V}^\top Z_\mathcal{C}&0&0\\A_\mathcal{I}^\top Z_\mathcal{C}&0&0\\0&I_{n_\mathcal{V}}&0\end{bmatrix}.\end{aligned}$$

Step 4: We show that $\mathcal{Z}_3\subset\mathcal{Z}_2$: Let $z=(z_1^\top\ z_2^\top\ z_3^\top)^\top\in\mathcal{Z}_3$ with $z_1\in\mathbb{R}^{n_e},z_2\in\mathbb{R}^{n_\mathcal{L}},z_3\in\mathbb{R}^{n_\mathcal{V}}$ Then $\begin{bmatrix}A^\top\\B^\top\end{bmatrix}z\in\mathcal{W}_2$ together with Step 3, leads to the existence of vectors w_1, w_2, w_3 with

$$\begin{pmatrix}A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top z_1\\\mathcal{L}z_2\\0\\0\\0\end{pmatrix}=\begin{pmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}w_1+A_\mathcal{V}w_2+A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}w_3\\A_\mathcal{L}^\top Z_\mathcal{C}w_1\\A_\mathcal{V}^\top Z_\mathcal{C}w_1\\A_\mathcal{I}^\top Z_\mathcal{C}w_1\\-w_2\end{pmatrix}.\tag{18}$$

Then $Z_{\mathcal{RCVI}}^\top A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}}w_3=0$ by a multiplication of the first row with $Z_{\mathcal{RCVI}}^\top$, and $\mathcal{L}>0$ gives $A_\mathcal{L}^\top Z_{\mathcal{RCVI}}w_3=0$. Altogether, we have

$$\begin{pmatrix}A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top z_1\\\mathcal{L}z_2\\0\\0\\0\end{pmatrix}=\begin{pmatrix}A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}w_1+A_\mathcal{V}w_2\\A_\mathcal{L}^\top Z_\mathcal{C}w_1\\A_\mathcal{V}^\top Z_\mathcal{C}w_1\\A_\mathcal{I}^\top Z_\mathcal{C}w_1\\-w_2\end{pmatrix}.$$

This is exactly the situation in (17), and we can follow the argumentation in Step 2 to conclude $z\in Z_2$.

Step 5: We conclude the statement of Thm. 8: We have $\mathcal{W}_3=\begin{bmatrix}A^\top\\B^\top\end{bmatrix}\mathcal{Z}_3\subset\begin{bmatrix}A^\top\\B^\top\end{bmatrix}\mathcal{Z}_2=\mathcal{W}_2$ by Step 4. Thus, by Lem. 11, $\mathcal{V}_2=\mathcal{W}_2^\perp\subset\mathcal{W}_3^\perp=\mathcal{V}_3$, whence, by Lem. 9, the system space reads $\mathcal{V}_2=\mathcal{W}_2^\perp$. Now using Step 3, we obtain

$$\mathcal{V}_2 = \mathcal{W}_2^\perp = \left(\operatorname{im} \begin{bmatrix} A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C} & A_\mathcal{V} & A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top Z_{\mathcal{RCVI}} \\ A_\mathcal{L}^\top Z_\mathcal{C} & 0 & 0 \\ A_\mathcal{V}^\top Z_\mathcal{C} & 0 & 0 \\ A_\mathcal{I}^\top Z_\mathcal{C} & 0 & 0 \\ 0 & I_{n_\mathcal{V}} & 0 \end{bmatrix} \right)^\perp$$

$$= \ker \begin{bmatrix} Z_\mathcal{C}^\top A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top & Z_\mathcal{C}A_\mathcal{L} & Z_\mathcal{C}^\top A_\mathcal{V} & Z_\mathcal{C}^\top A_\mathcal{I} & 0 \\ A_\mathcal{V}^\top & 0 & 0 & 0 & -I_{n_\mathcal{V}} \\ Z_{\mathcal{RCVI}}^\top A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top & 0 & 0 & 0 & 0 \end{bmatrix},$$

which completes the proof.

7. CONSISTENT INITIAL VALUES AND CONTROLLABILITY AT INFINITY

Here we analyze the space of consistent initial values, which is the space of all $x_0 \in \mathbb{R}^n$ for which there exists some control u for which there is a weakly differentiable solution x of (3) with initial condition $x(0) = x_0$. If this space is the entire $\mathbb{R}^n$, then the system (3) is called *controllable at infinity*. It is proven in (Berger and Reis, 2013, Sec. 5) that controllability at infinity is equivalent to $\operatorname{rk}[E\ B] = \operatorname{rk}[E\ A\ B]$. For $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in the circuit model (4) with assumption (5), we can conclude from (10) that $\operatorname{rk}[E\ A\ B] = n$, whence the analysis of controllability at infinity for MNA equations reduces to check whether $\operatorname{rk}[E\ B] = n$. By using $\mathcal{C} > 0$, $\mathcal{L} > 0$, we obtain that $\operatorname{im} E = \operatorname{im} A_\mathcal{C} \times \mathbb{R}^{n_\mathcal{L}} \times \{0\}$, whence

$$\operatorname{im}[E\ B] = \operatorname{im} A_\mathcal{C} \times \mathbb{R}^{n_\mathcal{L}} \times \{0\} + \operatorname{im} A_\mathcal{I} \times \{0\} \times \mathbb{R}^{n_\mathcal{V}}.$$

Controllability at infinity is therefore guaranteed if, and only if, $\operatorname{im}[A_\mathcal{C}\ A_\mathcal{I}] = \mathbb{R}^{n_e}$ or, equivalently, $\ker[A_\mathcal{C}\ A_\mathcal{I}]^\top = \{0\}$. We summarize these findings in the following result.

Proposition 13. Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Then the system (3) is controllable at infinity if, and only if, the underlying circuit does not contain any $\mathcal{RLV}$-cuts; equivalently (by Prop. 1)

$$\ker [A_\mathcal{C}\ A_\mathcal{I}]^\top = \{0\}.$$

It can be concluded from (Reis and Voigt, 2019, Lem. 3.7) that the system space $\mathcal{V}_\text{sys}$ and the space $\mathcal{V}_\text{init}$ of consistent initial values of the system (3) fulfill the identity

$$\mathcal{V}_\text{init} = \{x \in \mathbb{R}^n\ :\ \exists u \in \mathbb{R}^m \text{ s.t. } \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{V}_\text{sys}\}. \tag{19}$$

This identity is the essential ingredient in the proof of the following result which contains an expression of the space of consistent initial values for the MNA system.

Theorem 14. Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Let $Z_{\mathcal{RCVI}}$ and $Z_{\mathcal{CI}}$ be basis matrices of $\ker[A_\mathcal{C}\ A_\mathcal{R}\ A_\mathcal{L}\ A_\mathcal{V}\ A_\mathcal{I}]^\top$ and, resp., $\ker[A_\mathcal{C}\ A_\mathcal{I}]^\top$. Then the space of consistent initial values of (3) is given by

$$\ker \begin{bmatrix} Z_{\mathcal{RCVI}}^\top A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top & 0 & 0 \\ Z_{\mathcal{CI}}^\top A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top & Z_{\mathcal{CI}}^\top A_\mathcal{L} & Z_{\mathcal{CI}}^\top A_\mathcal{V} \end{bmatrix}.$$

Proof. Analogous to Lem. 12, there exists a basis matrix $Z_{\mathcal{I}-\mathcal{C}}$ of $\ker A_\mathcal{I}^\top Z_\mathcal{C}$, such that $Z_{\mathcal{CI}} = Z_\mathcal{C} Z_{\mathcal{I}-\mathcal{C}}$.

"$\subset$": Let $x = (x_1^\top\ x_2^\top\ x_3^\top)^\top$ with $x_1 \in \mathbb{R}^{n_e}$, $x_2 \in \mathbb{R}^{n_\mathcal{L}}$, $x_3 \in \mathbb{R}^{n_\mathcal{V}}$ be a consistent initial value. Then by (19), there exist $u_1 \in \mathbb{R}_\mathcal{I}$, $u_2 \in \mathbb{R}^{n_\mathcal{V}}$ such that for $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ holds that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)$ is in the system space of (3). Then Thm. 8 gives

$$\begin{aligned} Z_\mathcal{C}^\top(A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top x_1 + A_\mathcal{L}x_2 + A_\mathcal{V}x_3 + A_\mathcal{I}u_1) &= 0, \\ Z_{\mathcal{RCVI}}^\top A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top x_1 &= 0. \end{aligned} \tag{20}$$

Then a multiplication of the first equation with $Z_{\mathcal{I}-\mathcal{C}}^\top$ gives

$$\begin{aligned} \underbrace{Z_{\mathcal{C}-\mathcal{I}}^\top Z_\mathcal{C}^\top}_{=Z_{\mathcal{CI}}^\top}(A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top x_1 + A_\mathcal{L}x_2 + A_\mathcal{V}x_3) &= 0, \\ Z_{\mathcal{RCVI}}^\top A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^\top x_1 &= 0. \end{aligned} \tag{21}$$

"$\supset$": Let $x_1 \in \mathbb{R}^{n_e}$, $x_2 \in \mathbb{R}^{n_\mathcal{L}}$, $x_3 \in \mathbb{R}^{n_\mathcal{V}}$ such that (21) holds. The second equation implies

$$Z_\mathcal{C}^\top(A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top x_1 + A_\mathcal{L}x_2 + A_\mathcal{V}x_3) \in \ker Z_{\mathcal{C}-\mathcal{I}}^\top.$$

Using $\ker Z_{\mathcal{C}-\mathcal{I}}^\top = (\operatorname{im} Z_{\mathcal{C}-\mathcal{I}})^\perp = (\ker A_\mathcal{I}^\top Z_\mathcal{C})^\perp = \operatorname{im} Z_\mathcal{C}^\top A_\mathcal{I}$, we see that there exists some $u_1 \in \mathbb{R}^{n_\mathcal{I}}$ with

$$Z_\mathcal{C}^\top(A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top x_1 + A_\mathcal{L}x_2 + A_\mathcal{V}x_3) = -Z_\mathcal{C}^\top A_\mathcal{I}u_1.$$

Defining $u_2 = A_\mathcal{V}^\top x_1$ and using Thm. 8, we see that for $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$, it holds that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right)$ is in the system space of (3), and (19) implies that x is a consistent initial value. □

In the case where there are no $\mathcal{RLV}$-cuts, we can conclude from Prop. 1 that both $Z_{\mathcal{IC}}$ and $Z_{\mathcal{RCVI}}$ are trivial, i.e., these matrices have zero columns. Consequently, we also obtain from Thm. 14 that the absence of $\mathcal{RLV}$-cuts causes that any vector in $\mathbb{R}^n$ is a consistent initial value for the MNA system (cf. Prop. 13).

8. CONSISTENT INITIAL DIFFERENTIAL VALUES AND IMPULSE CONTROLLABILITY

We now consider another type of initialization, namely (3) with initial condition $Ex(0) = Ex_0$. $x_0 \in \mathbb{R}^n$ is called a *consistent initial differential value*, if there exists a control u for which a solution x of (3) with initial condition $Ex(0) = Ex_0$ exists. If this space equals to $\mathbb{R}^n$, then the system (3) is called *impulse controllable*. It is proven in (Berger and Reis, 2013, Sec. 5) that impulse controllability is equivalent to $\operatorname{rk}[E\ AZ\ B] = \operatorname{rk}[E\ A\ B]$ for some (and hence any) basis matrix Z of $\ker E$. By again using that the circuit model (4) with assumption (5) has the property $\operatorname{rk}[E\ A\ B] = n$, it is impulse controllable if, and only if, $\operatorname{rk}[E\ AZ\ B] = n$. By using that $\mathcal{C} > 0$ and $\mathcal{L} > 0$ by (5b), we obtain that a basis matrix of $\ker E$ is given by $Z = \operatorname{diag}(Z_\mathcal{C}, 0, I)$, where $Z_\mathcal{C}$ is a basis matrix of $\ker A_\mathcal{C}^\top$. Then

$$\begin{aligned} \operatorname{rk}[E\ AZ\ B] &= \operatorname{rk} \begin{bmatrix} A_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top & 0 & A_\mathcal{R}\mathcal{G}A_\mathcal{C}^\top Z_\mathcal{C} & A_\mathcal{V} & A_\mathcal{I} & 0 \\ 0 & \mathcal{L} & -A_\mathcal{L}^\top Z_\mathcal{C} & 0 & 0 & 0 \\ 0 & 0 & -A_\mathcal{V}^\top Z_\mathcal{C} & 0 & 0 & I_{n_\mathcal{V}} \end{bmatrix} \\ &= \operatorname{rk}[A_\mathcal{C}\ A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}\ A_\mathcal{V}\ A_\mathcal{I}] + n_\mathcal{L} + n_\mathcal{V}. \end{aligned} \tag{22}$$

If $\ker[A_\mathcal{C}\ A_\mathcal{R}\ A_\mathcal{V}\ A_\mathcal{I}]^\top \neq \{0\}$, (22) implies $\operatorname{rk}[E\ AZ\ B] < n$. Conversely, if $\ker[A_\mathcal{C}\ A_\mathcal{R}\ A_\mathcal{V}\ A_\mathcal{I}]^\top = \{0\}$ and $x_1 \in \ker[A_\mathcal{C}\ A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C}\ A_\mathcal{V}\ A_\mathcal{I}]^\top$, then $x_1 \in \ker A_\mathcal{C}$, i.e., $x_1 = Z_\mathcal{C} z_1$ for a vector z_1, and thus $Z_\mathcal{C}^\top A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top Z_\mathcal{C} z_1 = 0$. Then $\mathcal{G} + \mathcal{G} > 0$ leads to $A_\mathcal{R}^\top x_1 = A_\mathcal{R}^\top Z_\mathcal{C} z_1 = 0$, whence $x_1 \in \ker[A_\mathcal{C}\ A_\mathcal{R}\ A_\mathcal{V}\ A_\mathcal{I}]^\top = \{0\}$. We summarize these finding in the following result.

Proposition 15. Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Then the system (3) is impulse controllable if, and only if, the underlying circuit does not contain any $\mathcal{L}$-cuts; equivalently (by Prop. 1)

$$\ker [A_\mathcal{R}\ A_\mathcal{C}\ A_\mathcal{V}\ A_\mathcal{I}]^\top = \{0\}.$$

It is shown in (Berger and Reis, 2013, Lem. 2.3) that the space $\mathcal{V}_\text{init}$ of consistent initial values and the space $\mathcal{V}_\text{diff}$ of consistent initial differential values of the system (3) fulfill

$$\mathcal{V}_\text{diff} = \mathcal{V}_\text{init} + \ker E. \tag{23}$$

This identity is the essential ingredient in the proof of the following result on the space of consistent initial differential values for the MNA system. We will make use of the following preparatory result.

Lemma 16. For any subspace $\mathcal{V} \subset \mathbb{R}^l$ and $M \in \mathbb{R}^{k\times l}$ holds

$$M^{-1}(M\mathcal{V}) = \mathcal{V} + \ker M.$$

Proof. "$\subset$": Let $x \in M^{-1}(M\mathcal{V})$. Then $Mx = My$ for some $y \in \mathbb{R}^l$, whence $x = (x-y)+y \in \ker M + \mathcal{V}$.
"$\supset$": Let $x \in \mathcal{V} + \ker M$, i.e., $x = v+e$ for some $v \in \mathcal{V}$ and $e \in \ker M$. Thus $Mx = Mv$, whence $x \in M^{-1}(M\mathcal{V})$. $\square$

Theorem 17. Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ as in (4) and assume that (5) holds. Let $Z_{\mathcal{RCVI}}$ be a basis matrix of $\ker[A_{\mathcal{C}}\, A_{\mathcal{R}}\, A_{\mathcal{L}}\, A_{\mathcal{V}}\, A_{\mathcal{I}}]^\top$. Then the space of consistent initial differential values of (3) is given by

$$\ker \left[0 \; Z_{\mathcal{RCVI}}^\top A_{\mathcal{L}} \; 0\right].$$

Proof. Let $Z_{\mathcal{CI}}$ be a basis matrix of $\ker[A_{\mathcal{C}}\; A_{\mathcal{I}}]^\top$. Analogous to Lem. 12, there exists a basis matrix $Z_{\mathcal{RV}-\mathcal{CI}}$ of $\ker[A_{\mathcal{R}}\; A_{\mathcal{V}}]^\top Z_{\mathcal{CI}}$, such that $Z_{\mathcal{RCVI}} = Z_{\mathcal{CI}} Z_{\mathcal{RV}-\mathcal{CI}}$.
Step 1: We show that the space $\mathcal{V}_{\text{diff}}$ of consistent initial differential values fulfills

$$E^{-1}\mathcal{V}_{\text{diff}}^\perp = \ker A_{\mathcal{C}}^\top \times \operatorname{im} \mathcal{L}^{-1} A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} \times \mathbb{R}^{n_{\mathcal{V}}} : \qquad (24)$$

"$\subset$": let $x = (x_1^\top\; x_2^\top\; x_3^\top)^\top \in E^{-1}\mathcal{V}^\perp$ with $x_1 \in \mathbb{R}^{n_e}$, $x_2 \in \mathbb{R}^{n_{\mathcal{L}}}$, $x_3 \in \mathbb{R}^{n_{\mathcal{V}}}$. Now using Thm. 14 together with (13), we obtain that there exist vectors z_1, z_2 with

$$\begin{pmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top x_1 \\ \mathcal{L}x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} z_1 + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top Z_{\mathcal{CI}} z_2 \\ A_{\mathcal{L}}^\top Z_{\mathcal{CI}} z_2 \\ A_{\mathcal{V}}^\top Z_{\mathcal{CI}} z_2 \end{pmatrix}. \qquad (25)$$

A multiplication of the first equation in (25) with $Z_{\mathcal{RCVI}}^\top$ yields $Z_{\mathcal{RCVI}}^\top A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} z_1 = 0$, and the positive definiteness of $\mathcal{L}$ now gives rise to $A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} z_1 = 0$. It follows that $Z_{\mathcal{RCVI}} z_1 \in \ker A_{\mathcal{L}}^\top$. Then

$$Z_{\mathcal{RCVI}} z_1 \in \ker[A_{\mathcal{R}}\; A_{\mathcal{C}}\; A_{\mathcal{L}}\; A_{\mathcal{V}}\; A_{\mathcal{I}}]^\top \overset{(5a)}{=} \{0\},$$

and $\ker Z_{\mathcal{RCVI}} = \{0\}$ gives $z_1 = 0$. Then a multiplication of the first equation of (25) with $Z_{\mathcal{CI}}^\top$ gives $0 = Z_{\mathcal{CI}}^\top A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top Z_{\mathcal{CI}} z_2$, and we again make use of (5b) to infer that $A_{\mathcal{R}}^\top Z_{\mathcal{CI}} z_2 = 0$. Then the first equation in (25) gives $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top x_1 = 0$, and (5b) gives $A_{\mathcal{C}}^\top x_1 = 0$. The previous findings imply $z_2 \in \ker[A_{\mathcal{R}}\; A_{\mathcal{V}}]^\top Z_{\mathcal{CI}}$. Hence, there exists a vector y with $Z_{\mathcal{R}-\mathcal{CI}} y_2 = z_2$. Plugging this into the second equation of (25) and using $Z_{\mathcal{RCVI}} = Z_{\mathcal{CI}} Z_{\mathcal{RV}-\mathcal{CI}}$, it follows that $x_2 = \mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} y_2$, and we obtain $x \in \ker A_{\mathcal{C}}^\top \times \operatorname{im} \mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} \times \mathbb{R}^{n_{\mathcal{V}}}$.
"$\supset$": Consider $x = (x_1^\top\; x_2^\top\; x_3^\top)^\top$ with $x_1 \in \ker A_{\mathcal{C}}^\top$, $x_2 \in \operatorname{im} \mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}}$, $x_3 \in \mathbb{R}^{n_{\mathcal{V}}}$. Then $x_2 = \mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} y_2$ for a vector y_2, and we obtain

$$Ex = \begin{pmatrix} 0 \\ A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} y \\ 0 \end{pmatrix} = \begin{bmatrix} A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} & A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top Z_{\mathcal{CI}} \\ 0 & A_{\mathcal{C}}^\top Z_{\mathcal{CI}} \\ 0 & A_{\mathcal{V}}^\top Z_{\mathcal{CI}} \end{bmatrix} \begin{pmatrix} 0 \\ Z_{\mathcal{RV}-\mathcal{CI}} y \end{pmatrix}.$$

By again applying (13), Thm. 14 leads to $x \in E^{-1}\mathcal{V}_{\text{diff}}^\perp$.
Step 2: We conclude from Step 1 that

$$\begin{aligned} & E \cdot (E^{-1}\mathcal{V}_{\text{diff}}^\perp) \\ \overset{(24)}{=}\; & \operatorname{diag}(A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top, \mathcal{L}, 0) \cdot \left(\ker A_{\mathcal{C}}^\top \times \mathcal{L}^{-1}A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} \times \mathbb{R}^{n_{\mathcal{V}}}\right) \\ =\; & \operatorname{im} \begin{bmatrix} 0 \\ A_{\mathcal{L}}^\top Z_{\mathcal{RCVI}} \\ 0 \end{bmatrix}. \end{aligned}$$

Step 3: We conclude the statement of Thm. 17: By using the symmetry of E, we obtain

$$\begin{aligned} \ker \left[0 \; Z_{\mathcal{RCVI}}^\top A_{\mathcal{L}} \; 0\right] &\overset{\text{Step 2}}{=} \left(E \cdot (E^{-1}\mathcal{V}_{\text{init}}^\perp)\right)^\perp \\ &\overset{\text{Lem. 10}}{=} E^{-1} \cdot (E\mathcal{V}_{\text{init}}) \\ &\overset{\text{Lem. 16}}{=} \mathcal{V}_{\text{init}} + \ker E \overset{(23)}{=} \mathcal{V}_{\text{diff}}. \qquad \square \end{aligned}$$

In the case where the circuit does not contain any $\mathcal{L}$-cuts, we can conclude from Prop. 1 that $Z_{\mathcal{RCVI}}$ are trivial, i.e., it has zero columns. As a consequence, we also obtain from Thm. 17 that in the case of absence of $\mathcal{L}$-cuts, any vector in $\mathbb{R}^n$ is a consistent initial differential value for the MNA system (cf. Prop. 15).

REFERENCES

Bächle, S. (2007). Numerical solution of differential-algebraic systems arising in circuit simulation. Doctoral dissertation.

Basile, G. and Marro, G. (1992). *Controlled and Conditioned Invariants in Linear System Theory*. Prentice-Hall, Englewood Cliffs, NJ.

Berger, T. and Reis, T. (2014). Zero dynamics and funnel control for linear electrical circuits. *J. Franklin Inst.*, 351(11), 5099–5132.

Berger, T. and Reis, T. (2013). Controllability of linear differential-algebraic systems - a survey. In A. Ilchmann and T. Reis (eds.), *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, 1–61. Springer, Berlin-Heidelberg.

Diestel, R. (2017). *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, 5th edition.

Estévez Schwarz, D. (2002). A step-by-step approach to compute a consistent initialization for the mna. *Int. J. Circuit Theory Appl.*, 30(1), 1–16.

Estévez Schwarz, D. and Lamour, R. (2001). The computation of consistent initial values for nonlinear index-2 differential-algebraic equations. *Numer. Algorithms*, 26(1), 49–75.

Estévez Schwarz, D. and Tischendorf, C. (2000). Structural analysis for electric circuits and consequences for MNA. *Int. J. Circuit Theory Appl.*, 28(2), 131–162.

Freund, R. (2005). RCL circuit equations. In P. Benner, V. Mehrmann, and D. Sorensen (eds.), *Dimension Reduction of Large-Scale Systems*, volume 45 of *Lecture Notes in Computational Science and Engineering*, 367–371. Springer-Verlag, Berlin/Heidelberg.

Günther, M. and Feldmann, U. (1999a). CAD-based electric-circuit modeling in industry I. Mathematical structure and index of network equations. *Surv. Math. Ind.*, 8, 97–129.

Günther, M. and Feldmann, U. (1999b). CAD-based electric-circuit modeling in industry II. Impact of circuit configurations and parameters. *Surv. Math. Ind.*, 8, 131–157.

Ho, C.W., Ruehli, A., and Brennan, P. (1975). The modified nodal approach to network analysis. *IEEE Trans. Circuits Syst.*, CAS-22(6), 504–509.

Kunkel, P. and Mehrmann, V. (2006). *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, Switzerland.

Lamour, R., März, R., and Tischendorf, C. (2013). *Differential Algebraic Equations: A Projector Based Analysis*, volume 1 of *Differential-Algebraic Equations Forum*. Springer-Verlag, Heidelberg-Berlin.

März, R., Estévez Schwarz, D., Feldmans, U., Sturtzel, S., and Tischendorf, C. (2003). Finding beneficial dae structures in circuit simulation. In H.K. W. Jäger (ed.), *Mathematics – key technology for the future – Joint Projects between Universities and Industry*, 413–428. Springer-Verlag, Berlin.

Reis, T. (2014). Mathematical modeling and analysis of nonlinear time-invariant RLC circuits. In P. Benner, R. Findeisen, D. Flockerzi, U. Reichl, and K. Sundmacher (eds.), *Large-Scale Networks in Engineering and Life Sciences*, Modeling and Simulation in Science, Engineering and Technology, 125–198. Birkhäuser, Basel.

Reis, T., Rendel, O., and Voigt, M. (2015). The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra Appl.*, 86, 153–193.

Reis, T. and Voigt, M. (2015). The Kalman-Yakubovich-Popov inequality for differential-algebraic systems: Existence of nonpositive solutions. *Systems Control Lett.*, 86, 1–8.

Reis, T. and Voigt, M. (2019). Linear-quadratic optimal control of differential-algebraic systems: the infinite time horizon problem with zero terminal state. *SIAM J. Control Optim.*, 57(3), 1567–1596.

Riaza, R. (2006). Time-domain properties of reactive dual circuits. *Int. J. Circ. Theor. Appl.*, 34(3), 317–340.

Riaza, R. (2013). *Surveys in Differential-Algebraic Equations I*, volume 2 of *Differential-Algebraic Equations Forum*, chapter DAEs in Circuit Modelling: A Survey, 97–136. Springer.

Riaza, R. and Tischendorf, C. (2007). Qualitative features of matrix pencils and DAEs arising in circuit dynamics. *Dynamical Systems*, 22(2), 107–131.

Riaza, R. and Tischendorf, C. (2010). The hyperbolicity problem in electrical circuit theory. *Math. Meth. Appl. Sci.*, 33(17), 2037–2049.

Takamatsu, M. and Iwata, S. (2010). Index characterization of differential-algebraic equations in hybrid analysis for circuit simulation. *Int. J. Circuit Theory Appl.*, 38, 419–440.