

# OBSERVERS AND DYNAMIC CONTROLLERS FOR LINEAR DIFFERENTIAL-ALGEBRAIC SYSTEMS\*

THOMAS BERGER<sup>†</sup> AND TIMO REIS\*

**Abstract.** We study state-estimation and stabilization by dynamic feedback for linear differential-algebraic systems which are not necessarily regular. We show that the observer synthesis approach for behavioral systems in [M.E. Valcher and J.C. Willems, *IEEE Trans. Autom. Control*, 44 (1999), pp. 2297–2307] can be applied to differential-algebraic systems in a closed form, i.e., the observers and dynamic controllers are again differential-algebraic systems. The concept of an (asymptotic, exact) observer is introduced and existence is characterized. Since initialization of the observer is an important issue we investigate regular and freely initializable observers, whose existence can be characterized in terms of impulse observability. The observers are then exploited for the construction of dynamic controllers. We show that there exists a stabilizing controller if, and only if, the given system is both behaviorally stabilizable and behaviorally detectable.

**Key words.** differential-algebraic systems, descriptor systems, observer, dynamic controller

**AMS subject classifications.** 34A09, 93C05, 93B05, 93B07, 93B52

**1. Introduction.** We study state-estimation and stabilization of linear time-invariant systems (the *plant*) given by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \frac{d}{dt}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where  $E, A \in \mathbb{R}^{l \times n}$ ,  $B \in \mathbb{R}^{l \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times l}$ . Systems of that type are also called *descriptor systems*. The set of systems (1.1) is denoted by  $\Sigma_{l,n,m,p}$  and we write  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ . DAE systems of the form (1.1) naturally occur when modeling dynamical systems subject to algebraic constraints; for a further motivation we refer to [5, 22, 38, 40, 47] and the references therein. In the present paper we put special emphasis on the non-regular case, i.e., we *do not assume* that  $sE - A$  is *regular*, which would mean that  $l = n$  and  $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$ .

The functions  $u : \mathbb{R} \rightarrow \mathbb{R}^m$  and  $y : \mathbb{R} \rightarrow \mathbb{R}^p$  are called *input* and *output* of the system, resp. A trajectory  $(x, u, y) : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$  is said to be a *solution* of (1.1), if it belongs to the *behavior* of (1.1):

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x, u, y) \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \mid \begin{array}{l} Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l) \text{ and } (x, u, y) \\ \text{solves (1.1) for a.a. } t \in \mathbb{R} \end{array} \right\}.$$

For the notation we refer to the list at the end of this introductory section. Recall that  $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$  implies continuity of  $Ex$  (though  $x$  itself may be discontinuous). For the analysis of DAE systems in  $\Sigma_{l,n,m,p}$  we assume that the states, inputs and outputs of the system are fixed a priori by the designer. This is different from other approaches based on the behavioral setting [12, 24, 53].

In the present paper we aim to construct a stabilizing feedback controller that does not have direct access to the state of the plant, but only uses information about its output. This is motivated by practice: An operator of the system has only access to the external variables of the system. The state is an internal variable which in general cannot be measured directly. We follow the classical approach: First we construct a dynamical system whose input is

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<sup>†</sup>Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany, {thomas.berger, timo.reis}@uni-hamburg.de

composed of the input  $u$  and output  $y$  of the plant. The output of the to-be-built dynamical system will be a variable  $z$  which approximates the state  $x$  in a certain sense. Such a system will be called *observer* and we assume that it is itself a DAE,

$$\begin{aligned} \frac{d}{dt}E_o x_o(t) &= A_o x_o(t) + B_o \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \\ z(t) &= C_o x_o(t) + D_o \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \end{aligned} \quad (1.2)$$

with  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m+p, n}$ . Obvious applications for observers are diagnosis and error detection [34]. In the case of feedback control, the approximate state produced by the observer is used for stabilization of the system by feedback.

For linear ODE systems (i.e.,  $E = I$ ) the above described procedure is well-known (see the textbook [51]) and goes back to the work by Luenberger [43, 44]. In this case, roughly speaking, an observer is a dynamical system with the property that a zero initial observation error implies a zero observation error for all times. It can be shown that this leads to the general structure of an observer as depicted in Figure 1.1.

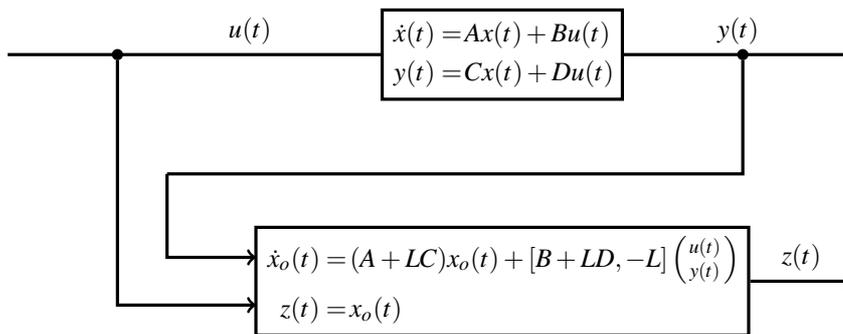


Fig. 1.1: Observer for ODE systems

Therefore, the observer only depends on the choice of the matrix  $L \in \mathbb{R}^{n \times p}$ . If  $L$  is chosen such that  $\sigma(A + LC) \subseteq \mathbb{C}_-$ , i.e., the pair  $[A, C]$  is detectable, then the observation error decays exponentially for any initial value; the observer is then called an asymptotic observer. If additionally  $[A, B]$  is stabilizable, then there exists  $F \in \mathbb{R}^{m \times n}$  such that  $\sigma(A + BF) \subseteq \mathbb{C}_-$ . Together with an asymptotic observer the feedback law  $u(t) = Fz(t)$  leads to asymptotically closed-loop system, see Figure 1.2, and thus solves the stabilization problem. Conversely, if there exists a stabilizing feedback controller, then  $[A, B, C]$  is stabilizable and detectable.

In the present paper we generalize the above theorem to case of DAE systems, see Theorem 4.2. This leads to a set of new results for DAE observers and dynamic controllers in the nonregular as well as in the regular case. We stress that for DAE systems the situation is more subtle than for ODE systems. Apart from the study of the compatibility of the interconnections, questions such as regularity and free initializability of the observers and dynamic controllers must be treated.

The paper is organized as follows: In Section 2 we collect the concepts used in the present paper and some results on their algebraic characterization. In Section 3 we introduce the concepts of (asymptotic, exact) observers for DAE systems and characterize their existence in terms of observability and detectability. We also give sufficient criteria for DAE observers being regular or freely initializable. The observers are exploited in Section 4 for the construction of stabilizing dynamic controllers for DAE systems; special care has to be

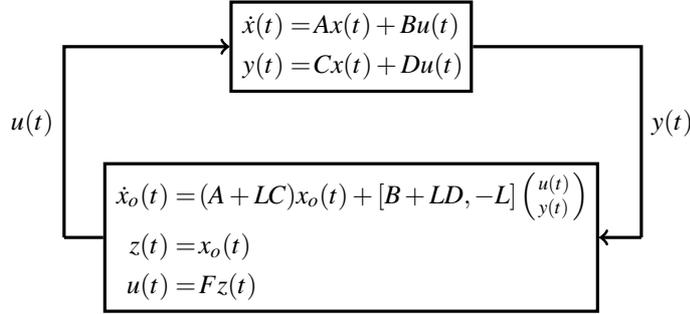


Fig. 1.2: Controller for regular systems

taken with the compatibility of the controllers. It is shown in Theorem 4.2 that there exists a stabilizing controller if, and only if, the given plant is both behaviorally stabilizable and behaviorally detectable. We show that in the regular case the controller structure simplifies to some well known results.

### Notation

$\mathbb{N}, \mathbb{N}_0$	the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\ell(\alpha),  \alpha $	length $\ell(\alpha) = l$ and absolute value $ \alpha  = \sum_{i=1}^l \alpha_i$ of a multi-index $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{C}_+, (\mathbb{C}_-)$	open set of complex numbers with positive (negative) real part, resp.
$\mathbb{R}[s], \mathbb{R}(s)$	the ring of polynomials with coefficients in $\mathbb{R}$ and its quotient field, resp.
$R^{n \times m}$	the set of $n \times m$ matrices with entries in a ring $R$
$\text{im}A, \ker A, \text{rk}_R A$	image, kernel and rank of the matrix $A \in R^{n \times m}$ , resp.
$\mathbf{GL}_n(R)$	the group of invertible matrices in $R^{n \times n}$
$\sigma(A)$	the spectrum of $A \in \mathbb{R}^{n \times n}$
$\ x\ $	$= \sqrt{x^\top x}$ , the Euclidean norm of $x \in \mathbb{R}^n$
$\overline{M}$	closure of the set $M$
$\mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$ , where $\int_K \ f(t)\  dt < \infty$ for all compact $K \subseteq \mathbb{R}$
$\mathcal{AC}(\mathbb{R}; \mathbb{R}^n)$	the set of absolutely continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$
$\dot{f} (f^{(i)})$	the ( $i$ -th) weak derivative of $f \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$ , $i \in \mathbb{N}_0$ , see [1, Chap. 1]
$f \stackrel{\text{a.e.}}{=} g$	the functions $f, g \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$ are equal “almost everywhere”, i.e., $f(t) = g(t)$ for almost all (a.a.) $t \in \mathbb{R}$
$\text{ess sup}_I \ f\ $	the essential supremum of the measurable function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ over $I \subseteq \mathbb{R}$

$f|_I$  the restriction of the function  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  to  $I \subseteq \mathbb{R}$

**2. Preliminaries.** We consider different notions of observability, detectability and stabilizability for DAE systems. For the definitions of these concepts in time domain and for a detailed discussion we refer to the surveys [11, 13]. In the following we give definitions using the algebraic characterizations in [11, Cor. 4.3] and [13, Prop. 6.1 & Cor. 9.5].

DEFINITION 2.1. A system  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  is called

- (i) impulse observable, if  $\ker_{\mathbb{R}} E \cap A^{-1}(\text{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C = \{0\}$ .
- (ii) behaviorally observable, if  $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$  for all  $\lambda \in \mathbb{C}$ .
- (iii) behaviorally detectable, if  $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$  for all  $\lambda \in \overline{\mathbb{C}_+}$ .
- (iv) strongly detectable, if  $\ker_{\mathbb{R}} E \cap A^{-1}(\text{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C = \{0\}$  and  $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$  for all  $\lambda \in \overline{\mathbb{C}_+}$ .
- (v) behaviorally stabilizable, if  $\text{rk}_{\mathbb{R}(s)}[sE - A, B] = \text{rk}_{\mathbb{C}}[\lambda E - A, B]$  for all  $\lambda \in \overline{\mathbb{C}_+}$ .

Note that behavioral detectability and behavioral stabilizability are not dual concepts, see also [13] for a comprehensive discussion of this issue.

For  $E, A \in \mathbb{R}^{l \times n}$  we consider the homogeneous system

$$\frac{d}{dt}Ex(t) = Ax(t); \quad (2.1)$$

the set of those systems is denoted by  $\Sigma_{l,n}$ . The behavior of (2.1) is given by

$$\mathfrak{B}_{[E,A]} := \left\{ x \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n) \mid Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l) \text{ and } x \text{ satisfies (2.1) for almost all } t \in \mathbb{R} \right\}.$$

From [5, 11] we recall the following concepts.

DEFINITION 2.2. A DAE  $[E, A] \in \Sigma_{l,n}$  is called

- (a) behaviorally stable

$$:\iff \forall x \in \mathfrak{B}_{[E,A]} : \lim_{t \rightarrow \infty} \text{ess sup}_{[t,\infty)} \|x\| = 0.$$

- (b) autonomous

$$:\iff \forall x_1, x_2 \in \mathfrak{B}_{[E,A]} : x_1|_{(-\infty, 0)} \stackrel{\text{a.e.}}{=} x_2|_{(-\infty, 0)} \implies x_1 \stackrel{\text{a.e.}}{=} x_2.$$

For a further discussion of autonomy of DAE systems see [12, Rem. 3.3]. Here we recall the important equivalent characterization that

$$[E, A] \in \Sigma_{l,n} \text{ is autonomous} \iff \forall x \in \mathfrak{B}_{[E,A]} : \left( Ex(0) = 0 \implies x \stackrel{\text{a.e.}}{=} 0 \right). \quad (2.2)$$

The following result is an immediate consequence of [11, Cor. 5.2].

LEMMA 2.3. Let  $[E, A] \in \Sigma_{l,n}$ . Then the following holds true:

- (i)  $[E, A]$  is behaviorally stable if, and only if,  $\text{rk}_{\mathbb{C}}(\lambda E - A) = n$  for all  $\lambda \in \overline{\mathbb{C}_+}$ .
- (ii)  $[E, A]$  is autonomous if, and only if,  $\text{rk}_{\mathbb{R}(s)}(sE - A) = n$ .
- (iii)  $x \stackrel{\text{a.e.}}{=} 0$  for all  $x \in \mathfrak{B}_{[E,A]}$  if, and only if,  $\text{rk}_{\mathbb{C}}(\lambda E - A) = n$  for all  $\lambda \in \mathbb{C}$ .

**3. Observers.** In this section we first present rigorous definitions of the concept of an (asymptotic, exact) observer. To this end we use the approach in [53] for the more general class of behaviors described by linear constant coefficient differential equations of possibly higher order. Thereafter, we consider observer design and characterize the existence of (asymptotic, exact) observers. In principle, we present ‘‘DAE versions’’ of the results in [53]. Though we treat a smaller class than [53], there is a certain benefit and novelty of our results: Observers for DAE systems can be chosen to be DAE systems themselves. Thereafter we introduce the classes of ‘‘freely initializable’’ and ‘‘regular observers’’. The first means that the plant does not influence the set of consistent initial values of the observer. The latter means that the observer is neither under- nor overdetermined in a certain sense.

**3.1. Definitions.** First of all, an observer, i.e., a dynamical system which aims to reconstruct the state, should be able to process the signals of the plant without influencing the plant itself. This is subject of the following definition.

**DEFINITION 3.1 (Acceptor).** Consider a system  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$ . Then  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m_o, p_o}$  is called an acceptor for  $[E, A, B, C, D]$ , if for all  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$ , there exist  $x_o \in \mathcal{L}_{loc}^1(\mathbb{R}; \mathbb{R}^{n_o})$ ,  $z \in \mathcal{L}_{loc}^1(\mathbb{R}; \mathbb{R}^{p_o})$  such that

$$(x_o, \begin{pmatrix} u \\ y \end{pmatrix}, z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}.$$

The above definition means that there is a one-directed signal flow from  $[E, A, B, C, D]$  to its acceptor  $[E_o, A_o, B_o, C_o, D_o]$  via input and output (see Fig. 3.1). That is,  $[E, A, B, C, D]$  may influence  $[E_o, A_o, B_o, C_o, D_o]$  but not vice-versa. Also compare the general structure in Fig. 3.1 with the ODE case depicted in Fig. 1.1.

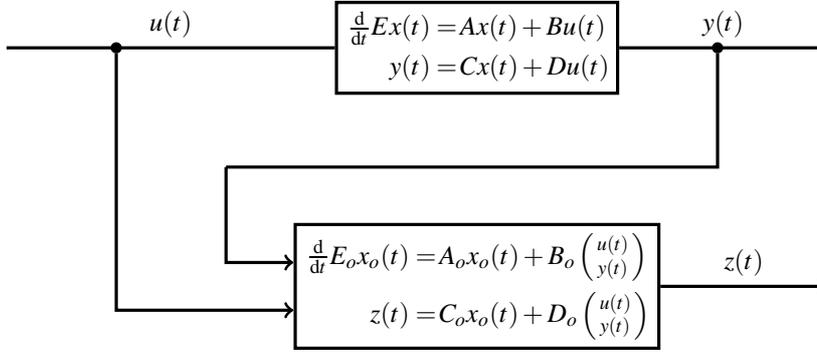


Fig. 3.1: Interconnection with an acceptor

**DEFINITION 3.2 (Observer).** Consider the system  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$ . Then  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m_o, p_o}$  is called

a) an observer for  $[E, A, B, C, D]$ , if it is an acceptor for  $[E, A, B, C, D]$ , and

$$\forall (x, u, y, x_o, z) \in \mathcal{L}_{loc}^1(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^{p_o}) : \\ \left( \begin{array}{l} (x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]} \wedge (x_o, \begin{pmatrix} u \\ y \end{pmatrix}, z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \\ \wedge Ez(0) = Ex(0) \end{array} \right) \implies z \stackrel{\text{a.e.}}{=} x.$$

b) an asymptotic observer for  $[E, A, B, C, D]$ , if it is an observer for  $[E, A, B, C, D]$ , and

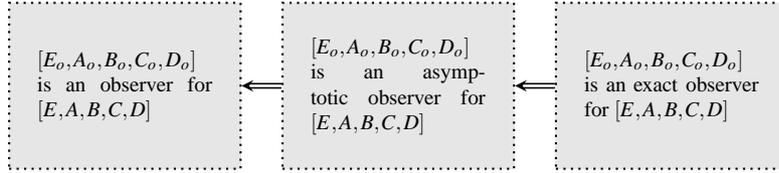
$$\forall (x, u, y, x_o, z) \in \mathcal{L}_{loc}^1(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^{p_o}) : \\ \left( (x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]} \wedge (x_o, \begin{pmatrix} u \\ y \end{pmatrix}, z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \right) \\ \implies \lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|z - x\| = 0.$$

c) an exact observer for  $[E, A, B, C, D]$ , if it is an acceptor for  $[E, A, B, C, D]$ , and

$$\forall (x, u, y, x_o, z) \in \mathcal{L}_{loc}^1(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^{p_o}) : \\ \left( (x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]} \wedge (x_o, \begin{pmatrix} u \\ y \end{pmatrix}, z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \right) \implies z \stackrel{\text{a.e.}}{=} x.$$

REMARK 3.3 (Observer).

- a) We have the following implications for  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  and  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m+p, n}$ :



- b) The property of an observer being an acceptor is natural: As the name suggests, an observer shall only observe and not influence the system. The further property says that once the observer matches the state of the plant, it does not loose track. For an asymptotic observer, the state trajectory of the plant is further attractive: Independent of the past of the observer, the observation error

$$e(t) = z(t) - x(t) \quad (3.1)$$

tends to zero for  $t \rightarrow \infty$ . Whereas, an exact observer matches the overall state trajectory.

- c) Our definition of an observer slightly differs from the one for behavioral systems by Valcher and Willems in [53, Def. 3.1], where, adapted to our DAE setup,  $[E_o, A_o, B_o, C_o, D_o]$  is called an observer for  $[E, A, B, C, D]$ , if

$$(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]} \wedge (x_o, \begin{pmatrix} u \\ y \end{pmatrix}, z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \wedge z|_{(-\infty, 0]} \stackrel{\text{a.e.}}{=} x|_{(-\infty, 0]} \implies z \stackrel{\text{a.e.}}{=} x. \quad (3.2)$$

Our definition therefore seems to be stronger at a glance. We will however see in Remark 3.6 that for DAE systems, our definition is equivalent to the one in [53].

**3.2. Observer design.** We now consider the construction of (asymptotic, exact) observers for a given system  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ . In [46, p. 351] Polderman and Willems give a preeminently nice and picturesque interpretation for observers, which we would like to quote completely at this place:

“How then should we choose the equations governing a state observer? The design that we put forward has a very appealing logic. The two central ideas are:

1. the observer contains a copy of the plant, called an *internal model*.
2. the observer is driven by the *innovations*, by the error feedback, that is, by a signal that expresses how far the actual observed output differs from what we would have expected to observe.

This logic functions not unlike what happens in daily life. Suppose that we meet a friend. How do we organize our thoughts in order to deduce his or her mood, or other latent properties, from the observed manifest ones? Based on past experience, we have an “internal model” of our friend in mind, and an estimate of the “associated state” of his/her mood. This tells us what reactions to expect. When we observe an action or hear a response, then this may cause us to update the state of this internal model. If the observed reaction agrees with what we expected from our current estimate, then there is no need to change the estimate. The more the reaction differs from our expectations, the stronger is the need to update. The difference between what we actually observe and what we had expected to observe is what we call the innovations. Thus it is logical to assume that the updating algorithm for the estimate of the internal model is driven by the innovations. We may also interpret the innovations as the *surprise factor*. ”

We propose a new observer which thoroughly matches this exegesis: Given a plant  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ , let  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$ , and consider the following observer design,

$$\begin{aligned} \frac{d}{dt} E z(t) &= A z(t) + B u(t) + L_x d(t), \\ y(t) &= C z(t) + D u(t) + L_y d(t), \end{aligned} \quad (3.3)$$

internal model
innovations

or, in terms of (1.2),

$$[E_o, A_o, B_o, C_o, D_o] = \left[ \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & L_x \\ C & L_y \end{bmatrix}, \begin{bmatrix} B & 0 \\ D & -I_m \end{bmatrix}, [I_n \ 0], 0_{n,m+p} \right] \in \Sigma_{l+p,n+k,m+p,n}. \quad (3.4)$$

The observer is additively composed of an internal model, i.e., a copy of the plant (or friend), and a further term which involves the variable  $d \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^k)$  and takes the role of the innovations term (or *surprise summand*). Loosely speaking, the smaller is  $d$ , the better are the variables in the internal model of the plant (which is part of the observer) in coincidence with the variables in the actual plant (and the better is the actual state matched by the approximate state). The only difference is that our innovations term is not an error feedback that is driven by a signal which expresses how far the actual observed output differs from what we would have expected to observe. The variable  $d(t)$  is rather a measure for the correctness of the overall internal model at time  $t$ . We will show in Remark 3.9 that, if  $sE - A$  is square, the innovations term is indeed a feedback.

The interconnection of  $[E, A, B, C, D]$  and  $[E_o, A_o, B_o, C_o, D_o]$  is described by the control system

$$\frac{d}{dt} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ C & -I & 0 & 0 \\ 0 & 0 & A & L_x \\ 0 & -I & C & L_y \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \end{pmatrix} + \begin{bmatrix} B \\ D \\ B \\ D \end{bmatrix} u(t). \quad (3.5)$$

Now considering the observation error  $e(t) = z(t) - x(t)$  and multiplying (3.5) from the left with

$$W = \begin{bmatrix} I_l & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ -I_l & 0 & I_l & 0 \\ 0 & -I_p & 0 & I_p \end{bmatrix},$$

we obtain

$$\frac{d}{dt} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ e(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ C & -I & 0 & 0 \\ 0 & 0 & A & L_x \\ 0 & 0 & C & L_y \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ e(t) \\ d(t) \end{pmatrix} + \begin{bmatrix} B \\ D \\ 0 \\ 0 \end{bmatrix} u(t). \quad (3.6)$$

In particular, the error satisfies the DAE

$$\frac{d}{dt} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & L_x \\ C & L_y \end{bmatrix} \begin{pmatrix} e(t) \\ d(t) \end{pmatrix}. \quad (3.7)$$

**THEOREM 3.4.** Consider the system  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  and let  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$  be such that

$$\text{rk} \begin{bmatrix} L_x \\ L_y \end{bmatrix} = k. \quad (3.8)$$

Then we have the following for the system  $[E_o, A_o, B_o, C_o, D_o]$  as in (3.4):

- a)  $[E_o, A_o, B_o, C_o, D_o]$  is an acceptor for  $[E, A, B, C, D]$ .  
b)  $[E_o, A_o, B_o, C_o, D_o]$  is an observer for  $[E, A, B, C, D]$  if, and only if,

$$\text{rk}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A & L_x \\ C & L_y \end{bmatrix} = n + k. \quad (3.9)$$

- c)  $[E_o, A_o, B_o, C_o, D_o]$  is an asymptotic observer for  $[E, A, B, C, D]$  if, and only if,

$$\text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & L_x \\ C & L_y \end{bmatrix} = n + k \quad \forall \lambda \in \overline{\mathbb{C}_+}. \quad (3.10)$$

- d)  $[E_o, A_o, B_o, C_o, D_o]$  is an exact observer for  $[E, A, B, C, D]$  if, and only if,

$$\text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & L_x \\ C & L_y \end{bmatrix} = n + k \quad \forall \lambda \in \mathbb{C}. \quad (3.11)$$

*Proof.*

- a) The system  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is an acceptor for  $[E, A, B, C, D]$ , since for all  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  we have

$$\left( \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix}, x \right) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}.$$

- b)  $\Rightarrow$ : Suppose that  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is an observer for  $[E, A, B, C, D]$ . Consider a solution  $\begin{pmatrix} e \\ d \end{pmatrix}$  of the DAE (3.7) with

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} e(0) \\ d(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.12)$$

By (3.5) and (3.6), we have

$$(0, 0, 0) \in \mathfrak{B}_{[E, A, B, C, D]} \quad \wedge \quad \left( \begin{pmatrix} e \\ d \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, e \right) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}. \quad (3.13)$$

The definition of an observer together with (3.12) implies  $e \stackrel{\text{a.e.}}{=} 0$ . Then we obtain from (3.7) and (3.8) that  $d \stackrel{\text{a.e.}}{=} 0$ . We may now conclude that the DAE (3.7) is autonomous, and hence we may infer from Lemma 2.3 that (3.9) holds.

$\Leftarrow$ : Assume that (3.9) is satisfied and consider  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and  $\left( \begin{pmatrix} z \\ d \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix}, z \right) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}$  with  $Ez(0) = Ex(0)$ . Then the definition of the observation error leads to  $Ee(0) = 0$  and thus  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} e(0) \\ d(0) \end{pmatrix} = 0$ . Again using Lemma 2.3, the assumption (3.9) gives autonomy of the DAE (3.7). Then it follows from (2.2) that  $e \stackrel{\text{a.e.}}{=} 0$  or, equivalently,  $x \stackrel{\text{a.e.}}{=} z$ . This means that  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is an observer for  $[E, A, B, C, D]$ .

- c)  $\Rightarrow$ : Assume that  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is an asymptotic observer for  $[E, A, B, C, D]$ . Consider a solution  $\begin{pmatrix} e \\ d \end{pmatrix}$  of the DAE (3.7). Then the relations in (3.13) again hold true. The definition of an asymptotic observer gives  $\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|e\| = 0$ . Hence, for all solutions  $\begin{pmatrix} e \\ d \end{pmatrix}$  of (3.7) we have

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \left\| \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} \right\| = 0.$$

Since (3.7) is furthermore autonomous by b), it follows from [11, Cor. 5.1] that (3.7) is behaviorally stable. Then we obtain from Lemma 2.3 that (3.10) holds true.

$\Leftarrow$ : Now assume that (3.10) is satisfied and consider  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and  $((\begin{smallmatrix} z \\ d \end{smallmatrix}), (\begin{smallmatrix} u \\ y \end{smallmatrix}), z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}$ . By (3.6),  $(\begin{smallmatrix} e \\ d \end{smallmatrix})$  satisfies (3.7). Using Lemma 2.3, we see that (3.10) implies

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|(\begin{smallmatrix} e \\ d \end{smallmatrix})\| = 0.$$

The system  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is therefore an asymptotic observer for  $[E, A, B, C, D]$ .

d)  $\Rightarrow$ : Assume that  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is an exact observer for  $[E, A, B, C, D]$ . Consider a solution  $(\begin{smallmatrix} e \\ d \end{smallmatrix})$  of the DAE (3.7). Then the relations in (3.13) again hold true. The definition of an exact observer yields  $e \stackrel{\text{a.e.}}{=} 0$ , whence, by (3.7) and (3.8), we have  $d \stackrel{\text{a.e.}}{=} 0$ . Hence, the solutions of the DAE (3.7) vanish almost everywhere, and we obtain from Lemma 2.3 that (3.11) holds true.

$\Leftarrow$ : Now assume that (3.11) is satisfied and consider  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and  $((\begin{smallmatrix} z \\ d \end{smallmatrix}), (\begin{smallmatrix} u \\ y \end{smallmatrix}), z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}$ . By (3.6),  $(\begin{smallmatrix} e \\ d \end{smallmatrix})$  satisfies (3.7). Lemma 2.3 together with (3.11) then implies in particular that  $e \stackrel{\text{a.e.}}{=} 0$ , i.e.,  $x \stackrel{\text{a.e.}}{=} z$ . In other words,  $[E_o, A_o, B_o, C_o, D_o]$  in (3.4) is an exact observer for  $[E, A, B, C, D]$ .  $\square$

Note that the properties (3.9)–(3.11) in Theorem 3.4 are related to the so called *zero dynamics* of the system  $[E, A, L_x, C, L_y] \in \Sigma_{n, l, k, p}$  (see [6–9] for linear DAEs). It is shown in [6] that (using the terminology of [6])

(3.9)  $\iff$  the zero dynamics of  $[E, A, L_x, C, L_y]$  are autonomous,

(3.10)  $\iff$  the zero dynamics of  $[E, A, L_x, C, L_y]$  are asymptotically stable,

(3.11)  $\iff$  the zero dynamics of  $[E, A, L_x, C, L_y]$  are trivial.

**3.3. Existence of observers.** Here we show that the special observer  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  in (3.4) has a universal property in a certain sense: If an (asymptotic, exact) observer exists, then it can be constructed to be of the form (3.4).

**THEOREM 3.5** (Characterization of existence of observers). *For  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$  the following holds true:*

a) *There exists an observer for  $[E, A, B, C, D]$  if, and only if,*

$$\text{rk}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A \\ C \end{bmatrix} = n. \quad (3.14)$$

b) *There exists an asymptotic observer for  $[E, A, B, C, D]$  if, and only if,*

$$\text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A \\ C \end{bmatrix} = n \quad \forall \lambda \in \overline{\mathbb{C}_+}. \quad (3.15)$$

c) *There exists an exact observer for  $[E, A, B, C, D]$  if, and only if,*

$$\text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}. \quad (3.16)$$

*Proof.* We start with proving “ $\Leftarrow$ ” for a), b) and c) together: Consider the acceptor  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  in (3.4) with  $k = 0$ ,  $L_x = 0_{l,0}$  and  $L_y = 0_{p,0}$ . Then, by Theorem 3.4 a) (resp. b), c)),  $[E_o, A_o, B_o, C_o, D_o]$  is an (asymptotic, exact) observer, if (3.14) (resp. (3.15), (3.16)) hold true.

It remains to prove “ $\Rightarrow$ ” for a), b) and c):

- a) Suppose that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m+p, n}$  is an observer for  $[E, A, B, C, D]$ . Consider  $x \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$  with  $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ ,  $Ex(0) = 0$  and

$$\frac{d}{dt} \begin{bmatrix} E \\ 0 \end{bmatrix} x = \begin{bmatrix} A \\ C \end{bmatrix} x. \quad (3.17)$$

Then  $(x, 0, 0) \in \mathfrak{B}_{[E, A, B, C, D]}$  and

$$(0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}. \quad (3.18)$$

Since  $[E_o, A_o, B_o, C_o, D_o]$  is an observer for  $[E, A, B, C, D]$  we obtain  $x \stackrel{\text{a.e.}}{=} 0$ . This proves that (3.17) is an autonomous DAE, whence Lemma 2.3 yields (3.14).

- b) Suppose that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m+p, n}$  is an asymptotic observer for  $[E, A, B, C, D]$ . Consider  $x \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$  with  $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$  which satisfies (3.17). Then  $(x, 0, 0) \in \mathfrak{B}_{[E, A, B, C, D]}$ . Again consider the trivial trajectory (3.18) of the observer. The assumption that  $[E_o, A_o, B_o, C_o, D_o]$  is an asymptotic observer leads to

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|x - 0\| = 0.$$

This shows that the DAE (3.17) is behaviorally stable. Then Lemma 2.3 implies (3.15).

- c) Suppose that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m+p, n}$  is an exact observer for  $[E, A, B, C, D]$ . Consider  $x \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$  with  $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$  which satisfies (3.17). Again we have  $(x, 0, 0) \in \mathfrak{B}_{[E, A, B, C, D]}$ , and we can consider the trivial trajectory (3.18) of the observer. Now using the assumption that  $[E_o, A_o, B_o, C_o, D_o]$  is an exact observer, we obtain  $x \stackrel{\text{a.e.}}{=} 0$ . This shows that all solutions of the DAE (3.17) vanish almost everywhere. Then we obtain from Lemma 2.3 that (3.16) holds true.  $\square$

Note that condition (3.14) is equivalent to  $\left[ \begin{bmatrix} E \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix} \right] \in \Sigma_{l+p, n}$  being autonomous, condition (3.15) is equivalent to  $[E, A, B, C, D]$  being behaviorally detectable and condition (3.16) is equivalent to  $[E, A, B, C, D]$  being behaviorally observable.

**REMARK 3.6 (Observers II).** *Recall from Remark 3.3 c) that the observer definition in [53] is slightly different from ours. Namely, it is characterized by (3.2) in the case where both the plant and observer behavior are represented by DAEs. As stated in Remark 3.3 c), an observer according to our Definition 3.2 is an observer according to [53, Def. 3.1]. Here we state that also the converse is true for the observer  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4): If  $Ee(0) = 0$ , then the autonomy of the DAE (3.7) implies that  $e \stackrel{\text{a.e.}}{=} 0$  (in the case of an observer according to [53, Def. 3.1], autonomy of (3.7) can be shown similar to the proof of Theorem 3.4 b)) and, in particular,  $e|_{(-\infty, 0]} \stackrel{\text{a.e.}}{=} 0$ . The general reason is that, for an autonomous DAE, an initial state completely describes the future behavior. This is no longer true for the behavior systems treated in [46, 53], since these are described by differential equations of possibly higher order.*

*Further note that our criteria for existence of (asymptotic, exact) DAE observers are equivalent to those obtained for behaviors in [53, Prop. 3.2].*

**3.4. Regular and freely initializable observers.** Though the name suggests, an exact observer is not ideal from a practical point of view: The typical situation is that an observer will be turned on at an initial moment. If  $[E_o, A_o, B_o, C_o, D_o]$  is an exact observer for  $[E, A, B, C, D]$ , then a consistent initialization of  $[E_o, A_o, B_o, C_o, D_o]$  requires the full information about the initial value of  $[E, A, B, C, D]$ . As a consequence, we have a certain redundancy

in the observation problem: The goal of an observer is to approximate the state trajectory  $x$  of  $[E, A, B, C, D]$  by means of  $u$  and  $y$ . On the other hand, by a combination of Lemma 2.3 and Theorem 3.5 a), the state trajectory is, in case of existence of an observer, completely determined by  $u$ ,  $y$  and  $E_x(0)$ . That is, initialization of an exact observer already consists of the problem that needs to be solved by the observer itself. In terms of the picturesque explanation in [46, p. 351] (see also p. 6), there is no space for innovations at all. An exact observer needs to have a complete picture of his/her friend's mood already at the beginning!

Another problem in the construction of the exact observer in the proof of Theorem 3.5 (i.e.,  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  in (3.4) with  $k=0$ ,  $L_x = 0_{l,0}$  and  $L_y = 0_{p,0}$ ) concerns robustness issues:  $[E_o, A_o, B_o, C_o, D_o]$  is not anymore an acceptor if the system  $[E, A, B, C, D]$  is slightly perturbed (in terms of the explanation in [46, p. 351], this may be a slightly false estimation of the character of the friend).

The above findings lead to the wish for a design of observers whose initialization is not influenced by the initial state of  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  at all.

**DEFINITION 3.7** (Regular/freely initializable observer). *Let a system  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$  be given and let  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{\bar{l}, \bar{n}, m+p, n}$  be an observer for  $[E, A, B, C, D]$ . Then we call  $[E_o, A_o, B_o, C_o, D_o]$*

- a) regular, if  $\bar{l} = \bar{n}$  and  $sE_o - A_o$  is regular;
- b) freely initializable, if for all  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and  $x_o^0 \in \mathbb{R}^{\bar{n}}$  there exist  $x_o \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^{\bar{n}})$ ,  $z \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$  such that

$$(x_o, \begin{pmatrix} u \\ y \end{pmatrix}, z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \quad \text{and} \quad E_o x_o(0) = E_o x_o^0.$$

In order to study the above concepts we need to introduce the notion of an index of a regular matrix pencil: The index  $\nu \in \mathbb{N}_0$  of a regular matrix pencil  $sE - A \in \mathbb{R}[s]^{n \times n}$  is defined via its (quasi-)Weierstraß form [10, 38, 40]: if for some  $S, T \in \mathbf{GL}_n(\mathbb{R})$

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix}, \quad \text{then } \nu := \begin{cases} 0, & \text{if } r = n, \\ \min \{ k \in \mathbb{N} \mid N^k = 0 \}, & \text{if } r < n, \end{cases}$$

where  $N$  is nilpotent. The index is independent of the choice of  $S, T$  and can be computed via the Wong sequences corresponding to  $sE - A$  as shown in [10].

Next we give sufficient conditions for the existence of regular and and freely initializable observers. In particular, it will turn out that an observer exists if, and and only if, a regular observer exists.

**THEOREM 3.8** (Existence of regular and freely initializable observers). *Let a system  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$  be given.*

- a) *If (3.14) holds true (equivalently, an observer exists, see Theorem 3.5 a)), then there exist  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$  such that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is a regular observer for  $[E, A, B, C, D]$ .*
- b) *If (3.15) holds true (equivalently, an asymptotic observer exists, see Theorem 3.5 b)), then there exist  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$  such that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is a regular asymptotic observer for  $[E, A, B, C, D]$ .*
- c) *If  $[E, A, B, C, D]$  is impulse observable, then there exist  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$  such that the pencil*

$$\begin{bmatrix} -sE + A & L_x \\ C & L_y \end{bmatrix} \tag{3.19}$$

*is square, regular and its index is at most one.*

*In this case, the observer  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is regular and freely initializable.*

d) If  $[E, A, B, C, D]$  is strongly detectable, then there exist  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$  such that the pencil (3.19) is square, regular, its index is at most one and it satisfies (3.10).

In this case,  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is a regular and freely initializable asymptotic observer for  $[E, A, B, C, D]$ .

*Proof.*

a) We show the existence of  $k \in \mathbb{N}_0$ ,  $L_x \in \mathbb{R}^{l \times k}$  and  $L_y \in \mathbb{R}^{p \times k}$  such that the pencil in (3.19) is square and regular. For the proof we introduce the following notation: For  $j \in \mathbb{N}$  let

$$N_j = \begin{bmatrix} 0 & & \\ & \parallel & \\ & & 1 \end{bmatrix} \in \mathbb{R}^{j \times j}, \quad K_j = \begin{bmatrix} 1 & 0 & & \\ & \parallel & & \\ & & & 1 \end{bmatrix}, \quad L_j = \begin{bmatrix} 0 & 1 & & \\ & & \parallel & \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{(j-1) \times j}.$$

Further, let  $e_i^{[j]} \in \mathbb{R}^j$  be the  $i$ th canonical unit vector, and, for some multi-index  $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ , we define

$$\begin{aligned} N_\alpha &= \text{diag}(N_{\alpha_1}, \dots, N_{\alpha_r}) \in \mathbb{R}^{|\alpha| \times |\alpha|}, \\ K_\alpha &= \text{diag}(K_{\alpha_1}, \dots, K_{\alpha_r}) \in \mathbb{R}^{(|\alpha| - \ell(\alpha)) \times |\alpha|}, \\ L_\alpha &= \text{diag}(L_{\alpha_1}, \dots, L_{\alpha_r}) \in \mathbb{R}^{(|\alpha| - \ell(\alpha)) \times |\alpha|}, \\ E_\alpha &= \text{diag}(e_{\alpha_1}^{[\alpha_1]}, \dots, e_{\alpha_r}^{[\alpha_r]}) \in \mathbb{R}^{|\alpha| \times \ell(\alpha)}. \end{aligned}$$

By [13, Thm. 4.4] there exist  $S \in \mathbf{GL}_l(\mathbb{R})$ ,  $T \in \mathbf{GL}_n(\mathbb{R})$ ,  $V \in \mathbf{GL}_p(\mathbb{R})$ ,  $L \in \mathbb{R}^{l \times p}$  such that

$$[SET, SAT - LCT, VCT] = \begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa^\top & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_\sigma} \end{bmatrix}, \begin{bmatrix} N_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_\sigma \end{bmatrix}, \begin{bmatrix} E_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (3.20)$$

for some multi-indices  $\alpha, \beta, \gamma, \varepsilon, \kappa$  and a matrix  $A_\sigma \in \mathbb{R}^{n_\sigma \times n_\sigma}$ .

Choosing  $F_\alpha = 0$ , the system

$$\frac{d}{dt} \begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix} z(t) = \begin{bmatrix} N_\alpha & F_\alpha^\top \\ E_\alpha^\top & -I_{\ell(\alpha)} \end{bmatrix} z(t) \quad (3.21)$$

is clearly regular and has index at most one. Furthermore, for

$$a_j = [a_{j0}, \dots, a_{j\beta_j-2}, 1]^\top \in \mathbb{R}^{\beta_j}$$

with the property that the polynomials

$$p_j(s) = s^{\beta_j} + a_{j\beta_j-1}s^{\beta_j-1} + \dots + a_{j0} \in \mathbb{R}[s]$$

are Hurwitz for  $j = 1, \dots, \ell(\beta)$ , the choice

$$B_\beta = \text{diag}(a_1, \dots, a_{\ell(\beta)}) \in \mathbb{R}^{|\beta| \times \ell(\beta)}$$

leads to the system

$$\frac{d}{dt} [K_\beta^\top, 0] \begin{pmatrix} z(t) \\ u(t) \end{pmatrix} = [L_\beta^\top, B_\beta] \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}. \quad (3.22)$$

We see that the input  $u$  is uniquely determined by  $u = -E_{\beta-1}^\top z$ , where  $\beta - 1 = (\beta_1 - 1, \dots, \beta_{\ell(\beta)} - 1)$  and if  $\beta_j = 1$  for some  $j$ , then the respective  $x$ -component does not exist and the equation simply reads  $u_j = 0$ . With  $B_{\beta-1} = \text{diag}(\tilde{a}_1, \dots, \tilde{a}_{\ell(\beta)})$ , where  $\tilde{a}_j = [a_{j0}, \dots, a_{j\beta_j-2}]^\top$ , a permutation of rows in (3.22) and insertion of  $u$  gives

$$\begin{aligned} \dot{z}(t) &= (N_{\beta-1} - B_{\beta-1}E_{\beta-1}^\top)z(t), \\ u(t) &= E_{\beta-1}^\top z(t). \end{aligned}$$

It is now clear, that the pencil  $s[K_\beta^\top, 0] - [L_\beta^\top, B_\beta]$  in system (3.22) is regular and has index at most one. Furthermore, the characteristic polynomial of  $N_{\beta-1} + B_{\beta-1}E_{\beta-1}^\top$  (which is a block diagonalization of companion matrices) is given by

$$\det(sI - (N_{\beta-1} + B_{\beta-1}E_{\beta-1}^\top)) = \prod_{j=1}^{\ell(\beta)} p_j(s),$$

which is Hurwitz, since all  $p_j(s)$  are Hurwitz. Therefore, (3.22) is also behaviorally stable.

Now, assumption (3.14) implies  $\ell(\varepsilon) = 0$  and hence the choice

$$\begin{aligned} k &= \ell(\alpha) + \ell(\beta) + (p - \ell(\alpha) - \ell(\gamma)), \\ \tilde{L}_x &= \begin{bmatrix} F_\alpha^\top & 0 & 0 \\ 0 & B_\beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{l \times k}, \quad \tilde{L}_y = \begin{bmatrix} -I_{\ell(\alpha)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{p - \ell(\alpha) - \ell(\gamma)} \end{bmatrix} \in \mathbb{R}^{p \times k}, \end{aligned} \quad (3.23)$$

leads to

$$\begin{aligned} n + k &= (|\alpha| + |\beta| - \ell(\beta) + |\gamma| + |\kappa| + n_\sigma) + (\ell(\alpha) + \ell(\beta) + (p - \ell(\alpha) - \ell(\gamma))) \\ &= (|\alpha| + |\beta| + |\gamma| - \ell(\gamma) + |\kappa| + n_\sigma) + p = l + p, \end{aligned}$$

by which the pencil

$$\begin{bmatrix} -sSET + SAT - LCT & \tilde{L}_x \\ VCT & \tilde{L}_y \end{bmatrix}$$

is square and regular. Therefore, with

$$L_x = S^{-1}\tilde{L}_x + LV^{-1}\tilde{L}_y, \quad L_y = V^{-1}\tilde{L}_y, \quad (3.24)$$

the pencil (3.19) is square and regular. We can further conclude from Theorem 3.4 b) that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is an observer for  $[E, A, B, C, D]$ .

- b) If (3.15) is true, then  $[E, A, B, C, D]$  is behaviorally detectable and by [13, Cor. 9.3] we find  $\ell(\varepsilon) = 0$  and  $\sigma(A_\sigma) \subseteq \mathbb{C}_-$  in (3.20). By [51, Thm. 4.20] there exists  $F_\alpha \in \mathbb{R}^{\ell(\alpha) \times |\alpha|}$  such that, using the same notation as in a),  $\sigma(N_\alpha^\top + E_\alpha F_\alpha) \subseteq \mathbb{C}_-$ . Therefore, system (3.21) is additionally behaviorally stable. Then, using the same choice as in (3.23) and (3.24), it follows from the fact that the systems (3.21) and (3.22) are regular and behaviorally stable, that the pencil in (3.19) is square and regular and it satisfies (3.10). Then, by Theorem 3.4 c),  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is an asymptotic observer for  $[E, A, B, C, D]$ .

- c) Impulse observability implies, invoking [13, Lem. 5.1], that  $|\gamma| = \ell(\gamma)$ ,  $\ell(\varepsilon) = 0$  and  $|\kappa| = \ell(\kappa)$  in (3.20). Then, using the same choice as in (3.23) and (3.24), it follows from the fact that the systems (3.21) and (3.22) are regular and of index at most one, that the pencil in (3.19) is square, regular and its index is at most one. Next we prove that  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is a freely initializable observer for  $[E, A, B, C, D]$ :

Let  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and  $z^0 \in \mathbb{R}^n$ . Since the pencil (3.19) is square, regular and its index is at most one, there exist  $e \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$  and  $d \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^k)$  such that  $Ee \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ ,  $Ee(0) = E(z^0 - x(0))$ , and the DAE (3.7) is satisfied for almost all  $t \in \mathbb{R}$ . Now consider  $z := x + e \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$ . Then, by  $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ , we obtain  $Ez \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$  and

$$E_o \begin{pmatrix} z(0) \\ d(0) \end{pmatrix} = \begin{pmatrix} Ex(0) + Ee(0) \\ 0 \end{pmatrix} = E_o z^0.$$

By  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and (3.7) we obtain that (3.6) holds true. Hence, the DAE (3.5) is satisfied. In particular, we have

$$\left( \begin{pmatrix} z \\ d \end{pmatrix}, (y), z \right) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]}.$$

- d) Strong detectability implies, invoking [13, Cor. 9.3], that  $|\gamma| = \ell(\gamma)$ ,  $\ell(\varepsilon) = 0$ ,  $|\kappa| = \ell(\kappa)$  and  $\sigma(A_{\bar{\sigma}}) \subseteq \mathbb{C}_-$  in (3.20). As in b) we may choose  $F_{\alpha} \in \mathbb{R}^{\ell(\alpha) \times |\alpha|}$  such that  $\sigma(N_{\alpha}^{\top} + E_{\alpha} F_{\alpha}) \subseteq \mathbb{C}_-$  and hence system (3.21) is behaviorally stable. Then, using the same choice as in (3.23) and (3.24), it follows from the fact that the systems (3.21) and (3.22) are regular, of index at most one and behaviorally stable, that the pencil in (3.19) is square, regular, its index is at most one and it satisfies (3.10). Then, by Theorem 3.4 c),  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p, n+k, m+p, n}$  as in (3.4) is an asymptotic observer for  $[E, A, B, C, D]$ . Regularity and free initializability of  $[E_o, A_o, B_o, C_o, D_o]$  follow from c). □

Note that it is an open problem as to whether the converse implications in Theorem 3.8 c) and d) hold true.

REMARK 3.9 (Regular observers for square systems). *Let a system  $[E, A, B, C, D] \in \Sigma_{n, n, m, p}$  be given.*

- a) *If  $sE - A \in \mathbb{R}[s]^{n \times n}$  is regular, then  $[E, A, B, C, D]$  has property (3.14), whence a regular observer exists by Theorem 3.8 a).*

- b) *By [13, Thm. 9.8] the following holds true:*

- (i) *If  $[E, A, B, C, D]$  is impulse observable, then there exists some  $L \in \mathbb{R}^{n \times p}$  such that  $sE - (A + LC)$  is regular and its index is at most one.*
- (ii) *If  $[E, A, B, C, D]$  is behaviorally detectable, then there exists some  $L \in \mathbb{R}^{n \times p}$  such that  $sE - (A + LC)$  is regular and  $[E, A + LC]$  is behaviorally stable.*
- (iii) *If  $[E, A, B, C, D]$  is strongly detectable, then there exists some  $L \in \mathbb{R}^{n \times p}$  such that  $sE - (A + LC)$  is regular, its index is at most one and  $[E, A + LC]$  is behaviorally stable.*

As a consequence, if  $sE - A$  is square, we may conclude from

$$\begin{bmatrix} I_n & L \\ 0 & I_p \end{bmatrix} \begin{bmatrix} -sE + A & -L \\ C & I_p \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C & I_p \end{bmatrix} = \begin{bmatrix} -sE + A + LC & 0 \\ 0 & I_p \end{bmatrix} \quad (3.25)$$

that we can make the choice  $L_y = I_p$ ,  $L_x = -L$  for the matrices in Theorem 3.8 and (3.19) is square and regular. Therefore, we have

$$d(t) = Cz(t) + Du(t) - y(t)$$

in the observer realization (3.3). Inserting this into the first equation in (3.3) we can eliminate the auxiliary variable  $d$ , and we obtain

$$\frac{d}{dt}Ez(t) = (A + LC)z(t) + (B + LD)u(t) - Ly(t) \quad (3.26)$$

Hence, we find that regular (asymptotic, freely initializable) observers for square systems can always be chosen of the form (3.26), i.e.,

$$[E_o, A_o, B_o, C_o, D_o] = [E, A + LC, [B + LD, -L], I_n, 0_{n,m+p}]. \quad (3.27)$$

**3.5. Notes and references.** Observers for differential-algebraic systems have been considered in various publications. The existing results (as well as ours) all rely on the principal idea by Luenberger in the seminal works [43, 44] for systems governed by ODEs. It has been first observed by Dai and Wang [54, 55] that the classical Luenberger observer straightforwardly generalizes to DAE systems  $[E, A, B, C, D] \in \Sigma_{n,n,m,p}$  with regular  $sE - A$  (this is a special case of Remark 3.9). Further aspects of observer design for  $[E, A, B, C, D] \in \Sigma_{n,n,m,p}$  with regular  $sE - A$  have been presented in [2, 31, 33, 48, 60]. These results have been applied to models for mechanical multibody systems in [36].

The articles [30, 35] treat observer design for general DAE systems  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  with the property  $\text{rk} \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + \text{rk} E$ , i.e., impulse observable systems are considered. It has been proved in [30] that systems with this property admit observers which can be realized by ODEs. This corresponds to our result in Theorem 3.8 c), where we have proved that observers  $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o, n_o, m+p, n}$  can be constructed with  $l_o = n_o$  and a regular pencil  $sE_o - A_o$  whose index is at most one. By resolving the algebraic constraints, this observer can indeed be reformulated as an ODE.

The results for the regular case have been generalized in [17–20] to input-output systems which are governed by DAEs with variable coefficients. Observer design for classes of nonlinear DAEs has been treated in [21, 23, 29, 39, 50, 61]. In particular, the article [39] gives criteria for the existence of observers with index at most one.

Numerical aspects of observer design for DAEs are presented in [14, 15, 32, 42, 45].

**4. Dynamic controllers.** In the present section we consider the problem of *dynamic compensation*, that is, a suitable interconnection with a controller system which only uses the knowledge of the output to stabilize a given plant  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ . We will consider design of stabilizing controllers and present equivalent conditions for their existence.

**4.1. Definitions.** We use concept of control in the behavioral sense which has its origin in the works by Willems, Polderman and Trentelman [4, 46, 52, 58, 59], where differential behaviors and their stabilization via *control by interconnection* is considered. The latter means a systematic addition of some further (differential) equations such that a desired behavior is achieved, see Fig. 4.1.

Note that, if  $y = x$  one could make the extreme choice  $E_c = 0, A_c = 0, B_c = I, C_c = 0, D_c = 0$  for the controller, which would result in an interconnected system where each trajectory vanishes. This, however, is not suitable from a practical point of view, since in this interconnection, the space of consistent initial differential variables is a proper subset of the initial differential variables which are consistent with the original system  $[E, A, B, C, D]$ . Consequently, the interconnected system does not have the causality property – that is, the implementation of the controller at a certain time  $t \in \mathbb{R}$  is not possible, since this causes jumps in the differential variables. To avoid this, we introduce the concept of *compatibility*. In order to define compatibility we need to introduce the space of consistent initial differential variables for  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ :

$$\mathcal{V}_{[E,A,B,C,D]}^{\text{diff}} = \{ x^0 \in \mathbb{R}^n \mid \exists (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} : Ex(0) = Ex^0 \}.$$

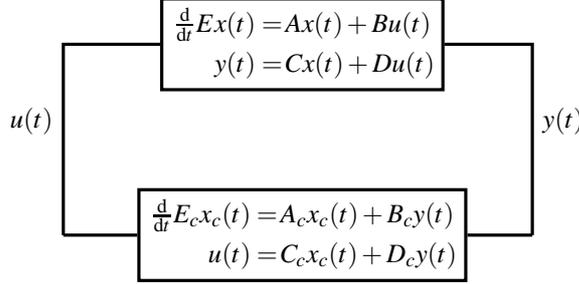


Fig. 4.1: Interconnection with a controller

**DEFINITION 4.1 (Compatible/stabilizing/freely initializable controller).** *Let a system  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  be given. Then we call a system  $[E_c, A_c, B_c, C_c, D_c] \in \Sigma_{l_c, n_c, p, m}$*

(a) *a compatible controller for  $[E, A, B, C, D]$ , if*

$$\forall x^0 \in \mathcal{Y}_{[E,A,B,C,D]}^{\text{diff}} \exists (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \exists x_c \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^{n_c}) : \\ Ex(0) = Ex^0 \wedge (x_c, y, u) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]}.$$

(b) *a stabilizing controller, if it is a compatible controller, and*

$$\forall (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \forall x_c \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^{n_c}) : \\ \left( (x_c, y, u) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]} \implies \lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \left\| \begin{pmatrix} x \\ u \\ y \\ x_c \end{pmatrix} \right\| = 0 \right).$$

(c) *a freely initializable controller, if*

$$\forall x^0 \in \mathcal{Y}_{[E,A,B,C,D]}^{\text{diff}} \forall x_c^0 \in \mathbb{R}^{n_c} \exists (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \exists x_c \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^{n_c}) : \\ Ex(0) = Ex^0 \wedge E_c x_c(0) = E_c x_c^0 \wedge (x_c, y, u) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]}.$$

Note that the above definition of compatibility is a slight modification of the concept introduced by Julius and van der Schaft in [37] where an interconnection is called compatible, if any trajectory of the system without control law can be concatenated with a trajectory of the interconnected system. This certainly implies that the space of initial differential variables of the interconnected system cannot be smaller than the corresponding set for the nominal system. The above compatibility definition also generalizes the compatibility concept introduced in [5, 11] for DAE control systems.

We like to stress that any freely initializable controller is in particular compatible.

**4.2. Controller design and existence.** In the following we show that the existence of a stabilizing controller is equivalent to behavioral stabilizability and behavioral detectability. We also investigate when a stabilizing controller is freely initializable. We start with the statement of the main result of this section.

**THEOREM 4.2 (Stabilizing controllers).** *Let  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  be given. Then there exists a stabilizing controller for  $[E, A, B, C, D]$  if, and only if,  $[E, A, B, C, D]$  is both behaviorally stabilizable and behaviorally detectable.*

*Proof of Necessity in Theorem 4.2.*

Let  $[E_c, A_c, B_c, C_c, D_c] \in \Sigma_{l_c, n_c, p, m}$  be a stabilizing controller for  $[E, A, B, C, D]$ .

*Step 1:* We prove that  $[E, A, B, C, D]$  is behaviorally stabilizable. Let  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$ , then  $x^0 \in \mathcal{V}_{[E, A, B, C, D]}^{\text{diff}}$ . By compatibility of  $[E_c, A_c, B_c, C_c, D_c]$  for  $[E, A, B, C, D]$ , there exists some  $(\tilde{x}, \tilde{u}, \tilde{y}) \in \mathfrak{B}_{[E, A, B, C, D]}$  with  $E\tilde{x}(0) = Ex(0)$  and some  $x_c \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^{n_c})$  such that  $(x_c, \tilde{y}, \tilde{u}) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]}$ . Since  $[E_c, A_c, B_c, C_c, D_c]$  is stabilizing we further obtain that

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|\tilde{x}\| = 0.$$

We have shown that

$$\forall (x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]} \exists (\tilde{x}, \tilde{u}, \tilde{y}) \in \mathfrak{B}_{[E, A, B, C, D]} : \\ E\tilde{x}(0) = Ex(0) \wedge \lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|\tilde{x}\| = 0.$$

Using the same arguments as in, for instance, [11, Rem. 3.7] it can be shown that the above property is equivalent to behavioral stabilizability of  $[E, A, B, C, D]$ .

*Step 2:* We prove that  $[E, A, B, C, D]$  is behaviorally detectable.

Let  $(x, 0, 0) \in \mathfrak{B}_{[E, A, B, C, D]}$ . Then, by using  $(0, 0, 0) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]}$  and the property that  $[E_c, A_c, B_c, C_c, D_c]$  is a stabilizing controller, we obtain

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|x\| = 0.$$

This proves that  $[E, A, B, C, D]$  is behaviorally detectable.  $\square$

The proof of sufficiency in Theorem 4.2 is based on a construction of a suitable controller for a given behaviorally stabilizable and behaviorally detectable system  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$ : If full information on the state is available (i.e.,  $C = I_n$ ), then a stabilizing controller can be constructed with  $E_c = 0$  (i.e., it is actually not dynamic anymore), see [11, Thm. 5.4]. To this end, let  $K_x \in \mathbb{R}^{l_c \times n}$  and  $K_u \in \mathbb{R}^{l_c \times m}$  be such that the DAE

$$\frac{d}{dt} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$

is behaviorally stable.

For the general case, we use the basic strategy in the classical approach [44]: We couple the plant with an observer to approximate the state. The approximate state is then used (as if it was the state) to determine an input which stabilizes the system.

More precisely, we add the static relation

$$K_x z(t) + K_u u(t) = 0 \quad (4.1)$$

to the model of the plant coupled with an asymptotic observer of the form (3.4), the output of which is the approximate state  $z$ , see Figure 4.2. Then we obtain the closed-loop system described by the DAE

$$\frac{d}{dt} \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & 0 & 0 & 0 & B \\ C & -I_p & 0 & 0 & D \\ 0 & 0 & A & L_x & B \\ 0 & -I_p & C & L_y & D \\ 0 & 0 & K_x & 0 & K_u \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \\ u(t) \end{pmatrix}. \quad (4.2)$$

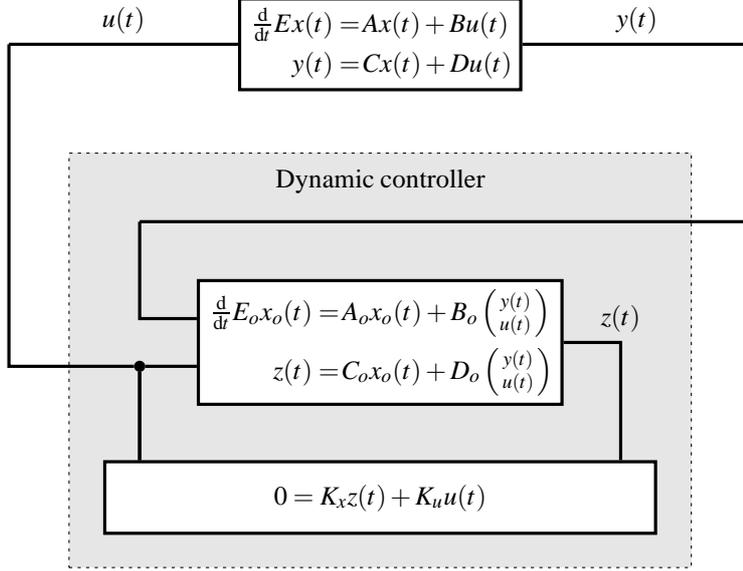


Fig. 4.2: Controller structure

By using

$$\begin{aligned}
 & \begin{bmatrix} I_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{l_c} \\ -I_l & 0 & I_l & 0 & 0 \\ 0 & -I_p & 0 & I_p & 0 \\ 0 & I_p & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -sE + A & 0 & 0 & 0 & B \\ C & -I & 0 & 0 & D \\ 0 & 0 & -sE + A & L_x & B \\ 0 & -I & C & L_y & D \\ 0 & 0 & K_x & 0 & K_u \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p \\ I_n & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 \\ 0 & I_m & 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} -sE + A & B & 0 & 0 & 0 \\ K_x & K_u & K_x & 0 & 0 \\ 0 & 0 & -sE + A & L_x & 0 \\ 0 & 0 & C & L_y & 0 \\ C & D & 0 & 0 & -I_p \end{bmatrix},
 \end{aligned}$$

we obtain that  $x, u, y, z, d$  solve (3.5) if, and only if,  $y(t) = Cx(t) + Du(t)$  and, using the observation error  $e(t) = z(t) - x(t)$ , we have

$$\frac{d}{dt} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \\ e(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & B & 0 & 0 \\ K_x & K_u & K_x & 0 \\ 0 & 0 & A & L_x \\ 0 & 0 & C & L_y \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \\ e(t) \\ d(t) \end{pmatrix}. \quad (4.3)$$

Next we analyze the properties of the previously introduced controller. To this end, for given  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ , consider

$$\begin{aligned}
 & [E_c, A_c, B_c, C_c, D_c] \\
 &= \left[ \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A & L_x & B \\ C & L_y & D \\ K_x & 0 & K_u \end{bmatrix}, \begin{bmatrix} 0 \\ -I_p \\ 0 \end{bmatrix}, [0 \ 0 \ I_m], 0_{m,p} \right] \in \Sigma_{l+p+l_c, n+k+m, p, m}. \quad (4.4)
 \end{aligned}$$

By an interconnection of this system with  $[E, A, B, C, D]$  as depicted in Figure 4.1, we see that the state of the controller contains a copy  $u_c$  of the input  $u$ . The closed-loop system is therefore described by the DAE (4.2).

In the following we analyze the properties of the controller (4.4) in terms of the following properties of  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  and the matrices  $K_x \in \mathbb{R}^{l_c \times n}$ ,  $K_u \in \mathbb{R}^{l_c \times m}$ ,  $L_x \in \mathbb{R}^{n \times k}$ ,  $L_y \in \mathbb{R}^{p \times k}$ .

- (C1)  $[0, K_u, K_x, I_m, 0]$  is a compatible controller for  $[E, A, B, I_n, 0]$ .
- (C2) The DAE  $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}$  is behaviorally stable.
- (C3) For all  $f \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^l)$ ,  $x^0 \in \mathcal{V}_{[E,A,B,C,D]}^{\text{diff}}$ , there exist  $x \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^n)$ ,  $u \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^m)$  with  $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ ,  $Ex(0) = Ex^0$  and, for almost all  $t \in \mathbb{R}$ ,

$$\frac{d}{dt} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

- (C4)  $[E_o, A_o, B_o, C_o, D_o]$  as in (3.4) (for some  $k \in \mathbb{N}_0$ ) with (3.8) is an asymptotic observer for  $[E, A, B, C, D]$ .
- (C5)  $[E_o, A_o, B_o, C_o, D_o]$  as in (3.4) (for some  $k \in \mathbb{N}_0$ ) with (3.8) is a freely initializable observer for  $[E, A, B, C, D]$ .

Note that (C4) is equivalent to behavioral detectability of  $[E, A, B, C, D]$  and (C5) implies strong detectability of  $[E, A, B, C, D]$ .

Before we present the main result on properties of the controller (4.4), we show that for a behaviorally stabilizable system we can always find  $K_x \in \mathbb{R}^{l_c \times n}$ ,  $K_u \in \mathbb{R}^{l_c \times m}$  with the properties (C1)–(C3).

LEMMA 4.3. *Let  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  be behaviorally stabilizable. Then there exist  $l_c \in \mathbb{N}_0$ ,  $K_x \in \mathbb{R}^{l_c \times n}$ ,  $K_u \in \mathbb{R}^{l_c \times m}$  with the properties (C1), (C2) and (C3).*

*Proof.* The properties (C1) and (C2) are an immediate consequence of [5, Thm. 3.4.10]. Property (C3) follows from the fact that in the construction used in the proof of [5, Thm. 3.4.10] the inhomogeneity  $f$  is only applied to a regular subsystem, thus solutions exist for all such inhomogeneities and all consistent initial values.  $\square$

THEOREM 4.4. *Let  $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$  and let  $k, l_c \in \mathbb{N}_0$  and  $K_x \in \mathbb{R}^{l_c \times n}$ ,  $K_u \in \mathbb{R}^{l_c \times m}$ ,  $L_x \in \mathbb{R}^{n \times k}$ ,  $L_y \in \mathbb{R}^{p \times k}$ . Then we have the following for  $[E_c, A_c, B_c, C_c, D_c]$  as in (4.4):*

- a) *If (C1) holds, then  $[E_c, A_c, B_c, C_c, D_c]$  is a compatible controller for  $[E, A, B, C, D]$ .*
- b) *If (C1), (C2) and (C4) hold, then  $[E_c, A_c, B_c, C_c, D_c]$  is a stabilizing controller for  $[E, A, B, C, D]$ .*
- c) *If (C1)–(C5) hold, then  $[E_c, A_c, B_c, C_c, D_c]$  is a freely initializable stabilizing controller for  $[E, A, B, C, D]$ .*

*Proof.*

- a) Assume that  $x^0 \in \mathcal{V}_{[E,A,B,C,D]}^{\text{diff}}$ . Then by (C1) there exists  $(x, u) \in \mathfrak{B} \left[ \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \right]$  with  $Ex(0) = Ex^0$ . Then we obtain that the DAE (4.2) is satisfied for  $y = Cx + Du$ ,  $z = x$  and  $d = 0$ . Therefore, we have  $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ ,  $Ex(0) = Ex^0$ , and  $(x_c, y, u) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]}$  for  $x_c = \begin{pmatrix} x \\ 0 \end{pmatrix}$ . This shows compatibility of the controller  $[E_c, A_c, B_c, C_c, D_c]$ .

- b) Assume that (C1), (C2) and (C4) hold true. Compatibility of  $[E_c, A_c, B_c, C_c, D_c]$  is a consequence of statement a). Next we show, using (C2) and (C4), that the closed-loop system is behaviorally stable. By (C2) together with Lemma 2.3 and (C4) together with Theorem 3.4 c), we have

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \quad \text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B \\ K_x & K_u \end{bmatrix} = n + m \wedge \text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & L_x \\ C & L_y \end{bmatrix} = n + k,$$

thus

$$\forall \lambda \in \overline{\mathbb{C}}_+ : \quad \text{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B & 0 & 0 \\ K_x & K_u & K_x & 0 \\ 0 & 0 & -\lambda E + A & L_x \\ 0 & 0 & C & L_y \end{bmatrix} = 2n + m + k.$$

Then Lemma 2.3 implies that the DAE (4.3) is behaviorally stable. Now, let  $(x, u, y) \in \mathfrak{B}_{[E, A, B, C, D]}$  and  $x_c \in \mathcal{L}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^{n_c})$  such that  $(x_c, y, u) \in \mathfrak{B}_{[E_c, A_c, B_c, C_c, D_c]}$ . Write  $x_c = \begin{pmatrix} z \\ d \\ u \end{pmatrix}$  according to the decomposition in (4.4). Then, for  $e = z - x$ ,  $(x, u, d, e)$  solves (4.3) and hence, by behavioral stability,

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \left\| \begin{pmatrix} x \\ u \\ d \\ e \end{pmatrix} \right\| = 0.$$

Therefore, invoking also  $y = Cx + Du$ , we further find

$$\lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|y\| = 0 \quad \wedge \quad \lim_{t \rightarrow \infty} \text{ess sup}_{[t, \infty)} \|x_c\| = 0.$$

This shows that  $[E_c, A_c, B_c, C_c, D_c]$  is a stabilizing controller.

- c) Assume that (C1)–(C5) hold true. By b), we obtain that  $[E_c, A_c, B_c, C_c, D_c]$  is a stabilizing controller for  $[E, A, B, C, D]$ . To prove statement c), it therefore suffices to show that for all  $x^0 \in \mathcal{V}_{[E, A, B]}^{\text{diff}}$  and  $z^0 \in \mathbb{R}^n$ , there exists a solution of the DAE (4.2) with  $Ex(0) = x^0$  and  $Ez(0) = Ez^0$ .

Assume that  $x^0 \in \mathcal{V}_{[E, A, B]}^{\text{diff}}$  and  $z^0 \in \mathbb{R}^n$ . By (C5),  $[E_o, A_o, B_o, C_o, D_o]$  as in (3.4) is a freely initializable observer for  $[E, A, B, C, D]$ . Thus, by an application of the definition of freely initializable observers to the trivial trajectory  $(0, 0, 0) \in \mathcal{B}_{[E, A, B, C, D]}$ , we obtain that there exists some solution  $(e, d)$  of the DAE (3.7) with  $Ee(0) = E(z^0 - x^0)$ . Moreover, by (C3), there exists a solution  $(x, u)$  of the DAE

$$\frac{d}{dt} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} 0 \\ K_x e(t) \end{pmatrix}, \quad Ex(0) = Ex^0.$$

Hence, the DAE (4.3) is solved by  $(x, u, e, d)$ . Therefore,  $x$ ,  $u$ ,  $z = x + e$ ,  $d$  and  $y = Cx + Du$  satisfy (4.2) with  $Ex(0) = Ex^0$  and

$$Ez(0) = Ex(0) + Ee(0) = Ex^0 + E(z^0 - x^0) = Ez^0.$$

This proves the desired result.  $\square$

We are now in the position to finish the proof of Theorem 4.2.

*Proof of Sufficiency in Theorem 4.2.*

The assertion follows from Theorem 4.4 b) together with Lemma 4.3 and Theorem 3.4 c).  $\square$

REMARK 4.5 (Controllers). *Assume that  $[E, A, B, C, D] \in \Sigma_{l, n, m, p}$  is behaviorally stabilizable. By the Kalman decomposition from [13, Thm. 13.1] (see also [3]) there exist  $W \in \mathbf{GL}_l(\mathbb{R})$ ,  $T \in \mathbf{GL}_n(\mathbb{R})$  such that*

$$W(sE - A)T = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ 0 & sE_{22} - A_{22} \end{bmatrix}, \quad WB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT = [0 \quad C_2],$$

where  $[E_{22}, A_{22}, B_2, C_2, D] \in \Sigma_{l_0, n_0, m, p}$  is completely observable (in the sense of [13]) and behaviorally stabilizable, and  $\text{rk} \begin{bmatrix} E_{22} \\ C_2 \end{bmatrix} = n_0$ . Therefore,  $[E_{22}, A_{22}, B_2, C_2, D]$  is in particular strongly detectable. From Theorem 3.8 d), Lemma 4.3 and Theorem 4.4 c) we may then conclude that there exists a freely initializable stabilizing controller  $[E_{c2}, A_{c2}, B_{c2}, C_{c2}, D_{c2}] \in \Sigma_{l_{c2}, n_{c2}, p, m}$  for  $[E_{22}, A_{22}, B_2, C_2, D]$ .

**REMARK 4.6** (Controllers for regular systems). *Let a behaviorally stabilizable and behaviorally detectable system  $[E, A, B, C, D] \in \Sigma_{n, n, m, p}$  be given such that  $sE - A \in \mathbb{R}[s]^{n \times n}$  is regular. Then, by Remark 3.9, we can make the choice  $L_y = I_p$ ,  $L_x = -L$  in the observer realization (3.4). Elimination of the variable  $d$  gives rise to an asymptotic observer (3.26). If  $[E, A, B, C, D]$  is additionally impulse observable, then, invoking Remark 3.9,  $L$  can be chosen such that (3.26) is a regular and freely initializable observer.*

*By regularity of  $sE - A$  and behavioral stabilizability of  $[E, A, B, C, D]$ , there exists some  $F \in \mathbb{R}^{m \times n}$  such that  $\text{rk}_{\mathbb{C}}(\lambda E - (A + BF)) = n$  for all  $\lambda \in \overline{\mathbb{C}}_+$  (in particular,  $sE - (A + BF)$  is regular). By using*

$$\begin{bmatrix} I_n & -B \\ 0 & I_p \end{bmatrix} \begin{bmatrix} -sE + A & B \\ -F & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F & I_p \end{bmatrix} = \begin{bmatrix} -sE + A + BF & 0 \\ 0 & I_p \end{bmatrix}$$

together with regularity of  $sE - (A + BF)$ , we now obtain that (C1)–(C3) hold true for  $K_x = -F$  and  $K_u = I_m$ . In other words, we add the feedback relation  $u(t) = Fz(t)$  to the observer (3.26), see Figure 4.3.

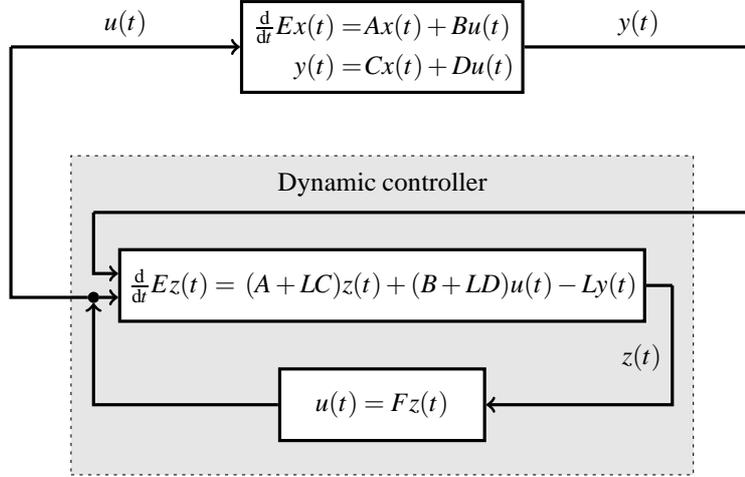


Fig. 4.3: Controller structure in the regular case

Altogether, this means that  $u(t) = Fz(t)$  and

$$d(t) = Cz(t) + Du(t) - y(t) = (C + DF)z(t) - y(t),$$

thus we can eliminate the variables  $d$  and the copy of the input in the controller realization (4.4), to obtain the following simplified controller realization

$$\begin{aligned} \frac{d}{dt}Ez(t) &= (A + LC + BF + LDF)z(t) - Ly(t), \\ u(t) &= Fz(t), \end{aligned}$$

see Figure 4.4.

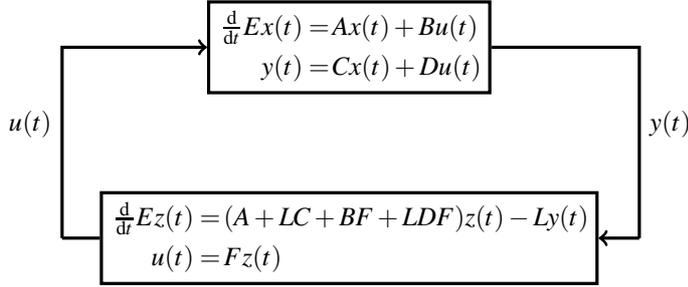


Fig. 4.4: Controller for regular systems

**4.3. Notes and references.** Dynamic controllers are also called *dynamic compensators* or *output regulators* in literature. Research on generalization of Luenberger's ideas for ODEs to the DAE case started in the 1980's.

Dai and Wang used the following approach for strongly stabilizable and strongly detectable systems  $[E, A, B, C, 0] \in \Sigma_{n,n,m,p}$  with regular  $sE - A \in \mathbb{R}[s]^{n \times n}$  (see [25, 26, 56, 57]): First, it is shown that there exists a proportional output feedback  $u(t) = Ky(t) + v(t)$  for some  $K \in \mathbb{R}^{m \times p}$ , such that for the closed-loop system

$$[E_K, A_K, B_K, C_K, D_K] = [E, A + BKC, B, C, 0]$$

we have that  $sE_K - A_K$  is regular and its index is at most one. Thereafter, a realization of this system by an ODE is considered and an ordinary stabilizing controller  $[I_{n_c}, A_c, B_c, 0] \in \Sigma_{n_c, n_c, p, m}$  according to Luenberger's approach is applied. A stabilizing controller is then given by  $[I_{n_c}, A_c, B_c, K]$ .

The more direct approach for regular systems as described in Remark 4.6 has been presented in [16, 16, 27, 28, 41, 49, 62].

To the authors' best knowledge, controller design for systems with singular  $sE - A$  has not been studied before.

**5. Conclusions.** In this paper we have studied existence and design of observers for linear time-invariant differential-algebraic systems which are not necessarily regular. Thereby we have followed the definition of (asymptotic, exact) observers for behavioral system from [M.E. Valcher and J.C. Willems, *IEEE Trans. Autom. Control*, 44 (1999), pp. 2297–2307]. In particular, we have been interested in existence and design of observers which are again an differential-algebraic system. Existence of such observers has been characterized in terms of behavioral detectability and observability of the to-be-observed system. Further, novel formulas for observers have been presented. We have moreover considered existence and design of (asymptotic) observers exist whose initial value can chosen arbitrarily. It turned out that a sufficient criterion for the existence of such observers is impulse observability and strong detectability, respectively.

After that, we have used our results on observers for design and existence of differential-algebraic stabilizing controllers for differential-algebraic systems. Existence of stabilizing controllers turned out to be equivalent to behavioral stabilizability and behavioral detectability of the to-be-controlled system. As well we have studied existence and design of compatible and freely initializable controllers. That is, a controller which can be interconnected to the system in arbitrary initial state and, respectively, a controller might have an arbitrary initial state.

## REFERENCES

- [1] R. A. ADAMS, *Sobolev Spaces*, no. 65 in Pure and Applied Mathematics, Academic Press, New York, London, 1975.
- [2] N. MINAMIDE N. ARII AND Y. UETAKE, *Design of observers for descriptor systems using a descriptor standard form*, Int. J. Control, 50 (1989), pp. 2141–2149.
- [3] A. BANASZUK, M. KOCIŃCKI, AND FRANK L. LEWIS, *Kalman Decomposition for Implicit Linear Systems*, IEEE Trans. Autom. Control, 37 (1992), pp. 1509–1514.
- [4] M. N. BELUR AND H. L. TRENTELMAN, *Stabilization, pole placement and regular implementability*, IEEE Trans. Autom. Control, 47 (2002), pp. 735–744.
- [5] T. BERGER, *On differential-algebraic control systems*, PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany, 2014.
- [6] T. BERGER, *Zero dynamics and funnel control of general linear differential-algebraic systems*, ESAIM Control Optim. Calc. Var., to appear, 2015.
- [7] T. BERGER, *Zero dynamics and stabilization for linear DAEs*, in Progress in Differential-Algebraic Equations, Sebastian Schöps, Andreas Bartel, Michael Günther, E. Jan W. ter Maten, and Peter C. Müller, eds., Differential-Algebraic Equations Forum, Springer-Verlag, Berlin-Heidelberg, 2014, pp. 21–45.
- [8] T. BERGER, A. ILCHMANN AND T. REIS, *Normal forms, high-gain, and funnel control for linear differential-algebraic systems*, in Control and Optimization with Differential-Algebraic Constraints, Lorenz T. Biegler, Stephen L. Campbell, and Volker Mehrmann, eds., vol. 23 of Advances in Design and Control, SIAM, Philadelphia, 2012, pp. 127–164.
- [9] T. BERGER, A. ILCHMANN AND T. REIS, *Zero dynamics and funnel control of linear differential-algebraic systems*, Math. Control Signals Syst., 24 (2012), pp. 219–263.
- [10] T. BERGER, A. ILCHMANN AND S. TRENN, *The quasi-Weierstraß form for regular matrix pencils*, Lin. Alg. Appl., 436 (2012), pp. 4052–4069.
- [11] T. BERGER AND T. REIS, *Controllability of linear differential-algebraic systems - a survey*, in Surveys in Differential-Algebraic Equations I, Achim Ilchmann and Timo Reis, eds., Differential-Algebraic Equations Forum, Springer-Verlag, Berlin-Heidelberg, 2013, pp. 1–61.
- [12] T. BERGER AND T. REIS, *Regularization of linear time-invariant differential-algebraic systems*, Syst. Control Lett., 78 (2015), pp. 40–46.
- [13] T. BERGER, T. REIS AND S. TRENN, *Observability of linear differential-algebraic systems - a survey*, to appear in Surveys in Differential-Algebraic Equations IV, Achim Ilchmann and Timo Reis, eds., Differential-Algebraic Equations Forum, Springer-Verlag, Berlin-Heidelberg, 2016.
- [14] N. BIEHN, S.L. CAMPBELL, F. DELEBECQUE, AND R. NIKOUKHAH, *Observer design for linear time varying descriptor systems: numerical algorithms*, in Proc. 37th IEEE Conf. Decis. Control, 1998, pp. 3801–3806.
- [15] N. BIEHN, S.L. CAMPBELL, R. NIKOUKHAH, AND F. DELEBECQUE, *Numerically constructible observers for linear time-varying descriptor systems*, Automatica, 37 (2001), pp. 445–452.
- [16] F. BLANCHINI, *Eigenvalue assignment via state observer for descriptor systems*, Kybernetika, 27 (1991), pp. 384–392.
- [17] K. BOBINEC, *Observer Construction for Systems of Differential Algebraic Equations using Completions*, phd dissertation, Department of Mathematics, North Carolina State University, Raleigh, NC, 2013.
- [18] K.S. BOBINEC AND S.L. CAMPBELL, *Linear differential algebraic equations and observers*, in Surveys in Differential-Algebraic Equations II, Achim Ilchmann and Timo Reis, eds., Differential-Algebraic Equations Forum, Springer-Verlag, Berlin-Heidelberg, 2014, pp. 1–67.
- [19] K. BOBINEC, S.L. CAMPBELL, AND P. KUNKEL, *Maximally reduced observers for linear time varying daes*, in Proc. 2011 IEEE Multi-Conference on Systems and Control, Denver, CO, 2011, pp. 1373–1378.
- [20] K. BOBINEC, S.L. CAMPBELL, AND P. KUNKEL, *Constructing observers for linear time varying daes*, in Proc. 51st IEEE Conf. Decis. Control, Maui, USA, 2012, pp. 5749–5754.
- [21] D. BOUTAT, G. ZHENG, L. BOUTAT-BADDAS AND M. DAROUACH, *Observers design for a class of nonlinear singular systems*, in Proc. 51st IEEE Conf. Decis. Control, Maui, USA, 2012, pp. 7407–7412.
- [22] K.E. BRENNAN, S.L. CAMPBELL AND L.R. PETZOLD, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, North-Holland, Amsterdam, 1989.
- [23] S.L. CAMPBELL, F. DELEBECQUE AND R. NIKOUKHAH, *Observers design for a class of nonlinear singular systems*, in Proc. Control Industrial Systems (CIS97), Belfort, France, 1997, pp. 507–512.
- [24] S. L. CAMPBELL, P. KUNKEL AND V. MEHRMANN, *Regularization of linear and nonlinear descriptor systems*, in Control and Optimization with Differential-Algebraic Constraints, Lorenz T. Biegler, Stephen L. Campbell, and Volker Mehrmann, eds., vol. 23 of Advances in Design and Control, SIAM, Philadelphia, 2012, pp. 17–36.
- [25] L. DAI, *Normal compensators for linear singular systems*. Master Thesis, Inst. of Systems Sci., Academia Sinica, Beijing, China, 1986.
- [26] L. DAI AND C. WANG, *Normal dynamical compensators for singular systems*, Acta Math. Scientia, 7 (1987),

- pp. 273–282.
- [27] L. DAI AND C. WANG, *Stable and structurally stable regulators for singular systems*, Acta Mathematicae Applicatae Sinica, 3 (1987), pp. 122–135.
- [28] L. DAI AND C. WANG, *Structurally stable normal compensators for singular systems*, J. Syst. Sci. & Math. Sci., 7 (1987), pp. 89–93.
- [29] M. DAROUACH AND L. BOUTAT-BADDAS, *Observers for a class of nonlinear singular systems*, IEEE Trans. Autom. Control, 53 (2008), pp. 2627–2633.
- [30] M. DAROUACH AND M. BOUTAYEB, *Design of observers for descriptor systems*, IEEE Trans. Autom. Control, 40 (2008), pp. 1323–1327.
- [31] M. DAROUACH, M. ZASADZINSKI, AND M. HAYAR, *Reduced-order observer design for descriptor systems with unknown inputs*, IEEE Trans. Autom. Control, 41 (1996), pp. 1068–1072.
- [32] M. EL-TOHAMI, V. LOVASS-NAGY, AND R. MUKUNDAN, *On the design of observers for generalized state space systems using singular value decomposition*, Int. J. Control, 38 (1983), pp. 673–683.
- [33] M. M. FAHMY AND J. O'REILLY, *Observers for descriptor systems*, Int. J. Control, 49 (1989), pp. 2013–2028.
- [34] Z. GAO AND H. WANG, *Descriptor observer approaches for multivariable systems with measurement noises and application in fault detection and diagnosis*, Syst. Control Lett., 55 (2006), pp. 304–313.
- [35] M. HOU AND P.C. MÜLLER, *Observer design for descriptor systems*, IEEE Trans. Autom. Control, 44 (1999), pp. 164–168.
- [36] M. HOU, T. SCHMIDT, R. SCHÜPPHAUS AND P. C. MÜLLER, *Normal form and luenberger observer for linear mechanical descriptor systems*, J. Dyn. Sys., Meas., Contr., 115 (1993), pp. 611–620.
- [37] A.A. JULIUS AND A.J. VAN DER SCHAFT, *Compatibility of behavior interconnections*, in Proc. 7th European Control Conf. 2003, Cambridge, UK, 2003.
- [38] P. KUNDEL AND V. MEHRMANN, *Differential-Algebraic Equations. Analysis and Numerical Solution*, EMS Publishing House, Zürich, Switzerland, 2006.
- [39] D. LABISCH AND U. KONIGORSKI, *Design of causal observers for nonlinear descriptor systems*, in Progress in Differential-Algebraic Equations, Sebastian Schöps, Andreas Bartel, Michael Günther, E. Jan W. ter Maten, and Peter C. Müller, eds., Differential-Algebraic Equations Forum, Springer-Verlag, Berlin-Heidelberg, 2014, pp. 47–79.
- [40] R. LAMOUR, R. MÄRZ AND C. TISCHENDORF, *Differential Algebraic Equations: A Projector Based Analysis*, vol. 1 of Differential-Algebraic Equations Forum, Springer-Verlag, Heidelberg-Berlin, 2013.
- [41] W. LIN AND L. DAI, *Solutions to the output regulation problem of linear singular systems*, Automatica, 32 (1996), pp. 1713–1718.
- [42] V. LOVASS-NAGY, R.J. MILLER AND R. MUKUNDAN, *On the application of matrix generalized inverses to the design of observers for time-varying and time-invariant linear systems*, IEEE Trans. Autom. Control, AC-25 (1980), pp. 1213–1218.
- [43] D.G. LUENBERGER, *Observing a state of a linear system*, IEEE Trans. Mil. Electron., MIL-8 (1964), pp. 74–80.
- [44] D.G. LUENBERGER, *An introduction to observers*, IEEE Trans. Autom. Control, 16 (1971), pp. 596–602.
- [45] R. NIKOUKHAH, S.L. CAMPBELL AND F. DELEBECQUE, *Observer design for general linear time-invariant systems*, Automatica, 34 (1998), pp. 575–583.
- [46] J.W. POLDERMAN AND J.C. WILLEMS, *Introduction to Mathematical Systems Theory. A Behavioral Approach*, Springer-Verlag, New York, 1998.
- [47] R. RIAZA, *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications*, World Scientific Publishing, Basel, 2008.
- [48] B. SHAFAI AND R. L. CARROLL, *Design of a minimal-order observer for singular systems*, Int. J. Control, 45 (1987), pp. 1075–1081.
- [49] M.A. SHAYMANN, *Pole placement by dynamic compensation for descriptor systems*, Automatica, 22 (1988), pp. 279–282.
- [50] J. ÅSLUND AND E. FRISK, *An observer for non-linear differential-algebraic systems*, Automatica, 42 (2006), pp. 959–965.
- [51] H.L. TRENTELMAN, A.A. STORVOGEL AND M.L.J. HAUTUS, *Control Theory for Linear Systems*, Communications and Control Engineering, Springer-Verlag, London, 2001.
- [52] H.L. TRENTELMAN AND J.C. WILLEMS, *The behavioral approach as a paradigm for modelling interconnected systems*, European J. Control, 9 (2003), pp. 296–306.
- [53] M.E. VALCHER AND J.C. WILLEMS, *Observer synthesis in the behavioral approach*, IEEE Trans. Autom. Control, 44 (1999), pp. 2297–2307.
- [54] C. WANG AND L. DAI, *The normal state observers in singular systems*, J. Syst. Sci. & Math. Sci., 6 (1986), pp. 307–313.
- [55] C. WANG AND L. DAI, *State observer structure in singular systems*, Act. Math. Appl. Sinica, 10 (1987), pp. 121–124.
- [56] Y. WANG, *Disturbance resistance and output regulation for linear singular systems*. Master Thesis, Inst. of

- Systems Science, Academia Sinica, Beijing, China, 1984.
- [57] Y. WANG, S. SHI AND Z. ZAAG, *Pole placement and compensator design of generalized systems*, Syst. Control Lett., 8 (1987), pp. 205–210.
  - [58] J.C. WILLEMS, *On interconnections, control, and feedback*, IEEE Trans. Autom. Control, 42 (1997), pp. 326–339.
  - [59] J.C. WILLEMS, *The behavioral approach to open and interconnected systems*, IEEE Control Systems Magazine, 27 (2007), pp. 46–99.
  - [60] A.-G. WU AND G.-R. DUAN, *Design of generalized pi observers for descriptor linear systems*, IEEE Trans. Circuits Syst. I Fund. Theory Appl., 53 (2006), pp. 2828–2837.
  - [61] C. YANG, Q. ZHANG AND T. CHAI, *Observer design for a class of nonlinear descriptor systems*, in Proc. Joint 48th IEEE Conf. Decis. Control and 28th Chinese Contr. Conf., Shanghai, P.R.China, vol. 1, Shanghai, China, 2009, pp. 8232–8237.
  - [62] B. ZHANG AND J. ZHU, *Eigenvalue assignment in linear descriptor systems using dynamic compensators*, International Journal of Control, Automation and Systems, 12 (2014), pp. 948–953.