Funnel control for nonlinear systems with known strict relative degree

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Abstract

We consider tracking control for nonlinear multi-input, multi-output systems which have arbitrary strict relative degree and input-to-state stable internal dynamics. For a given sufficiently smooth reference signal, our aim is to design a controller which achieves that the tracking error evolves within a prespecified performance funnel. To this end, we introduce a new controller which involves the first $r-1$ derivatives of the tracking error, where r is the strict relative degree of the system. We derive an explicit bound for the resulting input and discuss the influence of the controller parameters. We further present some simulations where our funnel controller is applied to a mechanical system with higher relative degree and a two-input, two-output robot manipulator. The controller is also compared with other approaches.

Keywords: nonlinear systems, relative degree, adaptive control, funnel control.

1. Introduction

In the present article we consider output trajectory tracking for nonlinear systems by funnel control. We assume knowledge of the strict relative degree of the system and that the internal dynamics are, in a certain sense, input-to-state stable, resembling the concept introduced by Sontag [32]. The concept of funnel control has been developed in [18] for systems with relative degree one, see also the survey [16] and the references therein. The funnel controller is an output-error feedback of high-gain type; it is an adaptive controller since the gain is adapted to the actual needed value by a time-varying (non-dynamic) adaptation scheme[1]. Note that no exact tracking is pursued, but a tracking error with prescribed transient behavior. Controllers of high-gain type have various advantages when it comes to "real world" applications; we like to quote from [7]:

> "Since only structural assumptions on the system are required, high-gain adaptive control is inherently robust and makes it attractive for industrial application."

In particular, the funnel controller proved to be the appropriate tool for tracking problems in various applications, such as temperature control of chemical reactor models [23], control of industrial servo-systems [12, 22] and rigid, revolute joint robotic manipulators [13], speed control of wind turbine systems [9, 11], current control for synchronous machines [10], DC-link power flow control [31], voltage and current control of electrical circuits [2], oxygenation control during artificial ventilation therapy [28] and control of peak inspiratory pressure [29].

A longstanding open problem in high-gain adaptive control is the treatment of systems with relative degree larger than one, see [14, 16, 27]. In [1], a "Prescribed Performance Controller" for systems with higher strict relative degree has been introduced by Bechlioulis and Rovithakis (and in [34] the influence of disturbances is discussed), however trivial internal dynamics are assumed. In [5], Bullinger and Allgöwer introduce an adaptive λ-tracker which achieves tracking with prescribed asymptotic accuracy $\lambda > 0$ for a class of systems which are affine in the control, of known relative degree, and with affine linearly bounded drift term. However, the drawback of this controller is that the transient behavior of the tracking error cannot be influenced. Ilchmann et al. [19, 20] developed a funnel controller for systems with higher strict relative degree by introducing a "backstepping" procedure in conjunction with a precompensator. This controller achieves tracking with prescribed transient behavior for a large class of systems governed by nonlinear (functional) differential equations. Unfortunately, this backstepping procedure is quite impractical, especially since it involves high powers of a gain function which typically takes very large values, cf. [8, Sec. 4.4.3]. Backstepping is also used for an adaptive λ-tracker in an earlier work by Ye [35].

For systems with relative degree two, a proportional-derivative (PD) funnel controller has been introduced in [12] (see also the modification in [7]), where the backstepping procedure is avoided. The only available generalization of this approach to systems with higher relative degree is the bang-bang funnel controller introduced by Liberzon and Trenn [26]. However, this controller is restricted to single-input, single-output systems and the involved compatibility conditions on the funnel boundaries, the safety distances and the settling times are quite complicated.

In the present paper we introduce a simple funnel controller for systems with arbitrary known relative degree r and (in a suitable sense) input-to-state stable internal dynamics. The con-

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[1] Note that often only controllers with dynamic gain adaptation are viewed as adaptive controllers of high-gain type.

troller is based on a simple recursion law and involves the first $r-1$ derivatives of the tracking error.

1.1. Nomenclature

$\mathbb{R}_{\geq 0}$	$:= [0,\infty)$
$\|x\|$	the Euclidean norm of $x \in \mathbb{R}^n$
$\mathscr{L}^\infty_{\text{loc}}(I \to \mathbb{R}^n)$	the set of locally essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval
$\mathscr{L}^\infty(I \to \mathbb{R}^n)$	the set of essentially bounded functions $f : I \to \mathbb{R}^n$ with norm
$\|f\|_\infty$	$:= \text{ess sup}_{t\in I}\|f(t)\|$
$\mathscr{W}^{k,\infty}(I \to \mathbb{R}^n)$	the set of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f,\ldots,f^{(k)} \in \mathscr{L}^\infty(I \to \mathbb{R}^n)$
$\mathscr{C}^k(V \to \mathbb{R}^n)$	the set of k-times continuously differentiable functions $f : V \to \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$; $\mathscr{C}(V \to \mathbb{R}^n) = \mathscr{C}^0(V \to \mathbb{R}^n)$
$f\|_W$	restriction of the function $f : V \to \mathbb{R}^n$ to $W \subseteq V$

1.2. System class

In the present paper we consider a class of non-linear systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big) \\ &\quad + \Gamma\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big)\, u(t) \\ y|_{[-h,0]} &= y^0 \in \mathscr{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{1}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

(P1): the "disturbance" satisfies $d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$;

(P2): $f \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$,

(P3): the "high-frequency gain matrix function" $\Gamma \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m\times m})$ takes values in the set of positive (negative) definite matrices[2];

(P4): $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm}) \to \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties:

a) T maps bounded trajectories to bounded trajectories, i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \leq c_1 \Rightarrow \sup_{t\in[0,\infty)} \|T(\zeta)(t)\| \leq c_2,$$

b) T is causal, i.e, for all $t \geq 0$ and all $\zeta, \xi \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \overset{\text{a.a.}}{=} T(\xi)|_{[0,t)},$$

where "a.a." stands for "almost all".

c) T is locally Lipschitz continuous in the following sense: for all $t \geq 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$ with $\Delta\zeta|_{[-h,t)} = 0$ and $\|\Delta\zeta|_{[t,t+\tau]}\|_\infty < \delta$ we have

$$\big\|\big(T(\zeta+\Delta\zeta) - T(\zeta)\big)|_{[t,t+\tau]}\big\|_\infty \leq c\|\Delta\zeta|_{[t,t+\tau]}\|_\infty.$$

The functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : [-h,\infty) \to \mathbb{R}^m$ are called *input* and *output* of the system (1), resp. Systems similar to (1) have been studied e.g. in [12, 17, 18, 20]. In the aforementioned references it is shown that the class of systems (1) encompasses linear and nonlinear systems with strict relative degree and input-to-state stable internal dynamics (zero dynamics in the linear case) and the operator T allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements, and combinations thereof. Note that the operator T is usually the solution operator of the differential equation describing the internal dynamics of the system and its property (P4a) thus amounts to the input-to-state stability of the internal dynamics. One important subclass of systems (1) are minimum-phase linear time-invariant systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= Cx(t), \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, which have *strict relative degree $r \in \mathbb{N}$ and positive (negative) definite high-frequency gain matrix*, i.e, $CB = CAB = \ldots = CA^{r-2}B = 0$ and $\Gamma := CA^{r-1}B \in \mathbb{R}^{m\times m}$ is positive (negative) definite. The minimum-phase assumption (equivalently, asymptotic stability of the zero dynamics, see [24]) is characterized by the condition

$$\forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq 0 : \det \begin{bmatrix} \lambda I_n - A & B \\ C & 0 \end{bmatrix} \neq 0.$$

It is known that systems of this type can be transformed into *Byrnes-Isidori normal form*, see [20],

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), \quad && y(0) = Cx^0 \\ \dot{\eta}(t) &= Py(t) + Q\eta(t), && \eta(0) = \eta^0 \in \mathbb{R}^{n-rm} \end{aligned}$$

where $R_i \in \mathbb{R}^{m\times m}$ for $i = 1,2,\ldots,r$, $S^\top, P \in \mathbb{R}^{(n-rm)\times m}$, and $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$ is a Hurwitz matrix, i.e., all eigenvalues of Q have negative real part. This is a system of type (1) with $\Gamma \equiv CA^{r-1}B$ and

$$\begin{aligned} f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big) &= T(y,\dot{y},\ldots,y^{(r-1)})(t) \\ &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + Se^{Qt}\eta^0 + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau. \end{aligned}$$

T is clearly causal, locally Lipschitz, and the Hurwitz property of Q implies that T has the bounded-input-bounded-output property (P4a). Note that T is parameterized by $\eta^0 \in \mathbb{R}^{n-rm}$.

[2] One may wonder why Γ is not assumed to be uniformly bounded away from zero. The reason is that in the closed-loop system this is established anyway due to the boundedness of the involved signals.

Finally, we like to stress that systems of the form

$$\begin{aligned} y^{(r)}(t) = f\big(d_1(t), T_1(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) \\ + \Gamma\big(d_2(t), T_2(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)\, u(t), \end{aligned}$$

where d_i is as in (P1) and T_i is as in (P4) for $i = 1, 2$ are included in the class (1). This can be achieved by setting $d := (d_1, d_2)$, $T := (T_1, T_2)$ and a suitable adjustment of f and Γ.

1.3. Control objective

The objective is to design an output error feedback $u(t) = F(t, e(t), \dot{e}(t), \ldots, e^{(r-1)}(t))$, where $e(t) = y(t) - y_{\text{ref}}(t)$ for some reference trajectory $y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, such that in the closed-loop system the tracking error $e(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{\, (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \,\}, \tag{3}$$

which is determined by a function φ belonging to

$$\Phi_r := \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau \to \infty} \varphi(\tau) > 0 \end{array} \right\}. \tag{4}$$

Furthermore, all signals $u, e, \dot{e}, \ldots, e^{(r-1)}$ should remain bounded.

The funnel boundary is given by the reciprocal of φ, see Fig. 1. It is explicitly allowed that $\varphi(0) = 0$, meaning that no restriction on the initial value is imposed since $\varphi(0)\|e(0)\| < 1$; the funnel boundary $1/\varphi$ has a pole at $t = 0$ in this case.

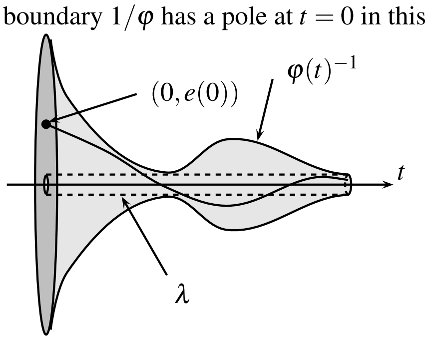

Figure 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

An important property of the class Φ_r is that the boundary of each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi_r$ is bounded away from zero, i.e., because of boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., when the reference trajectory changes strongly or the system is perturbed by some calibration so that a large tracking error would enforce a large input action. Therefore, a variety of different funnel boundaries are possible, see e.g. [15, Sec. 3.2].

1.4. Organization of the present paper

The paper is structured as follows. In Section 2, we introduce the funnel controller for the system class presented in Section 1.2. Feasibility of the control is proved in the main result in Section 3; in particular we show that our proposed funnel controller achieves the control objective described in Section 1.3. Additionally we derive an explicit bound on the input generated by the controller and discuss the influence of the design parameters. The performance of the funnel controller is illustrated by means of several examples in Section 4, where also our approach is compared to the feedback strategies in [12, 19, 20, 26].

2. Controller structure

We introduce the below funnel controller for systems of type (1):

$$\boxed{\begin{array}{l} \begin{array}{rclcl} e_0(t) & = & e(t) & = & y(t) - y_{\text{ref}}(t), \\ e_1(t) & = & \dot{e}_0(t) & + & k_0(t) \cdot e_0(t), \\ e_2(t) & = & \dot{e}_1(t) & + & k_1(t) \cdot e_1(t), \\ & \vdots & & & \\ e_{r-1}(t) & = & \dot{e}_{r-2}(t) & + & k_{r-2}(t) \cdot e_{r-2}(t), \end{array} \\ k_i(t) = \frac{1}{1 - \varphi_i^2(t)\|e_i(t)\|^2}, \quad i = 0, \ldots, r-1, \\ u(t) = \begin{cases} -k_{r-1}(t) \cdot e_{r-1}(t), & \text{if } \Gamma \text{ is pointwise pos. def.,} \\ k_{r-1}(t) \cdot e_{r-1}(t), & \text{if } \Gamma \text{ is pointwise neg. def.,} \end{cases} \end{array}} \tag{5}$$

where the reference signal and funnel functions have the following properties:

$$\boxed{\begin{array}{l} y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ \varphi_0 \in \Phi_r, \;\; \varphi_1 \in \Phi_{r-1}, \;\ldots,\; \varphi_{r-1} \in \Phi_1. \end{array}} \tag{6}$$

In the sequel we investigate existence of solutions of the initial value problem resulting from the application of the funnel controller (5) to a system (1). Even if (1) is a linear system derived from (2), some care must be exercised with the existence of a solution of (2), (5) since k_i introduces a pole on the right hand side of the closed-loop differential equation. By a *solution* of (1), (5) on $[-h, \omega)$ we mean a function $y \in \mathcal{C}^{r-1}([-h, \omega) \to \mathbb{R}^m)$, $\omega \in (0, \infty]$, with $y|_{[-h,0]} = y^0$ such that $y^{(r-1)}|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (1) with u defined in (5) for almost all $t \in [0, \omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [18] for instance.

Remark 2.1 (Funnel control for systems with $r \in \{1, 2, 3\}$)**.** In the following we determine the funnel controllers explicitly for the cases $r = 1, 2, 3$. We assume for convenience that the high-frequency gain matrix function Γ is pointwise positive definite.

- $r = 1$: The control law (5) reduces to the "classical" funnel controller $u(t) = -k(t)e(t)$ with $k(t) = \frac{1}{1 - \varphi^2(t)\|e(t)\|^2}$. Moreover, our assumptions on the reference signal and the funnel function φ reduce to those made in [18].

$r = 2$: We obtain the controller

$$
\begin{aligned}
u(t) &= -k_1(t)(\dot e(t) + k_0(t)e(t)),\\
k_0(t) &= \tfrac{1}{1-\varphi_0^2(t)\|e(t)\|^2},\\
k_1(t) &= \tfrac{1}{1-\varphi_1^2(t)\|\dot e(t)+k_0(t)e(t)\|^2}.
\end{aligned}
$$

$r = 3$: Here the controller (5) takes the form

$$
\begin{aligned}
u(t) = &-k_2(t)\cdot\big[\ddot e(t) + 2k_0(t)^2(\varphi_0^2(t)e^\top(t)\dot e(t)\\
&+\varphi_0(t)\dot\varphi_0(t)\|e(t)\|^2)e(t)\\
&+k_0(t)\dot e(t) + k_1(t)(\dot e(t)+k_0(t)e(t))\big],\\
k_0(t) = &\tfrac{1}{1-\varphi_0^2(t)\|e(t)\|^2},\\
k_1(t) = &\tfrac{1}{1-\varphi_1^2(t)\|\dot e(t)+k_0(t)e(t)\|^2},\\
k_2(t) = &\tfrac{1}{1-\varphi_2^2(t)\|\ddot e(t)+2k_0(t)^2(\varphi_0^2(t)e^\top(t)\dot e(t)+\varphi_0(t)\dot\varphi_0(t)\|e(t)\|^2)e(t)+k_0(t)\dot e(t)+k_1(t)(\dot e(t)+k_0(t)e(t))\|^2}.
\end{aligned}
$$

Remark 2.2 (The intuition behind the funnel controller (5))**.** The classical funnel controller for systems with relative degree one and input-to-state stable internal dynamics uses the "high gain property" [6], which states that such systems can can be stabilized by a proportional feedback law $u(t) = -ky(t)$ with a sufficiently large constant $k > 0$. This gives rise to the intuition of the funnel controller for relative degree one: If the error approaches the funnel boundary at t_0, then $k(t_0)$ takes a large value which stabilizes the error system.
To illustrate the functioning of the controller (5), we employ the following thought experiment for the single-input, single-output case $m = 1$, see Fig. 2: Assume that the error $e = e_0$ approaches the upper funnel boundary $1/\varphi_0$ at time $t_0 > 0$. Then $k_0(t_0)$, and consequently $k_0(t_0)\cdot e(t_0)$ will be very large. Since $e_1 = \dot e + k_0\cdot e$ evolves in the performance funnel $\mathscr{F}_{\varphi_1}$, we may infer that $\dot e(t_0) = e_1(t_0) - k_0(t_0)\cdot e(t_0)$ will take a large negative value. In other words, e will be decreasing enormously. That is, whenever the error e approaches the funnel boundary $1/\varphi_0$, the controller ensures a repelling effect.
This argumentation can be repeated for the functions $e_1,\dots,e_{r-2}$. Finally, since e_{r-1} includes the first $r-1$ derivatives of e, the system with artificial output e_{r-1} has relative degree one, and the classical high gain property applies to e_{r-1}.

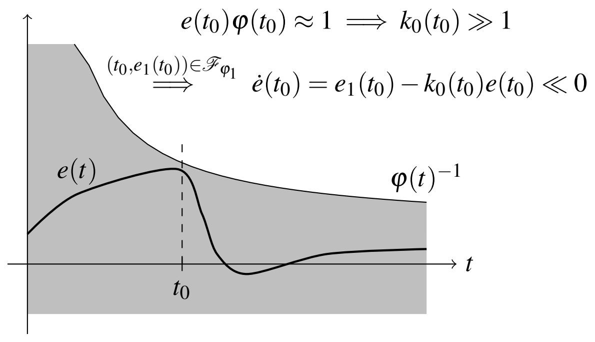

Figure 2: Error in the performance funnel $\mathscr{F}_{\varphi_1}$

Remark 2.3 (Funnel control by backstepping)**.** The works [19, 20] introduce a funnel controller based on a filter and backstepping construction for systems with higher relative degree. First consider a filter with

$$
\begin{aligned}
\dot\xi_i(t) &= -\xi_i(t) + \xi_{i+1}(t), \quad i = 1,\dots,r-2,\\
\dot\xi_{r-1}(t) &= -\xi_{r-1}(t) + u(t).
\end{aligned}
$$

Introduce the projections

$$
\pi_i : \mathbb{R}^{(r-1)m} \to \mathbb{R}^{im},\ \xi = (\xi_1,\dots,\xi_{r-1}) \mapsto (\xi_1,\dots,\xi_i)
$$

for $i = 1,\dots,r-1$ and functions

$$
\begin{aligned}
\gamma_1(k,e) &= k\cdot e\\
\gamma_i(k,e,\pi_{i-1}\xi) &:= \gamma_{i-1}(k,e,\pi_{i-2}\xi)\\
&\quad + \|D\gamma_{i-1}(k,e,\pi_{i-2}\xi)\|^2 k^4\cdot(1+\|\pi_{i-1}\xi\|^2)\\
&\quad \cdot(\xi_{i-1} + \gamma_{i-1}(k,e,\pi_{i-2}\xi))
\end{aligned}
$$

The controller in [20] takes the form

$$
\begin{aligned}
u(t) &= -\gamma_r(k(t),e(t),\xi(t)),\\
k(t) &= \tfrac{1}{1-\varphi(t)^2\|e(t)\|^2}.
\end{aligned}
$$

We stress that in [20] a much smaller class of systems than introduced in Section 1.2 is considered; in [20] T may only depend on y and Γ is assumed to be constant. The above presented controller works provided that $\Gamma \in \mathbb{R}^{m\times m}$ is positive definite. However, this approach can be modified such that it also works for systems in which it is not known whether Γ is positive or negative definite. In this case, the function γ_1 has to be modified by $\gamma_1(k,e) = \nu(k)\cdot e$, where $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is smooth and satisfies the "Nussbaum property" [20]. In the following we discuss the cases of relative degree two and three.

$r = 2$: Here the controller takes the form

$$
u = -ke - (\|e\|^2 + k^2)\cdot k^4(1+\|\xi\|^2)(\xi + ke),
$$

where we omit the argument t. This feedback law is dynamic and the gain occurs with $k(t)^7$. The presence of such a large power of the funnel gain $k(t)$ is problematic in practice; the controller produces inputs which might be impractical, cf. [8, Sec. 4.4.3].

$r = 3$: Here the controller reads, for $m = 1$,

$$
\begin{aligned}
u = &-ke - k^4(e^2+k^2)(1+\xi_1^2)(\xi_1+ke) - \Big\{\big[e + (1+\xi_1^2)\\
&\cdot\big[2k^5(\xi_1+ke) + 4k^3(e^2+k^2)(\xi_1+ke) + k^4(e^2+k^2)e\big]\big]^2\\
&+\big[k + k^4(1+\xi_1^2)\big[2e(\xi_1+ke) + k(e^2+k^2)\big]\big]^2\\
&+\big[k^4(e^2+k^2)\big[2\xi_1(\xi_1+ke) + (1+\xi_1^2)\big]\big]^2\Big\}k^4(1+\xi_1^2+\xi_2^2)\\
&\cdot\big[\xi_2 + ke + k^4(e^2+k^2)(1+\xi_1^2)(\xi_1+ke)\big].
\end{aligned}
\tag{7}
$$

An expansion of the above product gives that this controller contains the 25th power (!) of the funnel gain $k(t)$, and the problems depicted for $r = 2$ are present here a fortiori.

Remark 2.4 (Proportional-derivative funnel control for relative degree two)**.** Consider a system (1) with the properties (P1)–(P4) as in Section 1.2. Further assume that $m = 1$ and the high-frequency gain function Γ is pointwise positive definite. The work [12] introduces a funnel controller which feeds back the error e and its derivative. More precise, this controller reads

$$\begin{aligned} u(t) &= -k_0^2(t)e(t) - k_1(t)\dot e(t), \\ k_0(t) &= \tfrac{\varphi_0}{1-\varphi_0(t)|e(t)|}, \quad k_1(t) = \tfrac{\varphi_1}{1-\varphi_1(t)|\dot e(t)|}. \end{aligned} \tag{8}$$

Note that $k_i(t)$ in (8) is different from $k_i(t)$ in (5). The funnel functions φ_0 for the error and φ_1 for the derivative of the error have to satisfy $\varphi_0 \in \Phi_2$, $\varphi_1 \in \Phi_1$, and they have to fulfill the compatibility condition

$$\forall\, t > 0\ \exists\, \delta > 0:\ 1/\varphi_1(t) \geq \delta - \tfrac{\mathrm{d}}{\mathrm{d}t}(1/\varphi_0(t)) \qquad \forall\, t > 0. \tag{9}$$

This controller is simple and its practicability has been verified experimentally. However, there is no straightforward extension to systems with relative degree larger than two. We further emphasize that the funnel functions $\varphi_0, \ldots, \varphi_{r-1}$ in the funnel controller (5) do not have to satisfy any compatibility condition.

3. Main result

We show feasibility of the funnel controller (5).

Theorem 3.1. *Consider a system* (1) *with strict relative degree $r \in \mathbb{N}$ and properties (P1)-(P4). For Φ_i as defined in* (4)*, let*

$$\varphi_i \in \Phi_{r-i} \text{ for } i = 0, \ldots, r-1.$$

Let $y_{\mathrm{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ be a reference signal, and $y|_{[-h,0]} = y^0 \in \mathscr{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m)$ an initial value such that $e_0, \ldots, e_{r-1}$ as defined in (5) *fulfill*

$$\varphi_i(0)\|e_i(0)\| < 1 \text{ for } i = 0, \ldots, r-1. \tag{10}$$

Then the application of the funnel controller (5) *to* (1) *yields an initial-value problem, which has a solution, and every maximal solution $y : [-h, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, has the following properties*[3]*:*

(i) *The solution is global (i.e., $\omega = \infty$).*

(ii) *The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the gain functions $k_0, \ldots, k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $y, \ldots, y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are bounded.*

(iii) *The functions $e_0, \ldots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the following sense:*

$$\forall\, i = 0, \ldots, r-1\ \exists\, \varepsilon_i > 0\ \forall\, t > 0: \quad \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i. \tag{11}$$

In particular, the error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ evolves in the funnel $\mathscr{F}_{\varphi_0}$ as in (3) *and stays uniformly away from its boundary.*

[3] Note that maximal solutions are not unique in general.

Proof. We may, without loss of generality, assume that the high-frequency gain matrix function Γ of system (1) is pointwisely positive definite. We proceed in several steps.

Step 1: We show that a maximal solution $y : [-h, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, of (1), (5) exists. We aim at reformulating (1), (5) as an initial value problem

$$\begin{aligned} \dot x(t) &= F\big(t, x(t), T(x)(t)\big), \\ x|_{[-h,0]} &= \big(y^0, \dot y^0, \ldots, (\tfrac{\mathrm{d}}{\mathrm{d}t})^{r-1} y^0\big)|_{[-h,0]}, \end{aligned} \tag{12}$$

where

$$x = \big(y, \dot y, \ldots, y^{(r-1)}\big)$$

and F is some suitable continuous function.

Step 1a: Define, for $i = 0, \ldots, r-1$, the sets

$$\mathscr{D}_i := \left\{ (t, e_0, \ldots, e_i) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m \;\middle|\; \begin{array}{l} (t, e_j) \in \mathscr{F}_{\varphi_j}, \\ j = 0, \ldots, i \end{array} \right\},$$

where $\mathscr{F}_{\varphi_j}$ is as in (3), and the functions $K_i : \mathscr{D}_i \to \mathbb{R}^m$ recursively by

$$\begin{aligned} &K_0(t, e_0) := \tfrac{e_0}{1-\varphi_0^2(t)\|e_0\|^2}, \\ &K_i(t, e_0, \ldots, e_i) \\ &:= \tfrac{e_i}{1-\varphi_i^2(t)\|e_i\|^2} + \tfrac{\partial K_{i-1}}{\partial t}(t, e_0, \ldots, e_{i-1}) \\ &\quad + \sum_{j=0}^{i-1} \tfrac{\partial K_{i-1}}{\partial e_j}(t, e_0, \ldots, e_{i-1}) \left(e_{j+1} - \tfrac{e_j}{1-\varphi_j^2(t)\|e_j\|^2} \right). \end{aligned}$$

Choose some interval $I \subseteq \mathbb{R}_{\geq 0}$ with $0 \in I$ and let $(e_0, \ldots, e_{r-1}) : I \to \mathbb{R}^{rm}$ be such that, for all $t \in I$, $\big(t, e_0(t), \ldots, e_{r-1}(t)\big) \in \mathscr{D}_{r-1}$ and $(e_0, \ldots, e_{r-1})$ satisfies the relations in (5). Then $e = e_0$ satisfies, on the interval I,

$$e^{(i)} = e_i - \sum_{j=0}^{i-1} \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big)^{i-j-1} (k_j e_j) \quad \text{for all } i = 1, \ldots, r-1. \tag{13}$$

Step 1b: We show by induction that for all $i = 0, \ldots, r-1$ we have

$$\forall\, t \in I: \ \sum_{j=0}^{i} \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big)^{i-j} \big(k_j(t) e_j(t)\big) = K_i\big(t, e_0(t), \ldots, e_i(t)\big). \tag{14}$$

Equation (14) is obviously true for $i = 0$. Assume that $i \in \{1, \ldots, r-1\}$ and the statement holds for $i-1$. Then

$$\begin{aligned} &\sum_{j=0}^{i} \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big)^{i-j} \big(k_j(t) e_j(t)\big) \\ &= k_i(t) e_i(t) + \tfrac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{j=0}^{i-1} \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big)^{i-j-1} \big(k_j(t) e_j(t)\big) \right) \end{aligned}$$

$$= k_i(t)e_i(t) + \tfrac{\mathrm{d}}{\mathrm{d}t}K_{i-1}\big(t, e_0(t), \ldots, e_{i-1}(t)\big)$$
$$= K_i\big(t, e_0(t), \ldots, e_i(t)\big).$$

Step 1c: Define

$$\tilde{K}_0 : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m,\ (t,y) \mapsto y - y_{\mathrm{ref}}(t)$$

and the set

$$\tilde{\mathscr{D}}_0 := \left\{\ (t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \ \middle|\ \big(t, \tilde{K}_0(t,y)\big) \in \mathscr{D}_0\ \right\}.$$

Furthermore, recursively define for $i = 1, \ldots, r-1$ the maps

$$\begin{aligned}\tilde{K}_i : &\tilde{\mathscr{D}}_{i-1} \times \mathbb{R}^m \to \mathbb{R}^m,\ (t, y_0, \ldots, y_i)\\ &\mapsto y_i - y_{\mathrm{ref}}^{(i)}(t) + K_{i-1}\big(t, \tilde{K}_0(t,y_0), \ldots, \tilde{K}_{i-1}(t, y_0, \ldots, y_{i-1})\big)\end{aligned}$$

and the sets

$$\tilde{\mathscr{D}}_i := \left\{ (t, y_0, \ldots, y_i) \in \tilde{\mathscr{D}}_{i-1} \times \mathbb{R}^m \ \middle|\ \begin{array}{l}\big(t, \tilde{K}_0(t,y_0), \ldots,\\ \quad \tilde{K}_i(t, y_0, \ldots, y_i)\big) \in \mathscr{D}_i\end{array} \right\}.$$

It now follows from a simple induction, invoking (13) and (14) that, for all $t \in I$ and all $i = 0, \ldots, r-1$,

$$\begin{aligned}e_i(t) &= y^{(i)}(t) - y_{\mathrm{ref}}^{(i)}(t) + K_{i-1}\big(t, e_0(t), \ldots, e_{i-1}(t)\big)\\ &= \tilde{K}_i\big(t, y(t), \dot{y}(t), \ldots, y^{(i)}(t)\big).\end{aligned}$$

Therefore, the feedback u in (5) reads

$$u(t) = \frac{-\tilde{K}_{r-1}\big(t, y(t), \ldots, y^{(r-1)}(t)\big)}{1 - \varphi_{r-1}^2(t)\|\tilde{K}_{r-1}\big(t, y(t), \ldots, y^{(r-1)}(t)\big)\|^2}, \quad t \in I.$$

Step 1d: Define

$$\begin{aligned}F : &\tilde{\mathscr{D}}_{r-1} \times \mathbb{R}^q \to \mathbb{R}^{rm},\ (t, y_0, y_1, \ldots, y_{r-1}, \eta)\\ &\mapsto \left(y_1, \ldots, y_{r-1}, f\big(d(t), \eta\big) - \tfrac{\Gamma(d(t),\eta)\tilde{K}_{r-1}(t, y_0, \ldots, y_{r-1})}{1 - \varphi_{r-1}^2(t)\|\tilde{K}_{r-1}(t, y_0, \ldots, y_{r-1})\|^2}\right).\end{aligned}$$

Then the initial value problem (1), (5) is equivalent to (12). In particular, $(0, x(0)) \in \tilde{\mathscr{D}}_{r-1}$ and F is measurable in t, continuous in $(y_0, y_1, \ldots, y_{r-1}, \eta)$ and locally essentially bounded. Hence an application of [17, Thm. B.1][4] yields existence of solutions to (12) and every solution can be extended to a maximal solution. Furthermore, for a maximal solution $x = \big(y, \dot{y}, \ldots, y^{(r-1)}\big) : [-h, \omega) \to \mathbb{R}^{rm}$, $\omega \in (0, \infty]$, of (12), the closure of the graph of this solution is not a compact subset of $\tilde{\mathscr{D}}_{r-1}$. As a consequence, for $(e_0, \ldots, e_{r-1}) : [0, \omega) \to \mathbb{R}^{rm}$ defined by

$$e_i(t) := \tilde{K}_i\big(t, y(t), \dot{y}(t), \ldots, y^{(i)}(t)\big), \quad t \in [0, \omega),$$

it follows that the closure of the graph of $(e_0, \ldots, e_{r-1})$ is not a compact subset of $\mathscr{D}_{r-1}$.

Step 2: We show that $k_0, \ldots, k_{r-1}$ as in (5) are bounded on $[0, \omega)$. For all $i \in \{0, \ldots, r-1\}$, set $\psi_i(t) := \varphi_i(t)^{-1}$ for $t \in (0, \omega)$, let $\tau_i \in (0, \omega)$ be arbitrary but fixed and set $\lambda_i := \inf_{t \in (0,\omega)} \psi_i(t) > 0$. Since $\dot{\varphi}_i$ is bounded and $\liminf_{t\to\infty} \varphi_i(t) > 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t}\ \psi_i|_{[\tau_i, \infty)}$ is bounded and hence there exists a Lipschitz bound $L_i > 0$ of $\psi_i|_{[\tau_i, \infty)}$.

Step 2a: We show that k_i is bounded for $i \in \{0, \ldots, r-2\}$. Choose $\varepsilon_i > 0$ small enough so that

$$\begin{aligned}\varepsilon_i &\leq \min\left\{\tfrac{\lambda_i}{2}, \inf_{t \in (0, \tau_i]} \big(\psi_i(t) - \|e_i(t)\|\big)\right\}\\ \text{and} \quad L_i &\leq \tfrac{\lambda_i^2}{4\varepsilon_i} - \sup_{t \in [\tau_i, \infty)} |\psi_{i+1}(t)|.\end{aligned} \tag{15}$$

Using a standard procedure in funnel control, see e.g. [15], we show that for all $t \in (0, \omega)$ holds $\psi_i(t) - \|e_i(t)\| \geq \varepsilon_i$. By definition of ε_i this holds on $(0, \tau_i]$. Seeking a contradiction suppose that there exists some $t_{i1} \in [\tau_i, \omega)$ with $\psi_i(t_{i1}) - \|e_i(t_{i1})\| < \varepsilon_i$. Set $t_{i0} = \max\{\ t \in [\tau_i, t_{i1})\ |\ \psi_i(t) - \|e_i(t)\| = \varepsilon_i\ \}$. Then, for all $t \in [t_{i0}, t_{i1}]$, we have that

$$\begin{aligned}\psi_i(t) - \|e_i(t)\| &\leq \varepsilon_i,\\ \|e_i(t)\| &\geq \psi_i(t) - \varepsilon_i \geq \tfrac{\lambda_i}{2},\\ k_i(t) = \tfrac{1}{1 - \varphi_i^2(t)\|e_i(t)\|^2} &\geq \tfrac{\psi_i(t)}{2\varepsilon_i} \geq \tfrac{\lambda_i}{2\varepsilon_i}.\end{aligned}$$

Therefore, we find that by (5)

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_i(t)\|^2 &= e_i^\top(t)\,(e_{i+1}(t) - k_i(t)e_i(t))\\ &= -k_i(t)\|e_i(t)\|^2 + e_i^\top(t)e_{i+1}(t)\\ &\leq \left(-\tfrac{\lambda_i^2}{4\varepsilon_i} + \sup_{t \in [\tau_i, \infty)} |\psi_{i+1}(t)|\right)\|e_i(t)\|\\ &\overset{(15)}{\leq} -L_i\|e_i(t)\|\end{aligned}$$

for all $t \in [t_{i0}, t_{i1}]$. Then

$$\begin{aligned}\|e_i(t_{i1})\| - \|e_i(t_{i0})\| &= \int_{t_{i0}}^{t_{i1}} \tfrac{1}{2}\|e_i(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_i(t)\|^2\,\mathrm{d}t\\ &\leq -L_i(t_{i1} - t_{i0})\\ &\leq -|\psi_i(t_{i1}) - \psi_i(t_{i0})|\\ &\leq \psi_i(t_{i1}) - \psi_i(t_{i0}),\end{aligned}$$

and thus we obtain $\varepsilon_i = \psi_i(t_{i0}) - \|e_i(t_{i0})\| \leq \psi_i(t_{i1}) - \|e_i(t_{i1})\| < \varepsilon_i$, a contradiction.

Step 2b: We show that k_{r-1} is bounded. By (13) and Step 1 we have, invoking $x = (y, \dot{y}, \ldots, y^{(r-1)})$,

$$\begin{aligned}\dot{e}_{r-1}(t) = f\big(d(t), T(x)(t)\big) - k_{r-1}(t)\Gamma\big(d(t), T(x)(t)\big)e_{r-1}(t)\\ - y_{\mathrm{ref}}^{(r)}(t) + \sum_{i=0}^{r-2}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{r-i-1}[k_i(t)e_i(t)].\end{aligned}$$

In the following we will prove by induction that there exist con-

[4]In [17] a domain $\mathscr{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is straightforward.

stants $M_{i,j}, N_{i,j}, K_{i,j} > 0$ such that, for all $t \in [0,\omega)$,

$$\begin{aligned}
&\left\| \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^j [k_i(t)e_i(t)] \right\| \le M_{i,j}, \qquad \left\| \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^j e_i(t) \right\| \le N_{i,j},\\
&\qquad \left| \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^j k_i(t) \right| \le K_{i,j},\\
&\text{for } i = 0,\dots,r-2, j = 0,\dots,r-1-i.
\end{aligned} \tag{16}$$

First, we may infer from Step 2a that $k_0,\dots,k_{r-2}$ are bounded. Furthermore, $e_0,\dots,e_{r-1}$ are bounded since they evolve in the respective performance funnels, cf. (5). Therefore, (16) is true whenever $j=0$. We prove (16) for $i=r-2$ and $j=1$: We find that

$$\begin{aligned}
\dot e_{r-2}(t) &= e_{r-1}(t) - k_{r-2}(t)e_{r-2}(t),\\
\dot k_{r-2}(t) &= 2k_{r-2}^2(t)\big(\varphi_{r-2}^2(t)e_{r-2}^\top(t)\dot e_{r-2}(t)\\
&\qquad + \varphi_{r-2}(t)\dot\varphi_{r-2}(t)\|e_{r-2}(t)\|^2\big),\\
\tfrac{\mathrm{d}}{\mathrm{d}t}[k_{r-2}(t)e_{r-2}(t)] &= \dot k_{r-2}(t)e_{r-2}(t) + k_{r-2}(t)\dot e_{r-2}(t),
\end{aligned}$$

and all of these signals are bounded since $k_{r-2}, \varphi_{r-2}, \dot\varphi_{r-2}, e_{r-2}, e_{r-1}$ are bounded. Now let $p \in \{0,\dots,r-3\}$ and $q \in \{0,\dots,r-1-p\}$ and assume that (16) is true for all $i = p+1,\dots,r-2$ and all $j = 0,\dots,r-1-i$ as well as for $i=p$ and all $j=0,\dots,q-1$. We show that it is true for $i=p$ and $j=q$:

$$\begin{aligned}
\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^q e_p(t) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1}[e_{p+1}(t) - k_p(t)e_p(t)]\\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1} e_{p+1}(t) - \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1}[k_p(t)e_p(t)],
\end{aligned}$$

$$\begin{aligned}
\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^q k_p(t) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1}\Big(2k_p^2(t)\big(\varphi_p^2(t)e_p^\top(t)\dot e_p(t)\\
&\qquad + \varphi_p(t)\dot\varphi_p(t)\|e_p(t)\|^2\big)\Big),\\
\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^q [k_p(t)e_p(t)] &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{q-1}\big(\dot k_p(t)e_p(t) + k_p(t)\dot e_p(t)\big).
\end{aligned}$$

Then, successive application of the product rule and using the induction hypothesis as well as the fact that $\varphi_p, \dot\varphi_p, \dots, \varphi_p^{(r-p)}$ are bounded, yields that the above terms are bounded. Therefore, the proof of (16) is complete.
By (16) and (13) it follows that $e^{(i)}$ is bounded on $[0,\omega)$ and hence, invoking boundedness of $y_{\mathrm{ref}},\dots,y_{\mathrm{ref}}^{(r-1)}$, also $y^{(i)}$ is bounded on $[0,\omega)$ for all $i=0,\dots,r-1$. By the bounded-input, bounded-output property (P4a) of the operator T it follows that $T(x)$ is bounded, where $x = \left(y,\dot y,\dots,y^{(r-1)}\right)$. Since f is continuous and d is bounded, we may further infer that $f(d(\cdot),T(x)(\cdot))$ is bounded on $[0,\omega)$, i.e., there exists $M_F > 0$ such that

$$\text{for almost all } t \in [0,\omega): \quad \|f\big(d(t),T(x)(t)\big)\| \le M_F.$$

Define the compact set

$$\mathscr{M} := \left\{ (\delta,\eta,e) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \;\middle|\; \begin{array}{l} \|\delta\| \le \|d|_{[0,\omega)}\|_\infty \\ \|\eta\| \le \|T(x)|_{[0,\omega)}\|_\infty \\ \|e\| = 1. \end{array} \right\},$$

then, since Γ is pointwise positive definite and the map

$$\mathscr{M} \ni (\delta,\eta,e) \mapsto e^\top\Gamma(\delta,\eta)e \in \mathbb{R}_{>0}$$

is continuous, it follows that there exists $\gamma > 0$ such that

$$\forall\,(\delta,\eta,e) \in \mathscr{M}: \quad e^\top\Gamma(\delta,\eta)e \ge \gamma.$$

Therefore, we have

$$\begin{aligned}
&e_{r-1}(t)^\top\Gamma\big(d(t),T(x)(t)\big)e_{r-1}(t)\\
&= \left(\tfrac{e_{r-1}(t)^\top}{\|e_{r-1}(t)\|}\Gamma\big(d(t),T(x)(t)\big)\tfrac{e_{r-1}(t)}{\|e_{r-1}(t)\|}\right)\|e_{r-1}(t)\|^2\\
&\ge \gamma\|e_{r-1}(t)\|^2
\end{aligned}$$

for all $t \in [0,\omega)$. Now, choose $\varepsilon_{r-1} > 0$ small enough so that

$$\varepsilon_{r-1} \le \min\left\{\tfrac{\lambda_{r-1}}{2}, \inf_{t\in(0,\tau_{r-1}]}(\psi_{r-1}(t) - \|e_{r-1}(t)\|)\right\}$$

and

$$L_{r-1} \le \tfrac{\lambda_{r-1}^2}{4\varepsilon_{r-1}}\gamma - M_F - \sup_{t\in[0,\omega)}\|y_{\mathrm{ref}}^{(r)}(t)\| - \sum_{i=0}^{r-2} M_{i,r-1-i}. \tag{17}$$

We show that

$$\forall\, t \in (0,\omega): \ \psi_{r-1}(t) - \|e_{r-1}(t)\| \ge \varepsilon_{r-1}.$$

By definition of ε_{r-1} this holds on $(0,\tau_{r-1}]$. Seeking a contradiction suppose that

$$\exists\, t_{r-1,1} \in [\tau_{r-1},\omega): \ \psi_{r-1}(t_{r-1,1}) - \|e_{r-1}(t_{r-1,1})\| < \varepsilon_{r-1}.$$

Define

$$t_{r-1,0} = \max\{\, t \in [\tau_{r-1}, t_{r-1,1}) \mid \psi_{r-1}(t) - \|e_{r-1}(t)\| = \varepsilon_{r-1} \,\},$$

then, for all $t \in [t_{r-1,0}, t_{r-1,1}]$, we have that

$$\begin{aligned}
\psi_{r-1}(t) - \|e_{r-1}(t)\| &\le \varepsilon_{r-1},\\
\|e_{r-1}(t)\| \ge \psi_{r-1}(t) - \varepsilon_{r-1} &\ge \tfrac{\lambda_{r-1}}{2},\\
k_{r-1}(t) = \tfrac{1}{1-\varphi_{r-1}^2(t)\|e_{r-1}(t)\|^2} \ge \tfrac{\psi_{r-1}(t)}{2\varepsilon_{r-1}} &\ge \tfrac{\lambda_{r-1}}{2\varepsilon_{r-1}}.
\end{aligned}$$

We obtain, for all $t \in [t_{r-1,0}, t_{r-1,1}]$, that

$$\begin{aligned}
&\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_{r-1}(t)\|^2 = e_{r-1}^\top(t)\dot e_{r-1}(t)\\
&= e_{r-1}^\top(t)\Big(f\big(d(t),T(x)(t)\big) - k_{r-1}(t)\Gamma\big(d(t),T(x)(t)\big)e_{r-1}(t)\\
&\qquad - y_{\mathrm{ref}}^{(r)}(t) + \sum_{i=0}^{r-2}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{r-1-i}[k_i(t)e_i(t)]\Big)\\
&\le \left(M_F - \tfrac{\lambda_{r-1}^2}{4\varepsilon_{r-1}}\gamma + \sup_{t\in(0,\omega)}\|y_{\mathrm{ref}}^{(r)}(t)\| + \sum_{i=0}^{r-2} M_{i,r-1-i}\right)\|e_{r-1}(t)\|\\
&\le -L_{r-1}\|e_{r-1}(t)\|,
\end{aligned}$$

and therefore,

$$\|e_{r-1}(t_{r-1,1})\|-\|e_{r-1}(t_{r-1,0})\|=\int_{t_{r-1,0}}^{t_{r-1,1}} \tfrac{1}{2}\|e_{r-1}(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_{r-1}(t)\|^2\,\mathrm{d}t$$
$$\begin{aligned}&\le -L_{r-1}(t_{r-1,1}-t_{r-1,0})\\ &\le -|\psi_{r-1}(t_{r-1,1})-\psi_{r-1}(t_{r-1,0})|\\ &\le \psi_{r-1}(t_{r-1,1})-\psi_{r-1}(t_{r-1,0}),\end{aligned}$$

and thus we obtain $\varepsilon_{r-1}=\psi_{r-1}(t_{r-1,0})-\|e_{r-1}(t_{r-1,0})\|\le\psi_{r-1}(t_{r-1,1})-\|e_{r-1}(t_{r-1,1})\|<\varepsilon_{r-1}$, a contradiction.

Step 3: We show that $\omega=\infty$. Assume that $\omega<\infty$. Then, since e_i,k_i, $i=0,\dots,r-1$ are bounded by Step 2, it follows that the closure of the graph of $(e_0,e_1,\dots,e_{r-1})$ is a compact subset of $\mathscr{D}_{r-1}$, a contradiction. Hence $\omega=\infty$ which shows (i). Statements (ii) and (iii) are then immediate consequences of Step 2. □

Note that it follows from Theorem 3.1 that the funnel controller 5 solves the *Prescribed Performance Control Problem* as formulated for the system class in [34]. Furthermore, the funnel controller 5 is of much lower complexity than the controller proposed in [34].

In the following we derive explicit formulas for the ε_i appearing in (11) and bounds for the input u and the derivatives $e^{(i)}$ of the tracking error. We use the notation and assumptions from Theorem 3.1. For simplicity we assume that we have "finite" funnel boundaries, i.e., $\varphi_i(0)>0$ for $i=0,\dots,r-1$.

For all $i\in\{0,\dots,r-1\}$, set $\psi_i(t):=\varphi_i(t)^{-1}$ for all $t\ge 0$ and $\lambda_i:=\inf_{t\ge 0}\psi_i(t)>0$. Since $\dot\varphi_i$ is bounded and $\liminf_{t\to\infty}\varphi_i(t)>0$ we find that $\dot\psi_i$ is bounded and hence there exists a Lipschitz bound $L_i>0$ of ψ_i. For $i=0,\dots,r-2$ set

$$\varepsilon_i:=\frac{\lambda_i^2}{4\max\left\{\frac{\lambda_i}{2},L_i+\|\psi_{i+1}\|_\infty\right\}}.$$

Then ε_i satisfies (15) and hence $\psi_i(t)-\|e_i(t)\|\ge\varepsilon_i$ for all $t>0$ and $i=0,\dots,r-2$ as shown in the proof of Theorem 3.1.

For $i=r-1$ we first need to define the following constants in an iterative way. Set $N_{i,0}:=\|\psi_i\|_\infty$ for $i=0,\dots,r-1$ and

$$K_{i,0}:=\frac{N_{i,0}}{\varepsilon_i},\qquad M_{i,0}:=N_{i,0}\cdot K_{i,0}$$

for $i=0,\dots,r-2$. Therefore, (16) holds for $i=0,\dots,r-2$ and $j=0$ since

$$\begin{aligned}k_i(t)&=\frac{1}{(1-\varphi_i(t)\|e_i(t)\|)(1+\varphi_i(t)\|e_i(t)\|)}\le\frac{1}{1-\varphi_i(t)\|e_i(t)\|}\\ &=\frac{\psi_i(t)}{\psi_i(t)-\|e_i(t)\|}\le\frac{\psi_i(t)}{\varepsilon_i},\qquad t\ge 0.\end{aligned}$$

Define, for $i=0,\dots,r-2$ and $j=0,\dots,r-i-1$

$$\begin{aligned}N_{i,j}&:=N_{i+1,j-1}+M_{i,j-1},\\ L_{i,0}&:=N_{i,0}^2,\\ L_{i,j}&:=2\sum_{l=0}^{j-1}\binom{j-1}{l}N_{i,l}\cdot N_{i,j-l},\end{aligned}$$

$$\begin{aligned}\Phi_{i,0}&:=\|\varphi_i\|_\infty^2,\\ \Phi_{i,j}&:=2\sum_{l=0}^{j-1}\binom{j-1}{l}\|\varphi_i^{(l)}\|_\infty\cdot\|\varphi_i^{(j-l)}\|_\infty,\\ \Sigma_{i,j}&:=\tfrac{1}{2}\left(\Phi_{i,0}\cdot L_{i,j+1}+\Phi_{i,1}\cdot L_{i,j}+\Phi_{i,j}\cdot L_{i,1}+L_{i,0}\cdot\Phi_{i,j+1}\right)\\ &\quad+\sum_{l_1=1}^{j-1}\binom{j}{l_1}\left(\Phi_{i,l_1}\sum_{l_2=0}^{j-l_1}\binom{j-l_1}{l_2}N_{i,l_2}\cdot N_{i,j-l_1-l_2}\right.\\ &\quad\left.+L_{i,j-l_1}\sum_{l_2=0}^{l_1}\binom{l_1}{l_2}\|\varphi_i^{(l_2)}\|_\infty\cdot\|\varphi_i^{(l_1-l_2)}\|_\infty\right),\\ K_{i,j}&:=K_{i,0}^2\cdot\Sigma_{i,j-1}\\ &\quad+\sum_{l_1=1}^{j-1}\binom{j-1}{l_1}\Sigma_{i,j-l_1-1}\left(\sum_{l_2=0}^{l_1-1}\binom{l_1-1}{l_2}K_{i,l_2+1}\cdot K_{i,l_1-l_2-1}\right),\\ M_{i,j}&:=\sum_{l=0}^{j}\binom{j}{l}K_{i,l}\cdot N_{i,j-l}.\end{aligned}$$

Then cumbersome but straightforward calculations show that the above defined constants $N_{i,j}$, $K_{i,j}$, $M_{i,j}$ satisfy (16). Set

$$\hat K_{-1}:=0,\quad \hat K_i:=\sum_{j=0}^{i}M_{j,i-j}\quad\text{for }\ i=0,\dots,r-2.$$

Using the notation from the proof of Theorem 3.1 we see that any maximal solution $y:[-h,\infty)\to\mathbb{R}^m$ of (5), (1) satisfies

$$y^{(i)}(t)=e_i(t)+y_{\mathrm{ref}}^{(i)}(t)-K_{i-1}\big(t,e_0(t),\dots,e_{i-1}(t)\big),\quad t\ge 0.$$

Therefore, using (14), it follows that

$$\|e^{(i)}(t)\|\le\psi_i(t)+\hat K_{i-1},\quad t\ge 0,$$

and

$$\|y^{(i)}\|_\infty\le\|\psi_i\|_\infty+\|y_{\mathrm{ref}}^{(i)}\|_\infty+\hat K_{i-1}$$

for $i=0,\dots,r-1$. Define the compact set

$$\mathscr{B}:=\left\{\zeta\in\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r\ \middle|\ \begin{array}{r}\|\zeta_i\|_\infty\le\|\psi_{i-1}\|_\infty+\|y_{\mathrm{ref}}^{(i-1)}\|_\infty\\ +\hat K_{i-2},\ i=1,\dots,r\end{array}\right\}$$

and

$$M_1:=\sup_{\zeta\in\mathscr{B}}\|T(\zeta)\|_\infty.$$

With this we may set

$$M_F:=\sup\{\ \|f(\delta,z)\|\ \mid\ \|z\|\le M_1,\ \wedge\ \|\delta\|\le\|d\|_\infty\ \}.$$

Furthermore, let the set $\mathscr{M}$ be as in Step 2b of the proof of Theorem 3.1 and set

$$\gamma := \min_{(\delta,\eta,e)\in\mathscr{M}} e^\top \Gamma(\delta,\eta)e > 0.$$

Now we are in the position to define

$$\varepsilon_{r-1} := \frac{\gamma\cdot\lambda_{r-1}^2}{4\max\left\{\frac{\lambda_{r-1}}{2}, L_{r-1}+M_F+\|y_{\text{ref}}^{(r)}\|_\infty+\sum_{i=0}^{r-2}M_{i,r-i-1}\right\}}$$

and

$$u_{\text{bd}} := \frac{\|\psi_{r-1}\|_\infty^2}{\varepsilon_{r-1}},$$

which is an upper bound for the input u in the closed-loop system as can be concluded from the proof of Theorem 3.1. Then ε_{r-1} satisfies (17) and $\varepsilon_{r-1}\le\frac{\lambda_{r-1}}{2}$ and hence $\psi_{r-1}(t)-\|e_{r-1}(t)\|\ge\varepsilon_{r-1}$ for all $t>0$. We may now also extend the definitions of the constants $K_{i,0}$, $M_{i,0}$ to $i=r-1$; in particular, $K_{r-1,0} := \frac{N_{r-1,0}}{\varepsilon_{r-1}}$ is a bound for k_{r-1}. We summarize our findings in the following result.

Proposition 3.2. *Use the notation and assumptions from Theorem 3.1 and assume that $\varphi_i(0)>0$ for $i=0,\dots,r-1$. Then the following statements are true for any maximal solution $y:[-h,\infty)\to\mathbb{R}^m$ of* (5), (1)*:*

(i) (11) *holds with*

$$\varepsilon_i = \frac{\lambda_i^2}{4\max\left\{\frac{\lambda_i}{2}, L_i+\|\psi_{i+1}\|_\infty\right\}},\quad i=0,\dots,r-2,$$
$$\varepsilon_{r-1} = \frac{\gamma\cdot\lambda_{r-1}^2}{4\max\left\{\frac{\lambda_{r-1}}{2}, L_{r-1}+M_F+\|y_{\text{ref}}^{(r)}\|_\infty+\sum_{i=0}^{r-2}M_{i,r-i-1}\right\}},$$

(ii) $k_i(t)\le K_{i,0}$ *and* $\|e^{(i)}(t)\|\le\varphi_i(t)^{-1}+\hat{K}_{i-1}$ *for all* $t\ge 0$ *and all* $i=0,\dots,r-1$,

(iii) $\|u\|_\infty\le u_{\text{bd}}$.

Proposition 3.2 may be exploited for the design of suitable funnel functions $\varphi_0,\dots,\varphi_{r-1}$ in the presence of control constraints in the following way: If a bound $\hat{u}$ is given so that the desired control $u(\cdot)$ (of the form as in (5)) must satisfy $\|u(t)\|\le\hat{u}$ for all $t\ge 0$, then, if possible, $\varphi_0,\dots,\varphi_{r-1}$ must be chosen such that $u_{\text{bd}}\le\hat{u}$. Of course, there is a minimum feasibility requirement on the control depending on the system parameters, i.e., a lower bound for u_{bd}. For instance, if $r=1$ and we choose φ_0 to be constant, then $L_0=0$,

$$\varepsilon_0 = \frac{\gamma\varphi_0^2}{4\max\left\{\frac{\varphi_0}{2}, M_F+\|\dot{y}_{\text{ref}}\|_\infty\right\}},$$
$$M_F = M_F(\varphi_0)$$
$$=\sup\left\{\|f(\delta,z)\|\,\middle|\,\|\delta\|\le\|d\|_\infty\wedge\|z\|\le\sup_{\|\zeta\|_\infty\le\varphi_0^{-1}+\|y_{\text{ref}}\|_\infty}\|T(\zeta)\|_\infty\right\}$$

and hence

$$u_{\text{bd}} = \frac{\varphi_0^2}{\varepsilon_0} = \frac{4\max\left\{\frac{\varphi_0}{2}, M_F(\varphi_0)+\|\dot{y}_{\text{ref}}\|_\infty\right\}}{\gamma} \ge \frac{4\left(M_F^*+\|\dot{y}_{\text{ref}}\|_\infty\right)}{\gamma},$$

where

$$M_F^* := \sup\left\{\|f(\delta,z)\|\,\middle|\,\|z\|\le\sup_{\|\zeta\|_\infty\le\|y_{\text{ref}}\|_\infty}\|T(\zeta)\|_\infty\wedge\|\delta\|\le\|d\|_\infty\right\}.$$

Obviously, $\frac{\varphi_0}{2}$ is monotonically increasing in φ_0 and $M_F(\varphi_0)$ is monotonically non-increasing in φ_0, thus in the choice of φ_0 there is trade-off between these two quantities.

4. Simulations

4.1. Mass on car system

To demonstrate the application of our controller, we consider an example of a mass-spring system mounted on a car from [30], see Fig. 3. The mass $m_2[kg]$ moves on a ramp which is inclined by the angle α [rad] and mounted on a car with mass $m_1[kg]$, for which it is possible to control the force $u=F[N]$ acting on it. The equations of motion for the system are given by

$$\begin{bmatrix} m_1+m_2 & m_2\cos\alpha \\ m_2\cos\alpha & m_2\end{bmatrix}\begin{pmatrix}\ddot{x}(t)\\ \ddot{s}(t)\end{pmatrix}+\begin{pmatrix}0\\ ks(t)+d\dot{s}(t)\end{pmatrix}=\begin{pmatrix}u(t)\\0\end{pmatrix}, \tag{18}$$

where $x[m]$ is the horizontal car position and $s[m]$ the relative position of the mass on the ramp. The constants $k[N/m]$, $d[Ns/m]$ are the coefficients of the spring and damper, resp. The output of the system is given by the horizontal position of the mass on the ramp,

$$y(t) = x(t)+s(t)\cos\alpha.$$

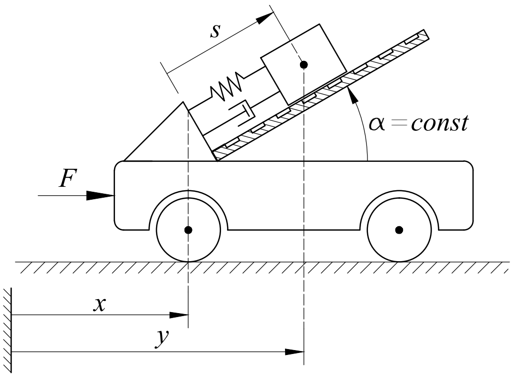

Figure 3: Mass on car system.

The reference trajectory is $y_{\text{ref}}(t)=\cos t[m]$. System (18) can be reformulated such that it belongs to the class (1), see [30], with a relative degree r depending on the angle $\alpha[rad]$ and the damping $d[Ns/m]$. We consider two cases.

Case 1: If $0 < \alpha < \frac{\pi}{2}$, see Fig. 3, then system (18) has relative degree $r = 2$ and the high-frequency gain matrix reads $\Gamma = \frac{\sin^2\alpha}{m_1+m_2\sin^2\alpha} > 0$; for the simulation, we choose the parameters $m_1 = 4[kg], m_2 = 1[kg], k = 2[N/m], d = 1[Ns/m]$, the initial values $x(0) = 0$, $\dot{x}(0) = 0$, $s(0) = 0$, $\dot{s}(0) = 0$ and $\alpha = \frac{\pi}{4}$. For the controller (5) we choose the funnel functions

$$\varphi_0(t) = (5e^{-2t} + 0.1)^{-1}, \qquad \varphi_1(t) = (10e^{-2t} + 0.5)^{-1},$$

and obviously the initial errors lie within the respective funnel boundaries, i.e., (10) is satisfied, thus Theorem 3.1 yields that funnel control is feasible. We compare the controller (5) with the proportional-derivative funnel controller (8) proposed in [12], which has been explained in Remark 2.4, and choose the same funnel functions φ_0, φ_1 for it. These functions satisfy the compatibility condition (9) and hence the controller (8) may be applied to (18) by [12].

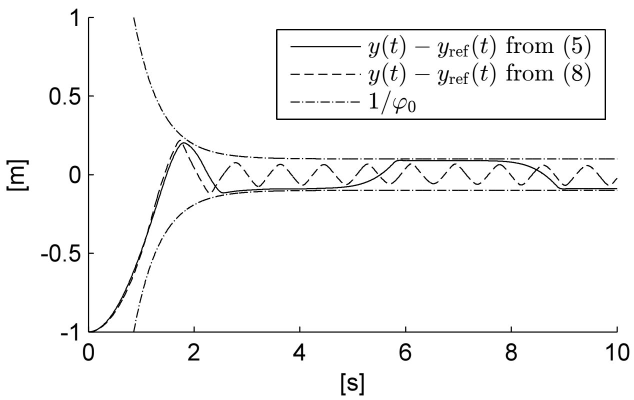

Fig. 4a: Funnel and tracking errors

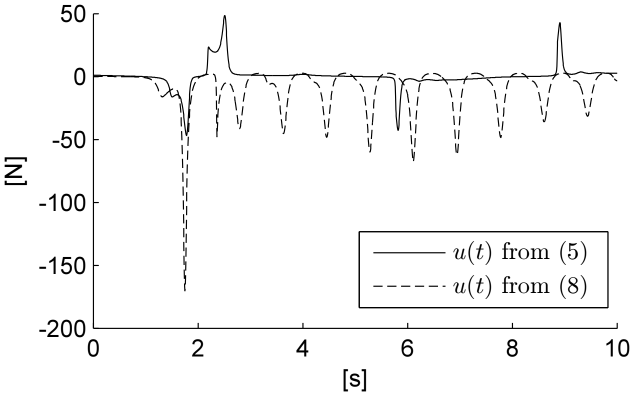

Fig. 4b: Input functions

Figure 4: Simulation of the controllers (5) and (8) for the mass on car system (18) with $\alpha = \frac{\pi}{4}$.

The simulation of the controllers (5) and (8) applied to (18) over the time interval $[0, 10]$ has been performed in MATLAB (solver: `ode45`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 4. Fig. 4a shows the tracking errors corresponding to the two different controllers applied to the system, while Fig. 4b shows the respective input functions generated by them. It can be seen that our proposed funnel controller (5) requires less input action than the controller (8), both in magnitude and over time. For instance, in the time interval $[3, 5.5]$ there is no input action generated by (5), but several (large) oscillations generated by (8). It seems that the controller (5) better exploits the inherent system properties and thus requires less input action than the controller proposed in [12].

Case 2: If $\alpha = 0$ and $d \neq 0$, see Fig. 5, then system (18) has relative degree $r = 3$ and high-frequency gain matrix $\Gamma = \frac{d}{m_1 m_2} > 0$. For the simulation, we choose the parameters $m_1 = 4[kg], m_2 = 1[kg], k = 2[N/m], d = 1[Ns/m]$ and the initial values $x(0) = 0$, $\dot{x}(0) = 0$, $s(0) = 0$, $\dot{s}(0) = 0$.

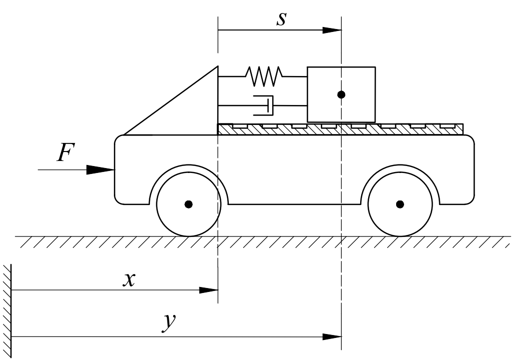

Figure 5: Mass on car system with $\alpha = 0$.

For the illustration of the controller (5) we choose the funnel functions

$$\varphi_0(t) = (5e^{-2t} + 2)^{-1}, \quad \varphi_1(t) = \varphi_2(t) = (ae^{-t} + b)^{-1}$$

with the three sets of parameter values

$$\begin{aligned} \text{C1}: &\quad a = 1.4,\ b = 0.05, \\ \text{C2}: &\quad a = 5,\ b = 0.05, \\ \text{C3}: &\quad a = 1.4,\ b = 0.5; \end{aligned}$$

the initial errors lie within the respective funnel boundaries, i.e., conditions (10) are satisfied, thus Theorem 3.1 yields that funnel control is feasible.

The simulation of the controller (5) with the different parameter sets C1–C3 applied to the relative degree 3 system (18) with $\alpha = 0$ over the time interval $[0, 10]$ has been performed in MATLAB (solver: `ode45`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 6. Fig. 6a shows the tracking errors corresponding to the different controllers applied to the system, while Fig. 6b shows the respective input functions generated by them. The difference in the performance of the controllers is discussed in the next subsection.

We did not provide the comparison of the controller (5) with the backstepping funnel controller (7) proposed in [20] here. A

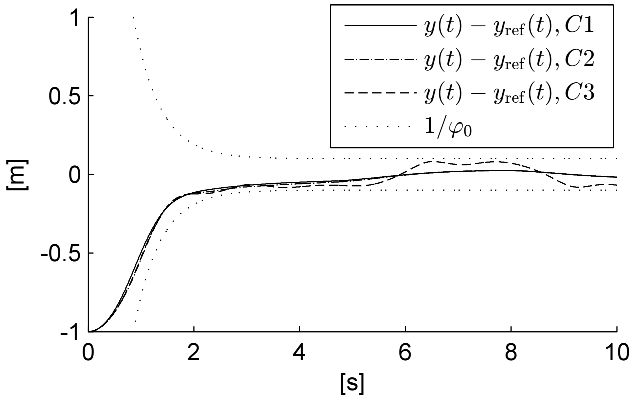

Fig. 6a: Funnel and tracking errors

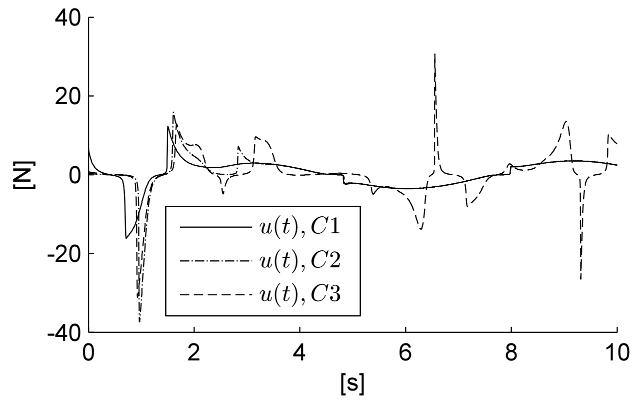

Fig. 6b: Input functions

Figure 6: Simulation of the controllers (5) for the mass on car system (18) with $\alpha = 0$ and different sets of funnel functions φ_1, φ_2.

simulation of (7) for the system (18) with funnel function $\varphi = \varphi_0$ is not feasible due to numerical issues, cf. the explanation in Remark 2.3.

4.2. Influence of design parameters

In this section we discuss the influence of the design parameters of the funnel controller (5). Of particular interest is the influence of the choice of the funnel functions φ_i in (5) on the controller performance, that means the maximal absolute value of the input u and its oscillation behavior. We assume that the choice of φ_0 is done by the designer based on specific objectives for the transient behavior of the tracking error such as desired tracking accuracy, and the choice of φ_i is free apart from the initial conditions (10) for $i = 1, \ldots, r-1$. In principle, based on the explicit formula for u_{bd} derived in Proposition 3.2, a minimization of this bound over all possible funnel functions could be performed. This is a highly complicated venture left for future research. However, as a rule of thumb, we may conclude that the performance funnels $\mathscr{F}_{\varphi_i}$ corresponding to φ_i should be chosen as tight as possible, i.e., starting as close to $\|e_i(0)\|$ as possible and then decaying to a small value.

In order to illustrate this we consider Case 2 of the mass on car system (18) and discuss the resulting controller performance for the choices of parameter values C1–C3. The case C1 represents an "optimal" choice of the parameters as far as the experiments show. It can be see in Fig. 6b that increasing the value of a as in C2 results in a peaking behavior of the input u for small t, while increasing the value of b as in C3 leads to possible peaks at later time instants, but smaller maximal input values than in C2 in general. Furthermore, the distance of the tracking error to the funnel boundary seems to depend on the parameter b; in case C3 (for larger b), the error gets closer to the boundary than in cases C1 and C2. These observations have been confirmed in several other experiments.

In order to improve the performance of the controller and reduce unnecessary large control actions one may use alternative gain functions in (5) as discussed e.g. in [21]. For instance, using the future distance to the future funnel boundary instead of the vertical distance to the funnel boundary as in (5) may increase the ability of the controller (5) to avoid large control values.

4.3. Nonlinear MIMO system

To illustrate the funnel controller (5) for a nonlinear multi-input, multi-output system we consider an example of a robotic manipulator from [13], see also [25, p. 77], as depicted Fig.7. The robotic manipulator is planar, rigid, with revolute joints and has two degrees of freedom.

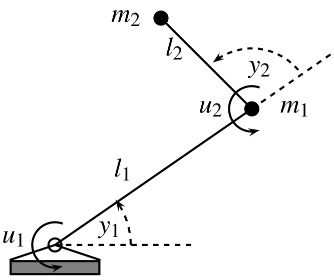

Figure 7: Planar rigid revolute joint robotic manipulator.

The two joints are actuated by $u_1[Nm]$ and $u_2[Nm]$. We assume that the links are massless, have lengths $l_1[m]$ and $l_2[m]$, resp., and point masses $m_1[kg]$ and $m_2[kg]$ are attached to their ends. The two outputs are the joint angles $y_1[rad]$ and $y_2[rad]$ and the equations of motion are given by (see also [33, pp. 259])

$$M(y(t))\ddot{y}(t) + C(y(t), \dot{y}(t))\dot{y}(t) + \mathbf{g}(y(t)) = u(t) \qquad (19)$$

with initial value $(y(0), \dot{y}(0)) = (0, 0)$, inertia matrix

$$M : \mathbb{R}^2 \to \mathbb{R}^{2\times 2},\ (y_1, y_2) \mapsto M(y_1, y_2) := \begin{bmatrix} m_1 l_1^2 + m_2(l_1^2 + l_2^2 + 2l_1 l_2 \cos(y_2)) & m_2(l_2^2 + l_1 l_2 \cos(y_2)) \\ m_2(l_2^2 + l_1 l_2 \cos(y_2)) & m_2 l_2^2 \end{bmatrix}$$

centrifugal and Coriolis force matrix

$$C:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^{2\times 2},\ (y_1,y_2,v_1,v_2)\mapsto C(y_1,y_2,v_1,v_2):=\begin{bmatrix}-2m_2l_1l_2\sin(y_2)v_1 & -m_2l_1l_2\sin(y_2)v_2\\ -m_2l_1l_2\sin(y_2)v_1 & 0\end{bmatrix},$$

and gravity vector

$$\mathbf{g}:\mathbb{R}^2\to\mathbb{R}^2,\ (y_1,y_2)\mapsto \mathbf{g}(y_1,y_2):= g\begin{pmatrix} m_1l_1\cos(y_1)+m_2(l_1\cos(y_1)+l_2\cos(y_1+y_2))\\ m_2l_2\cos(y_1+y_2)\end{pmatrix}$$

where $g=9.81[m/s^2]$ is the acceleration of gravity. If we multiply system (19) with $M(y(t))^{-1}$, which is pointwise positive definite, from the right we see that the resulting system belongs to the class (1) with $r=m=2$.

For the simulation, we choose the parameters $m_1=m_2=1[kg]$, $l_1=l_2=1[m]$ and the reference trajectories $y_{\text{ref},1}(t)=\sin t[rad]$ and $y_{\text{ref},2}(t)=\sin 2t[rad]$. For the controller (5) we choose the funnel functions

$$\varphi_0(t)=(e^{-2t}+0.1)^{-1},\quad \varphi_1(t)=(3e^{-2t}+0.1)^{-1}.$$

The initial errors lie within the respective funnel boundaries, i.e., conditions (10) are satisfied, thus Theorem 3.1 yields that funnel control is feasible. We compare the controller (5) with the MIMO funnel controller proposed in [13], that is (already fixing the gain scaling functions)

$$\begin{aligned} u(t)&=-M(y(t))\big(K_0(t)^2e(t)+K_0(t)K_1(t)\dot{e}(t)\big),\\ K_i(t)&=\operatorname{diag}\left(\frac{1}{1-\varphi_i(t)|e_1^{(i)}(t)|},\frac{1}{1-\varphi_i(t)|e_2^{(i)}(t)|}\right),\quad i=0,1\end{aligned}\tag{20}$$

and we choose the same funnel functions φ_0,φ_1 for it. The controller (20) is a modification of (8), first introduced in [7] for SISO systems and tailored to MIMO systems with mass matrix in [13]. We remark that there is a typo in the controller formula [13, (8)], the sign of the input u must be the opposite.

The simulation of the controllers (5) and (20) applied to (19) over the time interval [0,10] has been performed in MATLAB (solver: *ode45*, rel. tol: 10^{-14}, abs. tol: 10^{-10}) and is depicted in Fig. 8 (tracking error components) and Fig. 9 (input components). It can be seen that the funnel controller (5) outperforms the controller (20) as it generates a smaller maximal control action and does not "oscillate" as (20) does e.g. in the interval $[4,6]$. Moreover, we stress that the controller (20) requires knowledge of the mass matrix $M(\cdot)$ of the system (19) and is specifically constructed for systems with strict relative degree two. On the other hand, knowledge of $M(\cdot)$ is not necessary for the control strategy (5).

4.4. Comparison with the bang-bang funnel controller

We finally compare the funnel controller (5) with the bang-bang funnel controller developed in [26]. We consider the academic example presented in [26], that is the nonlinear relative

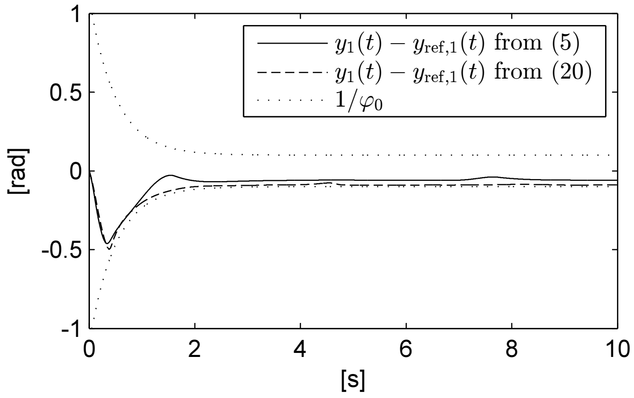

Fig. 8a: Funnel and first tracking error components

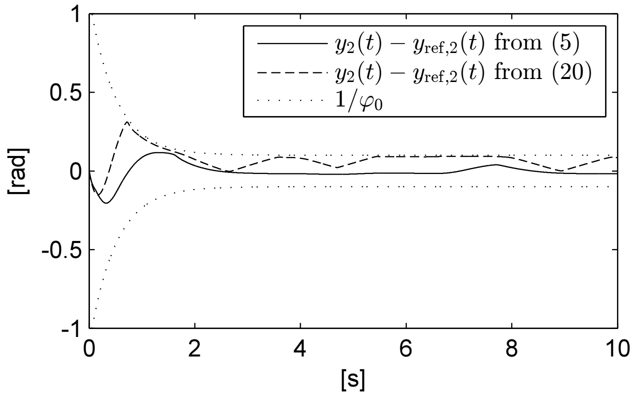

Fig. 8b: Funnel and second tracking error components

Figure 8: Funnel and tracking errors for the controllers (5) and (20) applied to (19).

degree 4 system

$$\begin{aligned} y^{(4)}(t)&=z(t)y^{(3)}(t)^2+e^{z(t)}u(t),\\ \dot{z}(t)&=z(t)\big(a-z(t)\big)\big(z(t)+b\big)-cy(t)\end{aligned}\tag{21}$$

with initial values

$$z(0)=0,\quad y^{(i)}(0)=y_{\text{ref}}^{(i)}(0),\ i=0,\dots,3,$$

where we choose the reference signal $y_{\text{ref}}(t)=5\sin t$. For the simulation we choose the parameters

$$a=0.09,\quad b=0.05,\quad c=0.008.$$

For the controller (5) we choose the constant funnel functions

$$\varphi_0(t)=1,\quad \varphi_1(t)=10,\quad \varphi_2(t)=10,\quad \varphi_3(t)=10.$$

The funnel φ_0 for the tracking error is the same as in [26], but apart from that we have chosen $\varphi_1,\dots,\varphi_3$ so that the corresponding performance funnels are tighter than in [26]; this is allowed in our framework, but in [26] several complicated compatibility assumptions require the funnel boundaries to be large

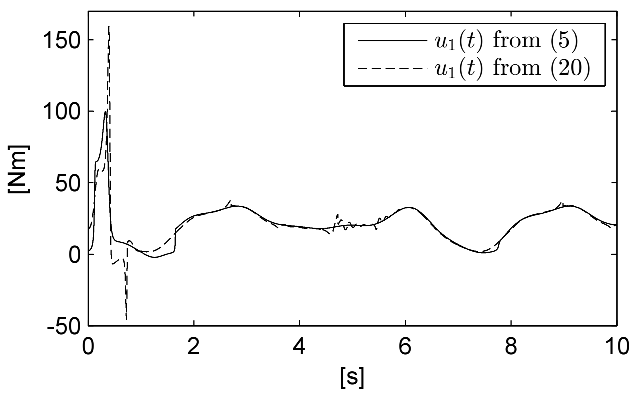

Fig. 9a: First input components

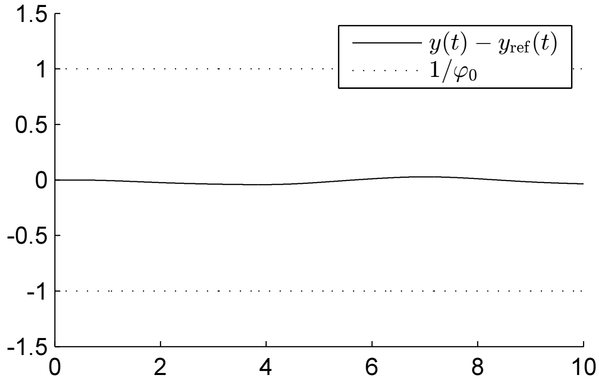

Fig. 10a: Funnel and tracking error

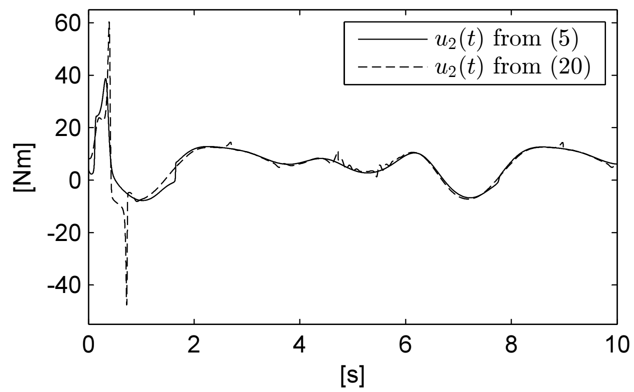

Fig. 9b: Second input components

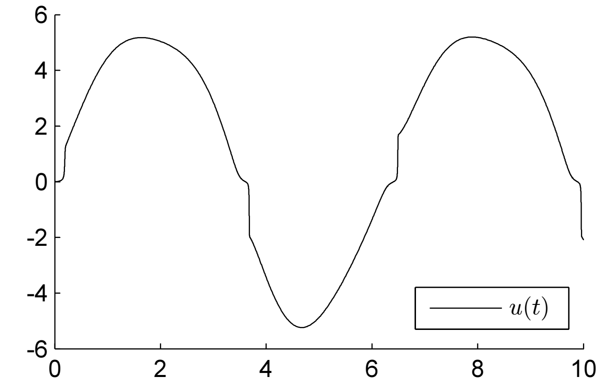

Fig. 10b: Input function

Figure 9: Input functions for the controllers (5) and (20) applied to (19).

Figure 10: Simulation of the controller (5) for the system (21).

enough. We also stress that the controller design (5) is quite different from the bang-bang funnel controller in [26].

The simulation of the controller (5) applied to (21) over the time interval [0,10] has been performed in MATLAB (solver: *ode15s*, rel. tol: 10^{-14}, abs. tol: 10^{-10}), see Fig. 10.

It can be seen that the funnel controller (5) generates a maximal control action of approximately 5, while for the bang-bang funnel controller in [26] the value is around 254. Obviously, the controller (5) achieves a better performance than the controller proposed in [26].

5. Conclusion

In the present paper, we proposed a new funnel controller for nonlinear systems with arbitrary known relative degree and input-to-state stable internal dynamics. We proved that this controller, which involves derivatives of the tracking error, achieves tracking of a sufficiently smooth reference trajectory with prescribed transient performance. An explicit upper bound for the input function resulting from the control law was derived and based on that the influence of the controller parameters was briefly discussed. We have illustrated the performance of our controller in comparison with other approaches by simulations of practically relevant mechanical systems. We stress that, although the backstepping funnel controller presented in [20] is proved to work for systems with arbitrary relative degree, it does not seem to be practically realizable for systems with relative degree larger than three, cf. [8, Sec. 4.4.3]. Furthermore, compared to the bang-bang funnel controller developed in [26], our approach is not restricted to SISO systems and does not involve complicated compatibility assumptions; additionally, the simulations reveal that our controller outperforms the bang-bang funnel controller. Therefore, the controller (5) seems to be a favorable choice for tracking with prescribed transient behavior for systems where the derivatives of the output are available.

Of course, in several applications the latter condition is not satisfied, and it may even be hard to obtain suitable estimates of the output derivatives. The solution of this issue is a topic of future research and a first approach to circumvent these problems using a “funnel pre-compensator” has been developed in [3, 4] for systems with relative degree $r = 2$ or $r = 3$.

Acknowledgement

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